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BULLETIN

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

VOLUME 8

1912

WASHINGTON
GOVERNMENT PRINTING OFFICE
1913

1991-3

CONTENTS OF VOLUME 8

NO. 1

[Issued January 1, 1912.]

Page.

169. FORMULAS AND TABLES FOR THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE (REVISED) ..... *E. B. Rosa and F. W. Grover* 1

NO. 2

[Issued June 15, 1912.]

170. THE CORRECTION FOR EMERGENT STEM OF THE MERCURIAL THERMOMETER ..... *Edgar Buckingham* 239

171. A DETERMINATION OF THE INTERNATIONAL AMPERE IN ABSOLUTE MEASURE ..... *E. B. Rosa, N. E. Dorsey, and J. M. Miller* 269

172 DEFLECTION POTENTIOMETERS FOR CURRENT AND VOLTAGE MEASUREMENTS ..... *H. B. Brooks* 395

173. OUTLINE OF DESIGN OF DEFLECTION POTENTIOMETERS, WITH NOTES ON THE DESIGN OF MOVING-COIL GALVANOMETERS ..... *H. B. Brooks* 419

NO. 3

[Issued November 1, 1912.]

174. THE DETERMINATION OF TOTAL SULPHUR IN INDIA RUBBER ..... *C. E. Waters and J. B. Tuttle* 445

175. THE MEASUREMENT OF THE INDUCTANCES OF RESISTANCE COILS ..... *F. W. Grover and H. L. Curtis* 455

176. LUMINOUS PROPERTIES OF ELECTRICALLY CONDUCTING HELIUM GAS. II. REPRODUCIBILITY ..... *P. G. Nutting* 487

177. RESISTANCE COILS FOR ALTERNATING CURRENT WORK ..... *H. L. Curtis and F. W. Grover* 495

178. THE HYDROLYSIS OF SODIUM OXALATE AND ITS INFLUENCE UPON THE TEST FOR NEUTRALITY ..... *William Blum* 519

179. WAVE-LENGTHS OF NEON ..... *I. G. Priest* 539

180. ON THE DEDUCTION OF WIEN'S DISPLACEMENT LAW .. *Edgar Buckingham* 545

181. THE FOUR-TERMINAL CONDUCTOR AND THE THOMSON BRIDGE ..... *Frank Wenner* 559

NO. 4

[Issued March 1, 1913.]

182. STANDARDIZATION OF POTASSIUM PERMANGANATE SOLUTION BY SODIUM OXALATE ..... *R. S. McBride* 611

183. BENZOIC ACID AS AN ACIDIMETRIC STANDARD ..... *G. W. Morey* 643

184. A TUBULAR ELECTRODYNAMOMETER FOR HEAVY CURRENTS .. *P. G. Agnew* 651

185. THERMOMETRIC LAG ..... *D. R. Harper, 3d* 659

186. DETERMINATION OF MANGANESE AS SULPHATE AND BY THE SODIUM BISMUTHATE METHOD ..... *William Blum* 715

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DEPARTMENT OF COMMERCE AND LABOR

Vol. 8 No. 1

BULLETIN

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

JANUARY 1, 1912

CONTENTS

Page

Formulas and Tables for the Calculation of Mutual and Self-Inductance (Revised) . . . . . *Edward B. Rosa and F. W. Grover* 1–237

WASHINGTON
GOVERNMENT PRINTING OFFICE
1912

DEPARTMENT OF COMMERCE AND LABOR

Vol. 8 No. 1

BULLETIN

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

Issued January 1, 1912

CONTENTS

Page

Formulas and Tables for the Calculation of Mutual and Self-Inductance (Revised) . . . . . *Edward B. Rosa and F. W. Grover* 1–237

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CORRECTION SHEET TO "FORMULAS AND TABLES FOR THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE"

By E. B. Rosa and Frederick W. Grover

[First published in the Bulletin of the Bureau of Standards, vol. 8, No. 1, pp. 1 to 237, 1911. Reprinted as separate publication, Scientific Paper No. 169, Jan. 1, 1911, and in revised form in 1912]

It is suggested that those who have received either the Bulletin or the separate publication, see that the following corrections are made:

CORRECTIONS TO 1911 EDITION (BOTH IN BULLETIN AND REPRINT)

Page 9:

In the denominator of k', equation [4] read $(A+a)^2$ instead of $(A+a^2)$.

Page 11:

In value of k, equation [9a], read d^2 instead of d.

Page 68:

In second line of equation [47] read $\frac{8\pi\ (A+a,}{3}$ instead of $\frac{32\ Aa}{3\ (A+a)}$.

Page 73:

For factor outside the souare brackets read $\frac{(2\ Aam)^{3/2}}{2}$ instead of $m^{3/2}\sqrt{2Aa}$

Page 129:

In tenth line from bottom read 101809990 cm.

55375°—16

CORRECTIONS TO ALL EDITIONS

Page 9:

In last equation (unnumbered) put $(n+2)$ for $(2n-1)$.

Page 33:

In the formula before equation (20) the second term should read $(b_1^2+b_2^2)\frac{d^2M_o}{dX^2}$.

Page 138:

Insert the following in place of the first paragraph:

The values of y_1 and y_2 are given in Table VI, page 196, as functions of the ratio of the breadth b to the depth c. The function y_1 is unchanged when b/c and c/b are interchanged. This is not, however, true in the case of the function y_2, which approaches infinity as $x=b/c$ approaches zero. Stefan gave values of y_2 for values of x greater than unity only. These values are reproduced in Table VI, where y_2 is given as a function of c/b, for values between zero and unity; that is, for coils having a breadth greater than their depth. For the case b/c less than unity, values have been calculated and are also included in the table.

Page 140:

Omit the last paragraph before "Maxwell's Correction Formula."

Page 142:

Second line from the bottom, read $n=100$, instead of $n=10$.

Page 168:

In the last row of the table, read 0.22313a and 0.22313 instead of 0.22315a and 0.22315, respectively.

Page 179:

In fifth line, move decimal point one place to the right, so that $x=0.2142\rho\sqrt{f}$.

Page 190:

Put as heading of first column

$\gamma=\sin^{-1}\frac{\sqrt{r_1^2-r_2^2}}{r_1}$.

***Page** 196:*

Replace Table VI by the following:

TABLE VI

Table of Constants for Stefan's Formula [90]

b/c or c/b	y_1	c/b	y_2	b/c	y_3
0.00	0.50000	0.00	0.1250	0.00	Infinite.
.05	.54899	.05	.1269	.05	239.43
.10	.59243	.10	.1325	.10	60.231
.15	.63102	.15	.1418	.15	27.020
.20	.66520	.20	.1548	.20	15.378
.25	.69532	.25	.1714	.25	9.9765
.30	.72172	.30	.1916	.30	7.0327
.35	.74469	.35	.2152	.35	5.2502
.40	.76454	.40	.2423	.40	4.0876
.45	.78154	.45	.2728	.45	3.2861
.50	.79600	.50	.3066	.50	2.7093
.55	.80815	.55	.3437	.55	2.2798
.60	.81823	.60	.3839	.60	1.9509
.65	.82648	.65	.4273	.65	1.6931
.70	.83311	.70	.4739	.70	1.4871
.75	.83831	.75	.5234	.75	1.3198
.80	.84225	.80	.5760	.80	1.1819
.85	.84509	.85	.6317	.85	1.0669
.90	.84697	.90	.6902	.90	.9698
.95	.84801	.95	.7518	.95	.8872
1.00	.84834	1.00	.8162	1.00	.8162

***Page** 200:*

For heading of Table X read:

Values of the correction term B_s depending on the number of turns of square conductor on the equivalent single layer coil shown in Fig. 40. In the headings of the columns of Table X, follow "number of turns" by b/c.

***Page** 201:*

Read for log 81; 4.394 449 155.

***Page** 229:*

For heading of first column read $\frac{2l}{\rho}$

FORMULAS AND TABLES FOR THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE

By Edward B. Rosa and Frederick W. Grover

CONTENTS

(See also Index)

	Page
INTRODUCTION	5
1. MUTUAL INDUCTANCE OF TWO COAXIAL CIRCLES:	
(a) Formulas—	
1, 2, Maxwell's formulas in elliptic integrals	6, 7
3, Formulas for the elliptic integrals F and E	8
4, Maxwell's formula in terms of F_0 and E_0	9
5, Maxwell's formula expressed as a k series	9
6, Maxwell's formula expressed as series in k_1	10
7, Weinstein's k' series formula	10
8, 9, Nagaoka's q series formulas	11
10, 11, Maxwell's series formulas for any two coaxial circles	13
12, Maxwell's series formula for circles of equal radii	13
13, Coffin's extension of (12)	14
14, Series formula for any two circles, extension of (10)	14
15, Formula for coplanar circles, derived from (14)	14
16, Havelock's formula (extended) for circles near together	16
17, Havelock's formula, for circles far apart	16
18, Mathy's formula (corrected)	17
19, Simple form of Mathy's formula for the case $r_1^2=2\,r_2^2$	18
(b) Choice of formulas	19
(c) Summary of formulas	19
(d) Examples 1–19, illustrating the formulas for the mutual inductance of coaxial circles	20
2. MUTUAL INDUCTANCE OF TWO COAXIAL COILS:	
(a) Formulas—	
20, Rowland's formula, equal radii, but unequal section	33
21, Rowland's formula, equal radii and equal section	33
22, Differential coefficients in (20) and (21)	34
23, Rayleigh's formula of quadratures	34
24, Same for coils of equal radii and equal section	35
25, 26, Formulas for errors of formulas (21) and (24)	36, 37
27, 28, Lyle's formulas	38
29–31, Rosa's formulas for two equal coils	39
32, Same for coils of equal radii, but unequal sections	39
33, The Rosa-Weinstein formula	40
34, Formula in terms of self-inductance	41
35, Geometric mean-distance formula	42
(b) Choice of formulas	43
(c) Examples 20–33, illustrating the formulas for the mutual inductance of two coaxial coils	44

3. MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS:
(a) Formulas— Page
36, 37, Maxwell's formulas for two concentric solenoids of equal length 53, 55
38, Havelock's formula for concentric solenoids of equal length 56
39, Ròiti's formula, inner solenoid the shorter, coils concentric 57
40, Gray's formula, coaxial coils not concentric 59
41, Formula for the Gray electrodynamometer 60
42, Searle and Airey's formula, inner coil the shorter, coils concentric 61
43, Rosa's extension of (42) 62
44, Cohen's absolute formula in elliptic integrals, coils concentric 64
45, Nagaoka's absolute formula for any coaxial solenoids 65
46, 47, Russell's formulas 67, 68
(b) Mutual inductance of a short secondary, on the outside of a long primary 68
(c) Formulas for single layer coils of equal radii—
48, Rosa's formula 70
49, Rosa's formula, coils of equal breadth 70
50, Rosa-Weinstein formula 71
51, 52, Mutual inductance by means of self-inductance formulas. 72
(d) Other formulas and Olshausen's absolute formula 73
(e) Choice of formulas 73
(f) Examples 34–47, illustrating the formulas for the mutual inductance of coaxial solenoids 77
4. MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SINGLE-LAYER COIL:
(a) Formulas—
53, Lorenz's series formula 98
54, Jones's formula in elliptic integrals 99
55, Campbell's form of (54) 100
56, 57, Rosa's series formulas 101, 102
(b) Examples, 48–51, illustrating the formulas for the mutual inductance of a circle and a coaxial single-layer coil 103
5. THE SELF-INDUCTANCE OF A CIRCULAR RING OF CIRCULAR SECTION:
(a) Formulas—
58, 59, Kirchhoff's formulas 110
60, 61, Maxwell's formulas 110
62, Maxwell's formula when the ring is a hollow, circular thin tube 111
63, Rayleigh and Niven's formula 111
64, Tube bent into a circle 112
65, Tube of infinitely thin walls 112
66, Wien's formula, current density proportional to the distance from the axis of the ring 113
67, Thomson's formula for a ring of elliptical section 113
(b) Examples 52–56, illustrating the formulas for the self inductance of a circular ring of circular section 114
6. THE SELF-INDUCTANCE OF A SINGLE LAYER COIL OR SOLENOID:
(a) Formulas—
68, Formula for infinitely long solenoid 116
69, 70, Rayleigh and Niven's formulas 116, 117
71, Coffin's series formula, extension of (69) 117
72, 73, Lorenz's absolute formula in elliptic integrals 118
74, Formula for use with Table IV, derived from (73) 118
75–78, Nagaoka's q series formulas 119–121
79, Webster-Havelock formula 121
80, Rosa's formula for obtaining L from L_s 122
81, Summation formula 123

6. THE SELF-INDUCTANCE OF A SINGLE LAYER COIL OR SOLENOID—Continued.
(*a*) Formulas—Continued. Page
82, Strasser's formula 124
83, Formula for L_s of a toroidal coil of rectangular section. 124
84, Rosa's formula for obtaining L from L_s for a toroidal coil 125
(*b*) Choice of formulas 125
(*c*) Examples 57–63, illustrating the formulas for the self-inductance of single layer coils or solenoids 126
7. THE SELF-INDUCTANCE OF A CIRCULAR COIL OF RECTANGULAR SECTION:
(*a*) Formulas—
85, 86, Maxwell's approximate formulas 135, 136
87, Perry's approximate formula 136
88, 89, Weinstein's formulas for L_u 137
90, Stefan's formula for L_u 137
91, Rosa's formula for obtaining L_u from L_s 138
92, Cohen's approximate formula 140
93, Maxwell's formula (revised) for obtaining L from L_u .. 140
(*b*) Choice of formulas 142
(*c*) Examples 64–69, illustrating the formulas for the self-inductance of a circular coil of rectangular section 142
8. SELF AND MUTUAL INDUCTANCE OF LINEAR CONDUCTORS:
(*a*) Formulas—
94–97, Self-inductance of a straight cylindrical wire 150, 151
98, 99, Mutual inductance of two parallel wires 151
100, 101, Self-inductance of a return circuit of two parallel conductors 151, 152
102, Mutual inductance of two linear conductors in the same straight line 152
103, 104, Self-inductance of a straight rectangular bar 152, 153
105, 106, Self-inductance of a square, circular section 154
107, Self-inductance of a rectangle, circular section 155
108, Self-inductance of a rectangle, rectangular section 155
109, 110, Self-inductance of a square, rectangular section 155
111, 112, Mutual inductance of two equal, parallel rectangles or squares 155, 156
113, 114, Self-inductance of a straight, thin tape 156
115, Mutual inductance of parallel tapes in the same plane .. 157
116, Self-inductance of return circuit of two parallel tapes in the same plane 157
117, 118, Self-inductance of return circuit of parallel tapes, not in the same plane 157
119, 120, Self-inductance of a return circuit of two concentric conductors 158, 159
121, 122, Self-inductance of multiple conductors 159
(*b*) Examples 70–81, illustrating the formulas for the self and mutual inductance of linear conductors 159
9. FORMULAS FOR GEOMETRICAL AND ARITHMETICAL MEAN DISTANCES:
123, G. M. D. of a straight line from itself 167
124, G. M. D. of a rectangular area 167
125, G. M. D. of a square area 167
126, G. M. D. of a circular area 167
127, G. M. D. of an elliptical area 167
128, Approximate formula for G. M. D. of a rectangle 167
129, G. M. D. of an annular ring 168
130, 131, G. M. D. of a straight line from another line of the same length in the same straight line 168
132, 133, G. M. D. of a line from an equal parallel line 169
134, G. M. D. of a circular line from itself 169
135, G. M. D. of a point from an annular area 169

9. FORMULAS FOR GEOMETRICAL AND ARITHMETICAL MEAN DISTANCES—Continued. Page

136, G. M. D. of two circular areas from one another 170
137, A. M. D. of a line from itself 171
138, A. M. S. D. of a line from itself 171
139, 140, A. M. D. and A. M. S. D. of a circular line from itself .. 171
141, A. M. D. of an external point from the circumference of a circle 172
142, A. M. D. of an external point from a circular area 172

10. HIGH-FREQUENCY FORMULAS AND EXAMPLES:

(a) Formulas—

143, 144, Kelvin's formula for the resistance and inductance of straight cylindrical wires 174
144a, Formulas defining W, Y, and Z 174
145–147, Formulas for the absolute and fractional change of inductance of straight wires with frequency 174
148, Defining formulas for *ber* and *bei* 175
149, Formulas for W, Y, and Z, small argument 176
150, Savidge's asymptotic formulas for W, Y, and Z 176
151, Rayleigh's formulas for straight wires, low frequency .. 177
152, 153, Rayleigh's limiting formulas 178
154, 155, Formulas for limiting change of inductance of straight wires 178
156, Formula for the argument x 178
157–160, Resistance and inductance of return circuit of two parallel wires 180
161, 162, Limiting change in inductance of circular ring 181
163–166, Resistance and inductance of circular ring 182

(b) Examples 82–85, illustrating the formulas for inductance and resistance at high frequency 183

NOTE: Recent formula for coaxial circles (Nagaoka) 187

APPENDIX

TABLES OF CONSTANTS AND FUNCTIONS USEFUL IN THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE

I. Maxwell's table of $\frac{M}{4\pi\sqrt{Aa}}=\left[\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right]$, for formula (1) .. 190

II. Log F and log E as functions of tan γ 193
III. Table of K's for formula 57 193
IV. Table of Q's for formula 74 194
V. Table of constants A and B for formula 82 195
VI. Table of constants y_1 and y_2 for formula 90 196
VII. Table of constants A for formula 80 197
VIII. Table of constants B for formula 80 199
IX. Table of constants A_s for formula 91 200
X. Table of constants B_s for formula 91 200
XI. Table of Napierian Logarithms for numbers 1–100 201
XII. Values of F and E, 0° to 90° 202
XIII. Values of log F and log E, 45° to 90° 204
XIV. Table of binomial coefficients for interpolation formula 213
XV. Values of q, q_1, and ϵ for use in formulas 8, 9, 45, 76, 77, and 78 215
XVI. Values of ϵ_1 and ϵ_1' as function of q_1, for use with formulas 6 and 6a. 218
XVII. Coefficients in hypergeometric series of formula 18 220
XVIII. Location and magnitude of maxima and minima of coefficients in Gray's and Searle and Airey's formulas 221
XIX. Values of coefficients in Gray's and Searle and Airey's formulas 40, 43, 56 222
XX. Nagaoka's table of the correction for the ends, as function of θ, formula 75 223

APPENDIX—Continued

TABLES OF CONSTANTS AND FUNCTIONS, ETC.—continued.

Page

XXI. Nagaoka's table of the correction for the ends as function of $\frac{\text{diameter}}{\text{length}}$ 224

XXII. Values of $\frac{W}{Y}, \frac{x}{2}\frac{W}{Y}, \frac{Z}{Y}$ and $\frac{4}{x}\frac{Z}{Y}$ for use in formulas for high frequency 226

XXIII. Values of limiting value of the change of inductance with frequency 229

XXIV. Values of argument x in high frequency formulas 230

INDEX 231

INTRODUCTION

A great many formulas have been given for calculating the mutual and self-inductance of the various cases of electrical circuits occuring in practice. Some of these formulas have subsequently been shown to be wrong, and of those which are correct and applicable to any given case there is usually a choice, because of the greater accuracy or greater convenience of one as compared with the others. For the convenience of those having such calculations to make we have brought together in this paper all the formulas with which we are acquainted which are of value in the calculation of mutual and self-inductance, particularly in nonmagnetic circuits where the frequency of the current is low enough to assure sensibly uniform distribution of current. In the last section some formulas are given for the variation of the self-inductance and resistance with the frequency. A considerable number of formulas which have been shown to be unreliable or which have been replaced by others that are less complicated or more accurate have been omitted, although in most cases we have given references to such omitted formulas. Where several formulas are applicable to the same case we have pointed out the especial advantage of each and indicated which one is best adapted to precision work.

In the second part of each section of the paper we give a number of examples to illustrate and test the formulas. We have given the work in many cases in full to serve as a guide in such calculations in order to make the formulas as useful as possible to students and others not familiar with such calculations, and also to facilitate the work of checking up the results by anyone going over the subject. We have been impressed with the importance of this in reading the work of others.

In the appendix to the paper are a number of tables that will be found useful in numerical calculations of inductance.

In most cases we have given the name of the author of a formula in connection with the formula. This is not merely for the sake of historical interest, or to give proper credit to the authors, but also because we have found it helpful to distinguish in this way the various formulas instead of denoting each merely by a number. The formulas of sections 8 and 9, which are taken largely from a paper by one of the present authors,[1] are, however, not so designated, although the authorship of those that are not new is indicated where known.

This paper includes practically all the matter contained in the 1907 paper under the same title by Rosa and Cohen, but in addition to a thorough revision in which some errors are corrected and some formulas extended, a large amount of new matter has been added both in the body of the paper and in the tables. We shall be grateful to anyone detecting any errors either in formulas or tables if he communicates the same to us.

1. MUTUAL INDUCTANCE OF TWO COAXIAL CIRCLES

MAXWELL'S FORMULAS IN ELLIPTIC INTEGRALS

The first and most important of the formulas for the mutual inductance of coaxial circles is the formula in elliptic integrals given by Maxwell:[2]

$$M = 4\pi\sqrt{Aa}\left\{\left(\frac{2}{k} - k\right)F - \frac{2}{k}E\right\} \qquad [1]$$

in which A and a are the radii of the two circles, d is the distance between their centers, and

$$k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + d^2}} = \sin\gamma = \frac{\sqrt{r_1^2 - r_2^2}}{r_1}$$

F and E are the complete elliptic integrals of the first and second kind, respectively, to modulus k. Their values may be obtained from the tables of Legendre (see Tables XII and XIII in the Appendix), or the values of $M \div 4\pi\sqrt{Aa}$ may be obtained from Table I in the appendix of this paper, the values of γ being the argument.

[1] This Bulletin, 4, p 301; 1907.

[2] Electricity and Magnetism, Vol. II, § 701.

The notation of Maxwell is slightly altered in the above expressions in order to bring it into conformity with the formulas to follow.

Formula (1) is an absolute one, giving the mutual inductance of two coaxial circles of any size at any distance apart. If the two circles have equal or nearly equal radii, and are very near each other, the quantity k will be very nearly equal to unity and γ will be near to 90°. Under these circumstances it may be difficult to obtain a sufficiently exact value of F and E from the tables, as the quantities are varying rapidly and the tabular differences are very large. Under such circumstances the following formula, also given by Maxwell[2] (derived by means of Landen's transformation), is more suitable:

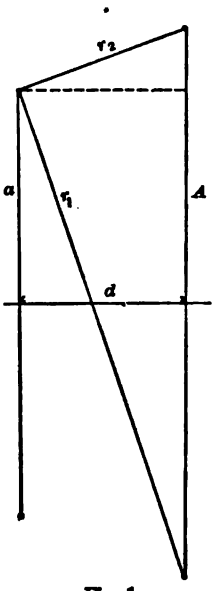

Fig. 1

$$M = 8\pi \frac{\sqrt{Aa}}{\sqrt{k_1}} \left\{ F_1 - E_1 \right\} \qquad [2]$$

in which F_1 and E_1 are complete elliptic integrals to modulus k_1, and

$$k_1 = \frac{r_1 - r_2}{r_1 + r_2} = \sin \gamma_1 = \frac{4Aa}{(r_1 + r_2)^2}$$

r_1 and r_2 are the greatest and least distances of one circle from the other (Fig. 1); that is,

$$r_1 = \sqrt{(A+a)^2 + d^2}$$
$$r_2 = \sqrt{(A-a)^2 + d^2}$$

The new modulas k_1 differs from unity more than k, hence γ_1 is not so near to 90° as γ and the values of the elliptic integrals can be taken more easily from the tables than when using formula (1) and the modulus k.

Another way of avoiding the difficulty when k is nearly unity is to calculate the integrals F and E directly, and thus not use the

tables of elliptic integrals, expanding F and E in terms of the complementary modulus k', where $k'=\sqrt{1-k^2}$. k' may usually be more accurately calculated by the formula $k'=\frac{r_2}{r_1}$. The expressions for F and E are very convergent when k' is small.

$$
\begin{aligned}
F = \log\frac{4}{k'} &+ \frac{1^2}{2^2}k'^2\left(\log\frac{4}{k'}-\frac{2}{1.2}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}k'^4\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}k'^6\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}-\frac{2}{5.6}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}\frac{7^2}{8^2}k'^8\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}-\frac{2}{5.6}-\frac{2}{7.8}\right)\\
&+\ \cdots\cdots\cdots\cdots\\
E = 1 &+ \frac{1}{2}k'^2\left(\log\frac{4}{k'}-\frac{1}{1.2}\right)\\
&+\frac{1^2}{2^2}\frac{3}{4}k'^4\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{1}{3.4}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5}{6}k'^6\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}-\frac{1}{5.6}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}\frac{7}{8}k'^8\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}-\frac{2}{5.6}-\frac{1}{7.8}\right)\\
&+\ \cdots\cdots\cdots\cdots
\end{aligned}
\qquad [3]
$$

The equations (3) are very convergent for $k'<0.1$, ($k\geqq 0.995$), and satisfactory accuracy will be attained down to $k=0.985$, thus covering the range of values for which interpolation in Legendre's tables becomes difficult.

For values of k near 0.985 it is perhaps more accurate to calculate M from elliptic integrals F_0 and E_0 with a modulus k_0 greater than k. The modulus k_0' which is complementary to k_0 is smaller than k', and the values of F_0 and E_0 calculated from the series formulas (3) putting k_0' in place of k' converge more rapidly than the values of F and E when calculated by the same series formulas. The formula for making the transformation is not quite so simple as (2). It is most conveniently written

$$\left.\begin{aligned} M &= 4\pi\sqrt{Aa}\left[\frac{F_0}{k(1+k)} - \frac{(1+k)}{k}E_0\right] \\ k_0' &= \frac{1-k}{1+k} = \frac{k'^2}{(1+k)^2} \\ k &= \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}} \qquad k' = \frac{\sqrt{(A-a)^2+d^2}}{\sqrt{(A+a^2)+d^2}} \end{aligned}\right\} \quad [4]$$

When the distance between the circles is large, formula (1) becomes unsuitable for calculation for two reasons, (*a*) because γ falls outside the range of Table XIII and (*b*) because the quantity $\left(\frac{2}{k} - k\right)F - \frac{2}{k}E$ comes out as the small difference of two large quantities. The use of formula (4) overcomes the first objection, but makes the matter still worse as far as the second is concerned. We may, however, express (1) in terms of a series by means of the well known expressions of Wallis[3]

$$F = \frac{\pi}{2}\left[1 + \sum_1^\infty \left\{\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right\}^2 k^{2n}\right]$$

$$E = \frac{\pi}{2}\left[1 - \sum_1^\infty \left\{\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right\}^2 \frac{k^{2n}}{(2n-1)}\right]$$

Substituting these values in (1) we find

$$M = \frac{\pi^2 k^2}{4}\sqrt{Aa}\left[1 + \frac{3}{4}k^2 + \frac{75}{128}k^4 + \frac{245}{512}k^6 + \cdots\right] \quad [5]$$

the general term in the brackets being

$$\left(\frac{3\cdot 5\cdot 7\cdot\cdots(2n+1)}{4\cdot 6\cdot 8\cdot\cdots(2n+2)}\right)^2\frac{(2n+2)}{(2n-1)}k^{2n} = \left[\frac{3\cdot 5\cdot 7\cdot\cdots(2n+1)}{4\cdot 6\cdot 8\cdot\cdots 2n}\right]^2\frac{k^{2n}}{(2n-1)(2n+2)}$$

For values of k up to 0.1 ($\gamma = 5°.7$) the series (5) is very convergent, and may be used for values of k up to 0.2 ($\gamma = 11°.5$) without serious labor. In the latter case and for still larger values of k, we may calculate M in terms of the smaller modulus k_1 of formula (2). This last expression becomes on expansion

[3] Greenhill's "Elliptic Functions," pp. 9, 176.

$$M = 2\pi^2 k_1^{\frac{3}{2}} \sqrt{Aa} \left[1 + \frac{3}{8} k_1^2 + \frac{15}{64} k_1^4 + \frac{175}{1024} k_1^6 + \cdot \cdot \cdot \cdot \right] \qquad [6]$$

the general term in the brackets being

$$\left(\frac{n+1}{2n+1} \right) \left[\frac{3 \cdot 5 \cdot 7 \cdot \cdot \cdot \cdot (2n+1)}{4 \cdot 6 \cdot 8 \cdot \cdot \cdot \cdot (2n+2)} \right]^2 k^{2n}$$

The series (6) converges more rapidly than (5), and may be used with ease for values of k_1 as great as $\frac{1}{4}$, ($\gamma_1 = 14^\circ.5$), which corresponds to $k = 0.8$, ($\gamma = 53^\circ.2$).

To recapitulate—

(1) For values of k between zero and 0.2 use (5).

(2) For values of k a little larger and up to 0.8 use (6).

(3) For values of k between about 0.7 and 0.985 the elliptic integrals in (1) may be conveniently taken by interpolation from Legendre's tables or from Table XIII.

(4) For values of k greater than about 0.7 we may use (4).

(5) For values of k greater than about 0.985 we may use (3).

It will be thus seen that the formulas overlap, so that it will be possible in every case to calculate the mutual inductance by at least two different formulas, the less accurate serving as a check on the more accurate.

The choice of formulas is considered more in detail on page 19.

WEINSTEIN'S FORMULA

Weinstein[4] gives an expression for the mutual inductance of two coaxial circles, in terms of the complementary modulus k' used in the preceding series (3). Substituting in equation (1) the values of F and E given above we have Weinstein's equation, which is as follows:

$$M = 4\pi \sqrt{Aa} \left\{ \left(1 + \frac{3}{4} k'^2 + \frac{33}{64} k'^4 + \frac{107}{256} k'^6 + \frac{5913}{16384} k'^8 + \cdot \right) \left(\log \frac{4}{k'} - 1 \right) - \left(1 + \frac{15}{128} k'^4 + \frac{185}{1536} k'^6 + \frac{7465}{65536} k'^8 + \cdot \cdot \cdot \right) \right\} \qquad [7]$$

[4] Wied. Ann., 21, p. 344; 1884.

This expression is rapidly convergent when k' is small, and hence will give an accurate value of M when the circles are near each other. Otherwise formula (1) may be more suitable.

NAGAOKA'S FORMULAS

Nagaoka[5] has given formulas for the calculation of the mutual inductance of coaxial circles, without the use of tables of elliptic integrals. These formulas make use of Jacobi's q-series, which is very rapidly convergent. The first is to be used when the circles are not near each other, the second when they are near each other. Either may be employed for a considerable range of distances between the extremes, although the first is more convenient. The *first* formula is as follows:

$$M = 16\pi^2\sqrt{Aa}\, q^{\frac{3}{2}}(1+\epsilon)$$
$$= 4\pi\sqrt{Aa}\{4\pi q^{\frac{3}{2}}(1+\epsilon)\} \qquad [8]$$

where A and a are the radii of the two circles. The correction term ϵ can be neglected when the circles are quite far apart.

$$\epsilon = 3q^4 - 4q^6 + 9q^8 - 12q^{10} + \cdots$$
$$q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \cdots$$
$$l = \frac{1-\sqrt{k'}}{1+\sqrt{k'}} \qquad k' = \frac{r_2}{r_1} = \frac{\sqrt{(A-a)^2+d^2}}{\sqrt{(A+a)^2+d^2}}$$

d being the distance between the centers of the circles, and k' the complementary modulus occurring in equations (3) and (7).

Nagaoka's *second* formula is as follows:

$$M = 4\pi\sqrt{Aa}.\frac{1}{2(1-2q_1)^2}\left\{[1+8q_1(1-q_1+4q_1^2-\cdots)]\log\frac{1}{q_1}-4\right\} \qquad [9]$$

$$= 4\pi\sqrt{Aa}.\frac{1}{2(1-2q_1)^2}\left\{[1+8q_1-8q_1^2+\epsilon_1]\log\frac{1}{q_1}-4\right\}$$
$$q_1 = \frac{l_1}{2} + 2\left(\frac{l_1}{2}\right)^5 + 15\left(\frac{l_1}{2}\right)^9 + \cdots$$
$$l_1 = \frac{1-\sqrt{k}}{1+\sqrt{k}} \qquad k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d}} \qquad [9a]$$
$$\epsilon_1 = 32q_1^3 - 40q_1^4 + 48q_1^5 - \cdots$$
$$-\epsilon_1' = 8q_1^2 - \epsilon_1$$

[5] Phil. Mag., 6, p. 19; 1903. Recently a third expression has been found by Nagaoka (Tokyo Math. Phys. Soc., 6, p. 10; 1911). (See p. 187 below.)

k is the modulus of equation (1), but is employed here to obtain the value of the q-series instead of the values of the elliptic integrals employed in (1). This formula is ordinarily simpler in use than it appears, because some of the terms in the expressions above are usually negligible. For a third formula see page 187.

Nagaoka has recently published[6] tables which materially reduce the labor of calculation with these formulas. These are reproduced as Tables XV and XVI of the appendix. From Table XV we obtain directly the small difference $q-\frac{l}{2}$ or $q_1-\frac{l_1}{2}$ with q or q_1 as argument. The same table gives also the corresponding values of ϵ and $\log_{10}(1+\epsilon)$ for use in the formula (8).

To calculate q or q_1 we enter the table with $\frac{l}{2}$ or $\frac{l_1}{2}$ as argument. The difference corresponding in the table when added to $\frac{l}{2}$ or $\frac{l_1}{2}$ gives the value of q or q_1 to a first approximation. This will be sufficient except for the larger values of q or q_1 which are tabulated here. For these it is sometimes necessary to use this first approximation as argument to obtain a more accurate value of q or q_1.

Table XVI gives the values of ϵ_1 and $-\epsilon_1'$ for different values of q_1 and is useful in calculations with formula (9).

For circles at some distance from one another q becomes small, and the expression for l given above becomes inconvenient, because k' is so nearly equal to unity. In this case we may calculate l from the somewhat more complicated expression

$$l=\frac{k^2}{(1+k')(1+\sqrt{k'})^2}$$

the values of k and k' being calculated from the formulas already given. The same applies to the calculation of l_1 in formula (9*a*), when the circles are very near together, and consequently q_1 is very small. For this case we use the expression

$$l_1=\frac{k'^2}{(1+k)(1+\sqrt{k})^2}$$

[6] Jour. of Coll. of Sci. Tokyo, vol. 27, art. 6; 1909.

MAXWELL'S SERIES FORMULA

Maxwell[7] obtained an expression for the mutual inductance between two coaxial circles in the form of a converging series which is often more convenient to use than the elliptical integral formula, and when the circles are nearly of the same radii and relatively near each other the value given is generally sufficiently exact. In the following formula a is the smaller of the two radii, c is their difference, $A-a$, d is the distance apart of the circles as before, and $r=\sqrt{c^2+d^2}$. The mutual inductance is then

$$M=4\pi a\left\{\log\frac{8a}{r}\left(1+\frac{c}{2a}+\frac{c^2+3d^2}{16a^2}-\frac{c^3+3cd^2}{32a^3}+\cdots\right)\right.$$
$$\left.-\left(2+\frac{c}{2a}-\frac{3c^2-d^2}{16a^2}+\frac{c^3-6cd^2}{48a^3}-\cdots\right)\right\} \quad [10]$$

When c and d are small compared with a, we have for an approximate value of the mutual inductance the following simple expression:[8]

$$M=4\pi a\left\{\log\frac{8a}{r}-2\right\} \quad [11]$$

When the two radii are equal, as is often the case in practice, the equation (10) is somewhat simplified, as follows:

$$M=4\pi a\left\{\log\frac{8a}{d}\left(1+\frac{3d^2}{16a^2}\right)-\left(2+\frac{d^2}{16a^2}\right)\right\} \quad [12]$$

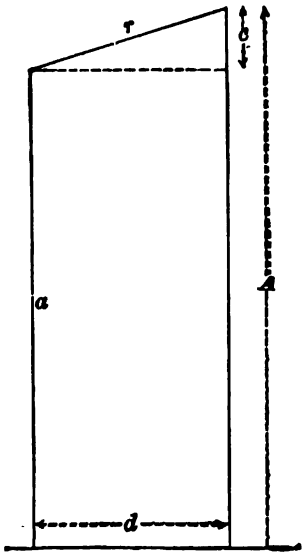

Fig. 2

The above formulas (10) and (12) are sufficiently exact for very many cases, the terms omitted in the series being unimportant when $\frac{c}{a}$ and $\frac{d}{a}$ are small. For example, if $\frac{d}{a}$ is 0.1, the largest term neglected in (12) is less than two parts in a million. If, however, $d=a$, this term will be more than one per cent, and the formula will be quite inexact.

[7] Electricity and Magnetism, Vol. II, § 705.

[8] This is equivalent to the approximate formula given by Wiedemann, $M=4\pi a\left\{\log\frac{2l}{c}-2.45\right\}$, where l is the circumference of the smaller circle and c is the same as r above.

Coffin[9] has extended Maxwell's formula (12) for two equal circles by computing three additional terms for each part of the expression. This enables the mutual inductance to be computed with considerable exactness up to $d=a$. Formula (1) is exact, as stated above, for all distances, and either it or (8) should be used in preference to (13) when d is large. Coffin's formula is as follows:

$$M=4\pi a\left\{\log\frac{8a}{d}\left(1+\frac{3d^2}{16a^2}-\frac{15d^4}{8\times128a^4}+\frac{35d^6}{128^2a^6}-\frac{1575d^8}{2\times128^3a^8}+\cdots\right)\right.$$
$$\left.-\left(2+\frac{d^2}{16a^2}-\frac{31d^4}{16\times128a^4}+\frac{247d^6}{6\times128^2a^6}-\frac{7795d^8}{8\times128^3a^8}+\cdots\right)\right\} \quad [13]$$

We have extended Maxwell's formula (10) for unequal circles as follows:[10]

$$M=4\pi a\left\{\log\frac{8a}{r}\left(1+\frac{c}{2a}+\frac{c^2+3d^2}{16a^2}-\frac{c^3+3cd^2}{32a^3}+\frac{17c^4+42c^2d^2-15d^4}{1024a^4}\right.\right.$$
$$\left.-\frac{19c^5+30c^3d^2-45cd^4}{2048a^5}+\cdots\right)-\left(2+\frac{c}{2a}-\frac{3c^2-d^2}{16a^2}+\frac{c^3-6cd^2}{48a^3}\right.$$
$$\left.\left.+\frac{19c^4+534c^2d^2-93d^4}{6144a^4}-\frac{379c^5+3030c^3d^2-1845cd^4}{61440a^5}\right)\right\} \quad [14]$$

Nagaoka[11] has confirmed this extension by expanding formula (9). He carried out the expansion, however, no further than terms in $\frac{c^4}{a^4}$ and $\frac{d^4}{a^4}$.

When $c=0$, this gives the first part of series (13). When $d=0$, the case of two circles in the same plane, with radii a and $a+c$, we have

$$M=4\pi a\left\{\log\frac{8a}{c}\left(1+\frac{c}{2a}+\frac{c^2}{16a^2}-\frac{c^3}{32a^3}+\frac{17c^4}{1024a^4}-\frac{19c^5}{2048a^5}+\cdots\right)\right.$$
$$\left.-\left(2+\frac{c}{2a}-\frac{3c^2}{16a^2}+\frac{c^3}{48a^3}+\frac{19c^4}{6144a^4}-\frac{379c^5}{61440a^5}+\cdots\right)\right\} \quad [15]$$

[9] J. G. Coffin, this Bulletin, 2, p. 113; 1906.
[10] This Bulletin, 2, p. 364; 1906.
[11] Loc. cit., p. 11.

These formulas (14) and (15) give the mutual inductance with great precision when the circles are not too far apart. The degree of convergence, of course, indicates approximately in any case the accuracy of the result.

HAVELOCK'S FORMULA

In 1908 Havelock[12] published a paper in which the calculation of mutual and self-inductance is made to depend on the evaluation of certain definite integrals of Bessel functions of the form $\int_0^\infty e^{-p\mu} J_1(\mu) J_1(\lambda\mu) \mu^{-n} d\mu$. These he expands in the form of series, which fall into two classes, those suitable for small values of p, and those suitable for large values of p. In the case of the latter, he gives the expressions for the general terms of the series, so that these may be extended as far as desired. In the case of the former only a few terms are given, and the derivation of further terms is very tedious.

He considers first the mutual inductance of two coaxial circles, and points out that the solution may be made to depend on either of two of his integrals. He does not, however, write out the formulas. It is a simple matter to carry out the necessary substitutions, and we find for circles near one another

$$M = 4\pi\sqrt{Aa}\left\{\left[1 + \frac{3}{16}\left(\frac{r^2}{Aa}\right) - \frac{15}{1024}\left(\frac{r^2}{Aa}\right)^2 + \cdots\right] \log \frac{8\sqrt{Aa}}{r} - \left(2 + \frac{1}{16}\left(\frac{r^2}{Aa}\right) - \frac{31}{2048}\left(\frac{r^2}{Aa}\right)^2 + \cdots\right)\right\}$$

This expression bears some resemblance to Maxwell's series formula (10); it is, however, simpler for use in calculation. To obtain the coefficients of further terms by Havelock's process would require a good deal of labor. We notice, however, that, putting $A = a$, the formula becomes the same as Coffin's formula (13) for equal circles. We may evidently, therefore, use the coefficients of the higher order terms in Coffin's formula to obtain an extension of the above, and find the expression

[12] Phil. Mag., **15**, p. 332; 1908.

$$M=4\pi\sqrt{Aa}\left\{\left[1+\frac{3}{16}\alpha-\frac{15}{1024}\alpha^2+\frac{35}{128^2}\alpha^3-\frac{1575}{2\times 128^3}\alpha^4+\cdots\right]\log\frac{8\sqrt{Aa}}{r}\right.$$
$$\left.-\left(2+\frac{1}{16}\alpha-\frac{31}{2048}\alpha^2+\frac{247}{6\times 128^2}\alpha^3-\frac{7795}{8\times 128^3}\alpha^4+\cdots\right)\right\} \qquad [16]$$

where

$$r^2=c^2+d^2 \qquad \alpha=\frac{a}{A}\frac{r^2}{a^2}$$

The expression thus extended[13] gives very accurate results for values of d almost as great as the radius a. For a given degree of convergence it requires only half as many terms to be calculated as does formula (14), and is much easier to calculate.

The second formula derived from Havelock's paper is not so generally useful, being rapidly convergent only for values of d greater than about $5A$. It is

$$M=\frac{2\pi^2a^2A^2}{d^2}\left[1-\frac{3}{2}\left(1+\frac{a^2}{A^2}\right)\frac{A^2}{d^2}+\frac{15}{8}\left(1+3\frac{a^2}{A^2}+\frac{a^4}{A^4}\right)\frac{A^4}{d^4}\right.$$
$$-\frac{35}{16}\left(1+6\frac{a^2}{A^2}+6\frac{a^4}{A^4}+\frac{a^6}{A^6}\right)\frac{A^6}{d^6}$$
$$+\frac{315}{128}\left(1+10\frac{a^2}{A^2}+20\frac{a^4}{A^4}+10\frac{a^6}{A^6}+\frac{a^8}{A^8}\right)\frac{A^8}{d^8} \qquad [17]$$
$$\left.-\frac{693}{256}\left(1+15\frac{a^2}{A^2}+50\frac{a^4}{A^4}+50\frac{a^6}{A^6}+15\frac{a^8}{A^8}+\frac{a^{10}}{A^{10}}\right)\frac{A^{10}}{d^{10}}+\cdots\right]$$

For the case of $d=10A$ and a as great as A, only three terms have to be calculated to obtain M to about one part in a million, and for a smaller value of $\frac{a}{A}$ the convergence would be more rapid still.

MATHY'S FORMULA

In an interesting paper in the Journal de Physique for 1901,[14] E. Mathy obtained a formula for the mutual inductance of two circles,

[13] Mr. T. J. Bromwich, of Cambridge, England, has recently communicated to us the same formula, without giving the proof, including however terms no higher than those in α^3.

[14] Jour. de Phys., **10**, p. 33; 1901.

in which the elliptic integral of the third kind, on which the mutual inductance depends, is expanded in a manner still different from that adopted in any of the preceding cases. It is expressed in terms of hypergeometric series involving the absolute invariant J of the Weierstrassian $\mathfrak{p}$ function. The final expression as found by Mathy is incorrect as regards the coefficients of the hypergeometric series. The corrected expression,[15] using the notation of this paper, is as follows:

$$\frac{M}{4\pi}=\left[\frac{x^3}{(x^4+12A^2a^2)^{\frac{1}{2}}}\left\{\frac{P}{\sqrt[4]{27}}F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2},\frac{J-1}{J}\right)-\frac{Q}{6\sqrt[4]{3}}\sqrt{\frac{J-1}{J}}\right.\right.$$
$$\left.F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2},\frac{J-1}{J}\right)\right\}-(x^4+12A^2a^2)^{\frac{1}{2}}\left\{\frac{Q}{\sqrt[4]{3}}F\left(-\frac{1}{12},\frac{7}{12},\frac{1}{2},\frac{J-1}{J}\right)\right.$$
$$\left.\left.+\frac{P}{6\sqrt[4]{27}}\sqrt{\frac{J-1}{J}}F\left(\frac{5}{12},\frac{13}{12},\frac{3}{2},\frac{J-1}{J}\right)\right\}\right] \qquad [18]$$

where

$$x^3=a^3+A^3+d^3$$

$$P=1.311028777\ \cdots\cdots$$

$$Q=0.599070117\ \cdots\cdots$$

$$\sqrt{\frac{J-1}{J}}=\frac{1-36\left(\frac{aA}{x^3}\right)^2}{\left[1+12\left(\frac{aA}{x^3}\right)^2\right]^{\frac{3}{2}}}$$

and

$$F(\alpha,\beta,\gamma,z)=1+\frac{\alpha\beta}{1\cdot\gamma}z+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}z^2$$
$$+\frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}z^3+\cdots\cdots$$

This formula is by no means so formidable to use as might be expected, since the constants which enter and the coefficients in the hypergeometric series may be calculated once for all. Using seven place logarithms we find

[15] Grover, this Bulletin, 6, p. 489; 1910.

$$\log_{10}\frac{P}{\sqrt[4]{27}}=9.7597712 \qquad \log_{10}\frac{P}{6\sqrt[4]{27}}=8.9816199$$

$$\log_{10}\frac{Q}{\sqrt[4]{3}}=9.6581974 \qquad \log_{10}\frac{Q}{6\sqrt[4]{3}}=8.8800461$$

The coefficients a_1, a_2, a_3 in each of the four series are given in Table XVII. For practical purposes the formula should be used only for values of $\sqrt{\frac{J-1}{J}}$ smaller than about 0.2.

For the special case $\sqrt{\frac{J-1}{J}}=0$, it is of interest to note that the mutual inductance is given by the simple expression

$$\frac{M}{4\pi}=\left[\frac{x^2}{(x^4+12A^2a^2)^{\frac{1}{4}}}\cdot\frac{P}{\sqrt[4]{27}}-(x^4+12A^2a^2)^{\frac{1}{4}}\cdot\frac{Q}{\sqrt[4]{3}}\right]$$

which, remembering that $x^4=36A^2a^2$ in this case, becomes

$$M=4\pi\sqrt{Aa}(P-2Q)=4\pi(0.112888542)\sqrt{Aa}$$

$$\doteq 1.418599262\sqrt{Aa} \qquad [19]$$

If we introduce the distances r_1 and r_2 (Fig. 1) into the formula for $\sqrt{\frac{J-1}{J}}$, we see that the necessary and sufficient condition that this remarkably simple formula [15a] may be used is that $r_1^2=2r_2^2$, or $k'=k=\frac{1}{\sqrt{2}}$. That is, the greatest distance between the two circumference must be $\sqrt{2}$ times the shortest distance between them. The most important cases satisfying this condition are

$\frac{a}{A}$	d	
1	$2A$	Equal circles.
$3-2\sqrt{2}$	0	Circles in the same plane.
$\frac{1}{2}$	$\frac{1}{2}\sqrt{7}\,A$	

[15a] Nagaoka has recently shown (Tokyo Math. Phys. Soc., 6, p. 10; 1911) that formula (19) may be derived from Maxwell's formula (1).

The convergence of the formula (18) will of course be satisfactory for moderate deviations on the either side of the ideal ratio of $\frac{d}{A}$, but the formula must be regarded as of more limited application than most of those above. It gives, however, a very rapid and accurate means of checking other formulas, since in the ideal case the mutual inductance can be calculated by (19) to any number of decimal places desired, according to the number of figures retained in Stirling's constants P and Q.

CHOICE OF FORMULAS

With so many to choose among, it is possible to select a favorable formula for any individual case. For this purpose r_1 and r_2, the longest and shortest distances between the circles, need to be considered, since on their relative values the convergence or convenience of the various formulas for calculation depends. The following table gives roughly the range of values of the ratio $\frac{r_2}{r_1}$ within which the different formulas are capable of giving the best results. Since, however, the determination of such limits is somewhat arbitrary, the values given here should not be regarded as more than a guide. In the case of those formulas which occur in the form of a series the limiting value of the ratio $\frac{r_2}{r_1}$ has been calculated which makes the last term included not greater than one ten-thousandth of the whole. The values of $\frac{r_2}{r_1}$ for Nagaoka's formulas have been calculated for the limits of his correction tables.

SUMMARY OF FORMULAS FOR CIRCLES

Formula		Range of values of $\frac{r_2}{r_1}$	Most favorable values of $\frac{d}{A}$ for equal circles
Weinstein's	(7)	0 to 0.25	0 to 0.5
Maxwell's	(10)	0 to 0.02	0 to 0.04
"	(12)	0 to 0.14	0 to 0.3
"	(14)	0 to 0.22	0 to 0.45
"	(3)	0 to 0.2	0 to 0.4
"	(2)	0.02 to 0.20	0.04 to 0.4

Formula		Range of values of $\frac{r_2}{r_1}$	Most favorable values of $\frac{d}{A}$ for equal circles
Havelock's	(16)	0 to 0.4	0 to 0.9
Coffin's	(13)		0 to 0.9
Nagaoka's	(9)	0.04 to 0.4	0.08 to 0.9
Maxwell's	(4)	0 to 0.75	0 to 2.25
"	(1)	0.2 to 0.7	0.4 to 2
Mathy	(18)	0.65 to 0.75	1.75 to 2.25
"	(19)	$\frac{1}{2}\sqrt{2}$	2
Nagaoka's	(8)	0.3 to 1	greater than 0.6
Maxwell's	(6)	0.6 to 1	" " 1.5
Havelock's	(17)	0.9 to 1	" " 4
Maxwell's	(5)	0.98 to 1	" " 10

EXAMPLES TO ILLUSTRATE AND TEST THE FORMULAS

EXAMPLE 1. MAXWELL'S FORMULA (1). FOR ANY COAXIAL CIRCLES

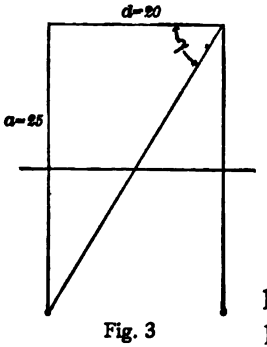

Fig. 3

Let $a = A = 25$ cm, Fig. 3,
$d = 20$ cm.

$$k = \frac{50}{\sqrt{2500+400}} = 0.9284766 = \sin\gamma$$

$$\gamma = 68^\circ\ 11'\ 54''88 = 68^\circ.198578.$$

From Legendre's tables, we obtain

$$\log F = 0.3852191 \qquad \left(\frac{2}{k} - k\right)F - \frac{2}{k}E = 0.5318500$$
$$\log E = 0.0547850$$

$$4a = 100 \qquad \therefore\ M = 167.08562 \text{ cm.}$$

To facilitate calculations in such problems as this, we have prepared Table II, which gives F and $\log F$, E and $\log E$, as functions of $\tan\gamma$. In the above case $\tan\gamma = \frac{50}{20} = 2.5$, and from Table II we can take the values of $\log F$ and $\log E$ directly, avoiding the calculation of γ and the interpolation for $\log F$ and $\log E$ in Legendre's tables (or Table XIII). This is only applicable for circles of equal radii, and is especially advantageous when $\tan\gamma$ is one of the values given in the table, when interpolation is unnecessary.

The above problem may also be calculated by means of Table I, taken from Maxwell, as follows:

$$\log_{10}\frac{M}{4\pi a} \text{ for } 68^\circ.1 \qquad = \bar{1}.7230634$$

$$\text{for } 68^\circ.2 \qquad = \underline{\bar{1}.7258281}$$

$$\text{for } 68^\circ.198578 = \bar{1}.7257888 = \log\frac{M}{4\pi a}$$

$\therefore M = 167.08546$ cm, agreeing almost exactly with the above value.

The calculation of mutual inductance by the above methods is simplest for circles not near each other, as then the values of log F, log E, and $\log\frac{M}{4\pi\sqrt{Aa}}$ are very exact when taken by simple interpolation. When γ is nearly 90°, however, second and third differences have to be used in interpolation.

EXAMPLE 2. MAXWELL'S SECOND EXPRESSION (2). FOR CIRCLES NEAR EACH OTHER

Let $a = A = 25$ cm, $d = 1$ cm

In this case $k = \sin\gamma = \frac{50}{\sqrt{2501}} = 0.9998002 \quad \gamma = 88^\circ\ 51'\ 14''$.

This value of γ is so nearly 90° that it is difficult to obtain accurate values of F and E from tables of elliptic integrals, or of $\frac{M}{4\pi a}$ from Maxwell's table.

We may therefore use formula (2) instead of (1).

$$r_1 = \sqrt{2501} = 50.01 \text{ nearly}, \quad r_2 = 1.0$$

$$\therefore k_1 = \sin\gamma_1 = \frac{4Aa}{(r_1+r_2)^2} = 0.9607920$$

$$\gamma_1 = 73^\circ\ 54'\ 9.''67 = 73^\circ.902687$$

From Legendre's tables or Table XIII, for $\gamma_1 = 73^\circ.902687$, $F_1 = 2.7024545$, $E_1 = 1.0852170$, $F_1 - E_1 = 1.6172375$

$$\frac{8\pi\sqrt{Aa}}{\sqrt{k_1}} = \frac{200\pi}{\sqrt{.9607920}} \quad \therefore \frac{8\pi\sqrt{Aa}}{\sqrt{k_1}}\left(F_1 - E_1\right) = M = 1036.6664 \text{ cm.}$$

EXAMPLE 3. FORMULA (3). SERIES FOR F AND E, CIRCLES NEAR EACH OTHER

Suppose that, in the last example, we calculate F and E by means of formula (3), instead of taking them from Table XIII.

$$A = a = 25,\ d = 1.$$

$$k'^2 = 1 - k^2 = 1 - \frac{2500}{2501} = \frac{1}{2501}$$

$$\therefore F = 5.2989471 \quad E = 1.0009594$$

If these values of F and E be substituted in formula (1), k being 0.9998002, we obtain $M = 1036.6652$, which is very closely the same value as by formula (2).

EXAMPLE 4. FORMULA (3). SECOND CASE, CIRCLES NOT NEAR

$$A = 25,\ a = 20,\ d = 10 \text{ cm.} \quad \text{(See Fig. 1.)}$$

$$k^2 = \frac{4 \times 20 \times 25}{(45)^2 + (10)^2} = \frac{16}{17} \quad \therefore k'^2 = \frac{1}{17}$$

$$\begin{aligned}
\log \frac{4}{k'} = \frac{1}{2} \log (16 \times 17) = \frac{1}{2} \log_e 272 \qquad &= 2.8029010 \\
\frac{k'^2}{4}\left(\log \frac{4}{k'} - 1\right) \qquad &= .0265132 \\
\frac{9k'^4}{64}\left(\log \frac{4}{k'} - \frac{7}{6}\right) \qquad &= .0007962 \\
\frac{25k'^6}{256}\left(\log \frac{4}{k'} - \frac{111}{90}\right) \qquad &= .0000312 \\
\frac{1225k'^8}{16384}\left(\log \frac{4}{k'} - 1.27\right) \qquad &= .0000014 \\
\therefore F \qquad &= 2.8302430 \\
1 + \frac{k'^2}{2}\left(\log \frac{4}{k'} - \frac{1}{2}\right) \qquad &= 1.0677324 \\
\frac{3k'^4}{16}\left(\log \frac{4}{k'} - \frac{13}{12}\right) \qquad &= .0011156 \\
\frac{15k'^6}{128}\left(\log \frac{4}{k'} - 1.20\right) \qquad &= .0000381 \\
\frac{175k'^8}{2048}\left(\log \frac{4}{k'} - 1.25\right) \qquad &= .0000017 \\
\therefore E \qquad &= 1.0688878
\end{aligned}$$

To find the value of M we now use equation (1).

$$\left\{\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right\}=0.885388$$

Multiplying by $4\pi\sqrt{Aa}=4\pi\sqrt{500}$ gives

$$M=248.7875 \text{ cm.}$$

EXAMPLE 5. FORMULA (4). CIRCLES NEAR TOGETHER

$$A=a=25 \qquad d=4$$

$$k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}}=\frac{50}{\sqrt{2516}}=0.9968154$$

$$k'^2=\frac{(A-a)^2+d^2}{(A+a)^2+d^2}=\frac{16}{2516}=0.0063593015$$

$$k_0'=\frac{k'^2}{(1+k)^2}=0.0015949004$$

$$\log_e\frac{4}{k_0'}=\log_e 2507.9937=7.8272373$$

$$\frac{k_0'^2}{4}\left(\log_e\frac{4}{k_0'}-1\right)=\frac{0.0000043}{7.8272416}=F_0$$

$$1+\frac{k_0'^2}{2}\left(\log_e\frac{4}{k_0'}-\frac{1}{2}\right)=1.0000093=E_0$$

$$\frac{F_0}{k(1+k)}=3.9323856$$

$$\frac{E_0}{k}(1+k)=\frac{2.0032134}{1.9291722}$$

Multiplying by $4\pi\sqrt{Aa}$ gives

$$M=606.0674 \text{ cm.}$$

If we calculate M by formula (3) we find that, to obtain the same precision, terms in k'^4 in the series for F and E have to be included, and we find

$$M=606.0678 \text{ cm.}$$

EXAMPLE 6. FORMULA (5). CIRCLES FAR APART

$$A = a = 10 \qquad d = 100$$

$$k = \frac{20}{\sqrt{10400}} = \frac{1}{\sqrt{26}} = 0.19611615$$

$$1 + \frac{3}{4}k^2 = 1.02884616$$

$$\frac{75}{128}k^4 = 0.00086684$$

$$\frac{245}{512}k^6 = 0.00002723$$

$$\frac{6615}{128^2}k^8 = \underline{0.00000088}$$

$$\text{Sum} = 1.02974111$$

$$\log \text{sum} = 0.0127281$$

$$\log k^2 = \bar{3}.8775400$$

$$\log \frac{\pi^2\sqrt{Aa}}{4} = \underline{1.3922398}$$

$$\log M = \bar{1}.2825079$$

$$M = 0.19164962 \text{ cm.}$$

If formula (1) be used, and the values of F and E be taken by interpolation from Table XII, the value $M = 0.191643$ is found, which is in error by more than 5 parts in 10000. Using the formula (6) terms in k_1^2 only need be calculated, and we find $M = 0.19164958$, which differs by only one part in five million from the value given by (5).

EXAMPLE 7. FORMULA (6). CIRCLES NOT NEAR TOGETHER

$$A = 25 \qquad a = 20 \qquad d = 40$$

$$r_1 = \sqrt{3625} \qquad r_2 = \sqrt{1625}$$

$$k_1 = \frac{4Aa}{(r_1 + r_2)^2} = 0.19793905$$

$$1+\frac{3}{8}k_1^2 = 1.01469245 \qquad \log \text{sum} = 0.0064929$$

$$\frac{15}{64}k_1^4 = 0.00035978 \qquad \log k_1^{\frac{5}{2}} = \bar{2}.9447972$$

$$\frac{175}{1024}k_1^6 = 0.00001028 \qquad \log \frac{\pi}{2}\sqrt{Aa} = \underline{1.5456049}$$

$$\frac{1225}{128^2}k_1^8 = \underline{0.00000032} \qquad \log \frac{M}{4\pi} = 0.4968950$$

$$\text{Sum} = 1.0150628$$

$$\therefore \frac{M}{4\pi} = 3.1397496 \text{ cm}$$

By formula (8) $\qquad \dfrac{M}{4\pi} = 3.1397486$

If formula (1) is used and the elliptic integrals be taken from Table XII by interpolation the value $\dfrac{M}{4\pi} = 3.1397656$ is found, which is only five in a million in error.

EXAMPLE 8. WEINSTEIN'S FORMULA (7). FOR ANY COAXIAL CIRCLES NOT TOO FAR APART

Take the same circles as in example 4.

$$A = 25,\ a = 20,\ c = 5,\ d = 10$$

$$k'^2 = \frac{1}{17},\ \log \frac{4}{k'} - 1 = 1.802901$$

$$1+\frac{3}{4}k'^2 = 1.0441176 \qquad 1+\frac{15}{128}k'^4 = 1.0004053$$

$$\frac{33}{64}k'^4 = .0017842 \qquad \frac{185}{1536}k'^6 = .0000245$$

$$\frac{107}{256}k'^6 = .0000851 \qquad \frac{7465}{65536}k'^8 = \underline{.0000012}$$

$$\frac{5913}{16384}k'^8 = \underline{.0000042} \qquad 1.0004310 = C$$

$$\text{Sum} = 1.0459911 = B$$

$$B \log\left(\frac{4}{k'} - 1\right) = 1.8858184;\ \left\{B \log\left(\frac{4}{k'} - 1\right) - C\right\} = 0.8853874$$

Multiplying by $4\pi\sqrt{500}$ gives $M = 248.7873$ cm, agreeing almost exactly with the value previously found, example 4.

EXAMPLE 9. NAGAOKA'S FORMULA (8). CIRCLES NOT NEAR TOGETHER

$$A = a = 25 \qquad d = 20 \qquad \text{(See Fig. 3.)}$$

$$\sqrt{k'} = \left(\frac{20}{\sqrt{2900}}\right)^{\frac{1}{2}} = 0.6094183$$

$$\frac{l}{2} = \frac{1-\sqrt{k'}}{2(1+\sqrt{k'})} = \frac{1}{2}\,\frac{0.3905817}{1.6094183} = 0.12134250$$

$$\begin{aligned}
\text{From Table XV, } q - \frac{l}{2} &= 0.00005269 \\
\therefore q &= 0.12139519 \\
\frac{3}{2}\log q &= \bar{2}.6263022 \\
\text{From Table XV, } \log(1+\epsilon) &= 0.0002775 \\
\log 400\pi^2 &= 3.5963598 \\
\log M &= 2.2229395 \\
\therefore M &= 167.08577 \text{ cm}
\end{aligned}$$

or about one in a million higher than the value found for the same circles in example 1.

EXAMPLE 10. NAGAOKA'S FORMULA (8). CIRCLES FAR APART

$$A = a = 10 \qquad d = 100$$

$$k' = \frac{100}{\sqrt{10400}} = 0.98058073$$

$$k = \frac{20}{\sqrt{10400}} = 0.19611615$$

$$1 + \sqrt{k'} = 1.9902427$$

$$\frac{l}{2} = \frac{1}{2}\,\frac{k^2}{(1+k')(1+\sqrt{k'})^2} = 0.0024512756$$

The differences $q - \frac{l}{2}$ and ϵ are negligible, so that we have

$$M = 16\pi^2\sqrt{Aa}\left(\frac{l}{2}\right)^{\frac{3}{2}} = 0.19164966 \text{ cm}$$

which is in very close agreement with the valves found by formulas (5) and (6) and Havelock's formula (17) for the same pair of circles. If we calculate $\frac{l}{2}$ by the formula $\frac{1}{2}\frac{(1-\sqrt{k'})}{(1+\sqrt{k'})}$ instead we find difficulty in obtaining $(1-\sqrt{k'})$ with sufficient precision. The value of M found by using this formula for $\frac{l}{2}$ and with seven place logarithms is in this case $M=0.19164980$, or about one part in a million different.

EXAMPLE 11. NAGAOKA'S SECOND FORMULA (9). FOR CIRCLES NEAR EACH OTHER

$$A=a=25 \qquad d=4$$

$$k=\sin\gamma=\frac{50}{\sqrt{2516}}=0.99681535 \qquad \sqrt{k}=0.99840640$$

$$k'^2=\frac{\sqrt{(A-a)^2+d^2}}{\sqrt{(A+a)^2+d^2}}=\frac{4}{\sqrt{2516}}=0.0063593014$$

$$\frac{l_1}{2}=\frac{k'^2}{(1+k)(1+\sqrt{k})^2}=0.00039872542=q_1$$

as $\left(\frac{l_1}{2}\right)^6$ and higher powers can be neglected.

$$\log_e\left(\frac{1}{q_1}\right)=\log_e 2507.9919=7.8272376$$

From Table XVI $\qquad -\epsilon_1'=0.00000128$

$$8q_1+\epsilon_1'=0.00318852$$

$$[1+8q_1+\epsilon_1']\log_e\left(\frac{1}{q_1}\right)-4=3.8521929 \quad =P$$

$$\frac{1}{2(1-2q_1)^2}=0.50079850=Q$$

$$\therefore\ M=4\pi\sqrt{Aa}\cdot PQ=606.0674 \text{ cm}$$

which is exactly the same value as was found for the same circles in example 5.

Using Table II for the above problem, where $\tan \gamma = 12.5$, we have $\log F = 0.5932708$ and $\log E = 0.0047004$. Using these values in formula (1) we obtain for the mutual inductance

$$M = 606.0666 \text{ cm}$$

which differs from the value by Nagaoka's formula by 1 part in a million.

EXAMPLE 12. MAXWELL'S SERIES FORMULA (10). FOR ANY TWO COAXIAL CIRCLES NEAR EACH OTHER

$$A = 26 \qquad a = 25 \qquad d = 1 \qquad c = 1 \qquad r = \sqrt{2}$$

Since $r = \sqrt{2}$, $\qquad \log_e \frac{8a}{r} = \log_e \frac{200}{\sqrt{2}} = 4.9517438$

$$1 + \frac{c}{2a} = 1.0200000 \qquad 2 + \frac{c}{2a} = 2.0200000$$

$$\frac{c^2 + 3d^2}{16a^2} = .0004000 \qquad -\frac{3c^2 - d^2}{16a^2} = -.0002000$$

$$-\frac{c^3 + 3cd^2}{32a^3} = -.0000080 \qquad +\frac{c^3 - 6cd^2}{48a^3} = -.0000067$$

$$1.0203920 = B \qquad 2.0197933 = C$$

$$B \log \frac{8a}{r} = 5.0527192$$

$$C = 2.0197933$$

$$\left\{B \log \frac{8a}{r} - C\right\} = 3.0329259 \text{ Multiply by } 4\pi a = 100\pi \text{ and}$$

$$M = 952.8218 \text{ cm.}$$

This formula would be less accurate for the circles of problem 4, but is accurate for circles close together, as this problem shows.

EXAMPLE 13. MAXWELL'S FORMULA (12). FOR CIRCLES OF EQUAL RADII NEAR EACH OTHER

$$A = a = 25 \qquad d = 1$$

$$\frac{8a}{d} = 200 \qquad \log_e 200 = 5.298317$$

$$\log_e \frac{8a}{d}\left(1 + \frac{3d^2}{16a^2}\right) = 1.000300 \times 5.298317 = 5.29990$$

$$\left(2 + \frac{d^2}{16a^2}\right) = \frac{2.00010}{3.29980}$$

Multiply by $4\pi a = 100\pi$

$$M = 1036.663 \text{ cm}$$

nearly agreeing with the more exact value found under problem 2.

This is a very simple and convenient formula for equal circles and gives approximate results for circles still farther apart than in this problem.

EXAMPLE 14. HAVELOCK'S FORMULA (16). FOR CIRCLES AT MODERATE DISTANCES

$$A=25 \qquad a=20 \qquad d=10 \qquad \therefore c=5$$

$$r=5\sqrt{5} \qquad \alpha=\frac{r^2}{Aa}=\frac{1}{4} \qquad \frac{8\sqrt{Aa}}{r}=16$$

$$\begin{aligned}
\log_e 16 &= 2.7725887\\
1+\frac{3}{16}\alpha &= 1.0468750\\
-\frac{15}{1024}\alpha^2 &= -0.0009155\\
\frac{35}{128^2}\alpha^3 &= 0.0000334\\
-\frac{1575}{2.128^3}\alpha^4 &= -0.0000015\\
\text{Sum} &= 1.0459914\\
\text{Multiplied by } \log_e 16 &= 2.9001037 = B
\end{aligned}$$

$$\begin{aligned}
2+\frac{1}{16}\alpha &= 2.0156250\\
-\frac{31}{2048}\alpha^2 &= -0.0009461\\
\frac{247}{6.128^2}\alpha^3 &= 0.0000393\\
-\frac{7795}{8.128^3}\alpha^4 &= -0.0000018\\
\text{Sum} &= 2.0147164 = C\\
B-C &= 0.8853873\\
\text{Multiplied by } 4\pi\sqrt{Aa} &= 248.7873 \text{ cm} = M
\end{aligned}$$

which agrees exactly with the value found in example 8.

If the example 12 be calculated by this formula, no terms of order higher than α^3 need be calculated, and

$$M = 952.8221 \text{ cm}$$
$$\text{Formula (10) } M = 952.8218$$
$$\text{Formula (3) } M = 952.8219$$

EXAMPLE 15. COFFIN'S FORMULA (13). EXTENSION OF FORMULA (12) FOR CIRCLES OF EQUAL RADII

$$A = a = 25 \qquad d = 16$$

$$\frac{8a}{d} = 12.5 \qquad \log_e 12.5 = 2.5257286$$

$$\text{First series of terms} = B = 1.074478$$
$$\text{Second series of terms} = C = 2.023220$$
$$\therefore \left\{B \log \frac{8a}{d} - C\right\} = 0.690620$$
$$4\pi a = 100\pi \qquad \therefore M = 216.9647 \text{ cm.}$$

This agrees with the value given by formula (1) within 1 part in 200,000. As the distance apart of the circles increases the accuracy by this formula of course gradually decreases.

EXAMPLE 16. FORMULA (14). EXTENSION OF MAXWELL'S FORMULA (10) FOR CIRCLES OF UNEQUAL RADII

$$A = 25 \qquad a = 20 \qquad c = 5 \qquad d = 10$$

$$r = \sqrt{c^2 + d^2} = 5\sqrt{5} \qquad \log_e \frac{8a}{r} = \log_e \frac{32}{\sqrt{5}} = 2.6610169$$

$$\text{First series of terms} = B \log_e \frac{8a}{r} = 3.112060$$

$$\text{Second series of terms} = C \qquad = \underline{2.122114}$$
$$0.989946$$

$$\text{multiplying by } 4\pi a = 80\pi \qquad M = 248.8006 \text{ cm.}$$

This result is correct to 1 part in 19,000 (see examples 4, 8, and 14). Using only the first three terms for B and C (that is, formula 10), the result would be too large by 1 part in 1750.

EXAMPLE 17. HAVELOCK'S FORMULA (17). CIRCLES FAR APART

$$a = 10 = A \qquad d = 100$$

$$\frac{a}{A} = 1 \qquad \frac{A}{d} = 0.1$$

$$1-\frac{3}{2}\cdot 2\cdot\frac{A^2}{d^2}= \quad 0.9700000$$

$$\frac{15}{8}\cdot 5\cdot\frac{A^4}{d^4}= \quad 0.0009375$$

$$-\frac{35}{16}\cdot 14\cdot\frac{A^6}{d^6}= -0.0000306$$

$$\frac{315}{128}\cdot 42\cdot\frac{A^8}{d^8}= \quad \underline{0.0000010}$$

$$\text{Sum}= \quad 0.9709079$$

$$\text{Multiplied by } \frac{2\pi^2A^2a^2}{d^3}= \quad 0.19164958 \text{ cm}=M$$

which is in exact agreement with the value found by formula (6).

EXAMPLE 18. MATHY'S FORMULA (18)

$$A=25 \qquad a=20 \qquad d=40$$

$$x^2=625+400+1600=2625$$

$$\frac{Aa}{x^2}=\frac{500}{5625}=\frac{4}{21}$$

$$x^4+12\,A^2a^2=9890625$$

$$\frac{1}{4}\log\,(x^4+12A^2a^2)=1.7488059$$

$$\log\left[\frac{x^2}{(x^4+12A^2a^2)^{\frac{1}{2}}}\right]=1.6703234$$

$$1-36\,\frac{A^2a^2}{x^4}=-0.3061225$$

$$1+12\,\frac{A^2a^2}{x^4}= \quad 1.4353742$$

$$\sqrt{\frac{J-1}{J}}=-0.17801131$$

$$\log z= \quad \bar{2}.5008952$$

Using the constants in Table XVII we calculate the four series

$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2},z\right)$	$F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2},z\right)$	$F\left(-\frac{1}{12},\frac{7}{12},\frac{1}{2},z\right)$	$F\left(\frac{5}{12},\frac{13}{12},\frac{3}{2},z\right)$
1.0022006	1.0112962	0.9969192	1.0095357
0.0000357	0.0002173	−0.0000473	0.0001784
0.0000008	0.0000049	−0.0000010	0.0000040
1.0022371	1.0115184	0.9968709	1.0097181

From these, using the values of the constants already calculated, we find the four terms in the formula for M

$$\begin{array}{rr} C = & 26.981438 \\ D = & -\ 0.639427 \\ \hline C-D = & 27.620865 \end{array} \qquad \begin{array}{rr} G = & 25.447327 \\ H = & -\ 0.966215 \\ \hline G-H = & 24.481112 \end{array}$$

$$\frac{M}{4\pi} = 27.620865 - 24.481112 = 3.139753$$

By Nagaoka's formula (8) we find $\frac{M}{4\pi} = 3.1397496$

" " (6) " " $\frac{M}{4\pi} = 3.1397486$

Mathy's formula suffers here under the inconvenience that M is given as the difference of two quantities considerably larger than itself.

EXAMPLE 19. FORMULA (19). FOR CIRCLES SATISFYING THE CONDITION $r_1^2 = 2r_2^2$ OR $k' = k = \frac{1}{\sqrt{2}}$

$$A = a = 25 \qquad d = 50$$

$$M = 1.41859262 \cdot \cdot \cdot \cdot \sqrt{Aa}$$
$$= 35.4649816 \cdot \cdot \cdot \cdot \text{ cm.}$$

By Nagaoka's formula (8), $M = 35.464975$
" " (6), $M = 35.464981$
" " (1), $M = 35.46481$

We see that the formulas (8) and (6) here give an accuracy limited only by that of the logarithm tables. The result found by formula (1), using Table XIII, is, however, affected by the fact that the value of the quantity in the parentheses (1.4239167 − 1.3110287), is only about a tenth as large as the numbers of which it is the difference.

2. MUTUAL INDUCTANCE OF TWO COAXIAL COILS

ROWLAND'S FORMULA

Let there be two coaxial coils of mean radii A and a, axial breadth of coils b_1 and b_2, radial depth c_1 and c_2, and distance apart of their mean planes d. Suppose them uniformly wound with n_1 and n_2 turns of wire. The mutual inductance M_0 of the two central turns of the coils (Fig. 4), will be given by formula (1) or (7), or any one of the foregoing formulas for the mutual inductance of coaxial circles adapted to the particular case may be used, and the mutual inductance M of the two coils of n_1 and n_2 turns will then be, to a *first approximation*,

$$M = n_1 n_2 M_0$$

The following second approximation was obtained by Rowland by means of Taylor's theorem, following Maxwell, § 700:

$$\frac{M}{n_1 n_2} = M_0 + \frac{1}{24}\left\{(b_1^2 + b_2^2)\frac{d\,M_0}{dx^2} + c_1^2\frac{d^2 M_0}{da^2} + c_2^2\frac{d^2 M_0}{dA^2}\right\}$$

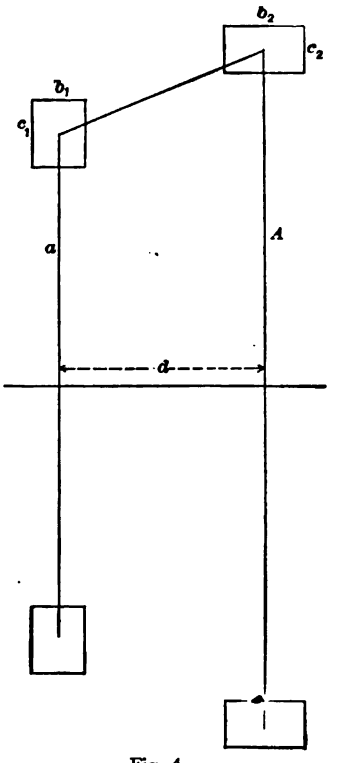

Fig. 4

If the two coils are of equal radii but unequal section,

$$\frac{M}{n_1 n_2} = M_0 + \frac{1}{24}\left\{(b_1^2 + b_2^2)\frac{d^2 M_0}{dx^2} + (c_1^2 + c_2^2)\frac{d^2 M_0}{da^2}\right\} \qquad [20]$$

If the two coils are of equal radii and equal section, this becomes

$$\frac{M}{n_1 n_2} = M_0 + \frac{1}{12}\left\{b^2\frac{d^2 M_0}{dx^2} + c^2\frac{d^2 M_0}{da^2}\right\} \qquad [21]$$

The correction terms will be calculated by means of the following:

$$\frac{d^2M_0}{da^2}=\pi\frac{k}{a}\left\{(2-k^2)F-\left(2-k^2\frac{1-2k^2}{1-k^2}\right)E\right\}$$
$$\frac{d^2M_0}{dx^2}=\pi\frac{k^3}{a}\left\{F-\frac{1-2k^2}{1-k^2}E\right\} \qquad [22]$$

The equation (21) is equivalent to Rowland's equation, where 2ξ and 2η are the breadth and depth of the section of the coil, instead of b and c, except that there is an error in the formula as printed in Rowland's[16] paper, ξ and η being interchanged. The equations (22) are equivalent to those given by Rowland, being somewhat simpler.[17]

Formula (21) gives a very exact value for the mutual inductance of two coils, provided the cross sections are relatively small and the distance apart d is not too small. But when b or c is large or d is small the fourth differential coefficients which have been neglected become appreciable and the expression may not be sufficiently exact.

RAYLEIGH'S FORMULA

Maxwell[18] gives a formula, suggested by Rayleigh, for the mutual inductance of two coils, which has a very different form from Rowland's, but is nearly equivalent to it when the coils are not near each other. It has been used by Rayleigh in calculating the mutual inductance of a Lorenz apparatus and by Glazebrook (Phil. Trans., 1883) in calculating the mutual inductance of parallel coils of rectangular section employed in a determination of the ohm. It may also be employed in calculating the attraction between two coils.[19] It is sometimes called the formula of quadratures, and is as follows:[20]

$$M=\frac{1}{6}\left(M_1+M_2+M_3+M_4+M_5+M_6+M_7+M_8-2M_0\right) \qquad [23]$$

[16] Collected Papers, p. 162. Am. Jour. Sci. [3], XV, 1878.
[17] Gray, Absolute Measurements, Vol. II, Part II, p. 322.
[18] Electricity and Magnetism, Vol. II, Appendix II, Chapter XIV.
[19] Gray, Absolute Measurements, Vol. II, Part II, p. 403.
[20] This Bulletin, 2, p. 370–372; 1906.

where M_1 is the mutual inductance of the circle O_2 and a circle through the point 1 of radius $A-\frac{c_1}{2}$, and similarly for the others, Fig. 5.

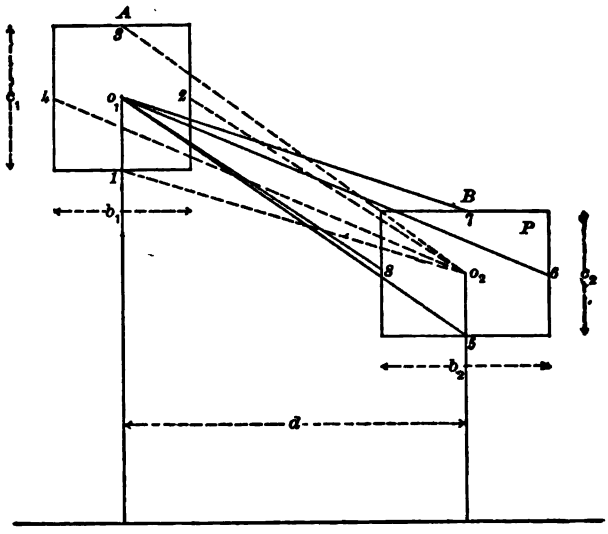

Fig. 5

For two coils of equal radii and equal section this becomes

$$M=\frac{1}{3}\left(M_1+M_2+M_3+M_4-M_0\right) \qquad [24]$$

Equation (23) is Rayleigh's formula, or the formula of quadratures. Instead of computing the correction to M_0 by means of the differential coefficients (20), eight additional values are computed, corresponding to the mutual inductances of the single turns at the eight numbered points indicated in Fig. 5, each with reference to the central turn of the other coil. These M's may be computed by any of the formulas for the mutual inductance of coaxial circles which may be best adapted to the particular case, and the values of the constants for the case of two coils of *equal radii* are given in the following table, the radius being a in every case.

	Axial distance	Radial distance	r
Using (10)	d	$-\frac{c_1}{2}$	$\sqrt{d^2+\frac{c_1^2}{4}}$
"	d	$+\frac{c_1}{2}$	$\sqrt{d^2+\frac{c_1^2}{4}}$
"	d	$-\frac{c_2}{2}$	$\sqrt{d^2+\frac{c_2^2}{4}}$
"	d	$+\frac{c_2}{2}$	$\sqrt{d^2+\frac{c_2^2}{4}}$
Using (12)	$d-b_1/2$	0	
"	$d+b_1/2$	0	
"	$d+b_2/2$	0	
"	$d-b_2/2$	0	

MAGNITUDE OF THE ERRORS IN ROWLAND'S AND RAYLEIGH'S FORMULAS

The error e_1 in equation (24), for two coils of equal radii a, distance between centers being d, and section $b \times c$ (Fig. 6), depends on the dimensions of the coil in a manner shown by the following expression:[21]

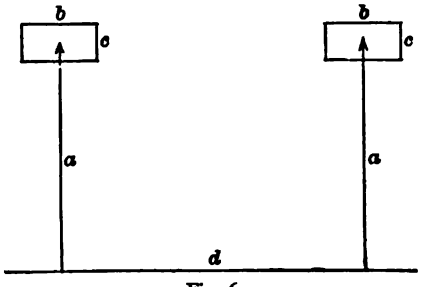

Fig. 6

$$e_1 \propto 4\pi a\left\{\frac{3b^4+3c^4-20b^2c^2}{480d^4}\right\} \quad [25]$$

For a square coil the correction is a negative quantity, showing that M by equation (24) is too large, and the error is proportional to the fourth power of $\frac{1}{d}$, the reciprocal of the distance between the mean planes of the coils. For a rectangular coil in which b is greater than c the correction is negative so long as b is not more than 2.5 times c. When b is still larger with respect to c the correction becomes plus, the value of M by (24) being too small.

[21] This Bulletin, 2, p. 373; 1906.

Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (25) as we vary the shape of cross section keeping $bc=4$.

Dimensions of coil		Error proportional to—
$b=2$	$c=2$	$-\ 224$
$b=2.5$	$c=1.6$	$-\ 183$
$b=3$	$c=1.33$	$-\ 67.5$
$b=4$	$c=1$	$+\ 451$
$b=8$	$c=0.5$	$+11{,}988$

Thus we see that the value of M as given by the formula of quadratures may be too large or too small according to the shape of the section, and that the error is proportional directly to the fourth power of the dimensions of the section and inversely to the fourth power of the distance between the mean planes of the coils. When the section is small and d large the error will become negligible.

The error by Rowland's formula is—[22]

$$\epsilon_2 \propto 4\pi a\frac{6}{d^4}\left\{\frac{b^4+c^4}{360}-\frac{b^2c^2}{144}\right\} \propto 4\pi a\left\{\frac{8b^4+8c^4-20b^2c^2}{480d^4}\right\} \qquad [26]$$

This is negative for a square coil, but smaller than ϵ_1. For a coil of section such that $b=c\sqrt{2}$, the error is zero, and for sections such that $\frac{b}{c}>\sqrt{2}$, the error is positive. Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (26) which is proportional to the error by Rowland's formula.

Dimensions of coil		Error proportional to—
$b=2$	$c=2$	$-\ 64$
$b=2.5$	$c=1.6$	$+\ 45$
$b=3$	$c=1.33$	$+\ 353$
$b=4$	$c=1$	$+\ 1{,}736$
$b=8$	$c=0.5$	$+32{,}448$

Thus the error is smaller by Rowland's formula for coils having square or nearly square section, but larger for coils having rectangular sections not nearly square.

[22] This Bulletin, 2, p. 373; 1906.

LYLE'S FORMULA

Professor Lyle [22] has recently proposed a very convenient method for calculating the mutual inductance of coaxial coils, which gives very accurate results for coils at some distance from each other.

The mutual inductance is calculated from formula (1) or any other formula for two coaxial circles, using, however, a modified radius r instead of the mean radius a, r being given by the following equation when the section is square, b being the side of the square section:

$$r = a\left(1 + \frac{b^2}{24a^2}\right) \qquad [27]$$

If the coil has a rectangular section not square, it can be replaced by two filaments (Fig. 7) the distance apart of the filaments being called the *equivalent breadth* or the *equivalent depth* of the coil.

$$\beta^2 = \frac{b^2 - c^2}{12}, \ 2\beta \text{ is the equivalent breadth of A}$$

$$\delta^2 = \frac{c^2 - b^2}{12}, \ 2\delta \text{ is the equivalent depth of B} \qquad [28]$$

The equivalent radius of A is given by the same expression which holds for a square coil, viz:

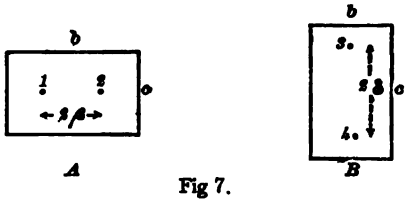

Fig 7.

$$r = a\left(1 + \frac{c^2}{24a^2}\right)$$

In the coil B the equivalent filaments have radii $r + \delta$ and $r - \delta$, respectively, where

$$r = a\left(1 + \frac{b^2}{24a^2}\right)$$

The mutual inductance of two coils may now be readily calculated. If each has a square section, it is necessary only to calculate the mutual inductance of the two equivalent filaments. For coils of rectangular sections, as A, B, the mutual inductance will be the sum of the mutual inductances of the two filaments of A on the two

[22] Phil. Mag., 3, p. 310; 1902. Also this Bulletin, 2, pp. 374-378; 1906.

filaments of B, counting $n/2$ turns in each. Or, it is $n_1 n_2$ times the mean of the four inductances M_{13}, M_{14}, M_{23}, M_{24}, where M_{13} is the mutual inductance of filament 1 on filament 3, etc.

Lyle's method is of special value in computing mutual inductances because it applies to coils of unequal as well as of equal radii.

ROSA'S FORMULA [24]

Writing the mutual inductance of two coaxial coils of equal radii and equal section as $\frac{M}{n_1 n_2} = M_0 + \Delta M$, where M_0 is the mutual inductance of the central circles of the two equal coils of sections $b \times c$, Fig. 5, and ΔM is the correction for the section of the coil, the value of ΔM is as follows:

$$\Delta M = 4\pi a\left\{\frac{3b^2+c^2}{96a^2}\cdot\log\frac{8a}{d} - \frac{11b^2-3c^2}{192a^2} + \frac{b^2-c^2}{12d^2} + \frac{2b^4+2c^4-5b^2c^2}{120d^4}\right.$$
$$+\frac{6b^4+6c^4+5b^2c^2}{5760a^2d^2} + \frac{3b^6-3c^6+14b^2c^4-14b^4c^2}{504d^6} + \frac{7c^2d^2}{1024a^4}\left(\log\frac{8a}{d} - \frac{163}{84}\right)$$
$$\left. - \frac{15b^2d^2}{1024a^4}\left(\log\frac{8a}{d} - \frac{97}{60}\right)\right\} \qquad [29]$$

For a square section, when $b = c$, this becomes

$$\Delta M = \frac{\pi b^2}{6a}\left\{\log\frac{8a}{d} - 1 - \frac{a^2b^2}{5d^4} - \frac{3d^2}{16a^2}\left(\log\frac{8a}{d} - \frac{4}{3}\right) + \frac{17b^2}{240d^2}\right\} \qquad [30]$$

The last two terms of equation (30) are relatively small, so that we may write, *approximately:*

$$\Delta M = \frac{\pi b^2}{6a}\left\{\log\frac{8a}{d} - 1 - \frac{a^2b^2}{5d^4}\right\} \qquad [31]$$

For coils of equal radii but unequal sections, the formula is, neglecting differentials of sixth order

$$\Delta M = 4\pi a\left\{\frac{3(b_1^2+b_2^2)+(c_1^2+c_2^2)}{192a^2}\log\frac{8a}{d} - \frac{11(b_1^2+b_2^2)-3(c_1^2+c_2^2)}{384a^2}\right.$$
$$+\frac{(b_1^2+b_2^2)-(c_1^2+c_2^2)}{24d^2} \qquad [32]$$
$$\left. + \frac{(3b_1^4+10b_1^2b_2^2+3b_2^4)+(3c_1^4+10c_1^2c_2^2+3c_2^4)-10(b_1^2+b_2^2)(c_1^2+c_2^2)}{960d^4}\right\}$$

[24] This Bulletin, 4, p. 348, equations (38) and (39).

These expressions for ΔM are very exact where the coils are near together or even where they are separated by a considerable distance, but become less exact as d is greater. They are therefore most reliable where formulas (21), (24), and (27) are least reliable. As formula (31) is exact enough for most purposes, it affords a very easy method of getting the correction for equal coils of square section.

Stefan's formula for the mutual inductance of two equal coaxial coils (originally published[25] without demonstration) is incorrect and is not given here. It resembles equation (29), but is seriously in error for coils at considerable distances.

THE ROSA-WEINSTEIN FORMULA

Weinstein's formula[26] for the mutual inductance of equal coaxial coils has been revised and corrected by Rosa, and the value of ΔM, the correction for section, expressed separately. The expression for ΔM is as follows:[26a]

$$\Delta M = 4\pi a \sin\gamma\left\{(F-E)\left(A+\frac{c^2}{24a^2}\right)+EB\right\} \qquad [33]$$

where F and E are the complete elliptic integrals to modulus $\sin\gamma$, Fig. 8 (as in equation 1),

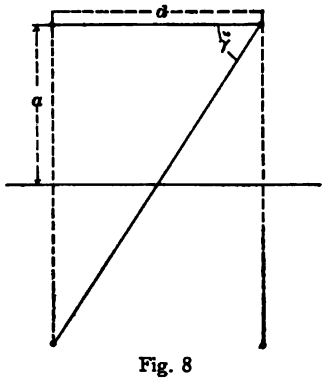

Fig. 8

[25] Wied. Annalen, **22**, p. 107; 1884.
[26] Wied. Annalen, **21**, p. 350; 1884.
[26a] This Bulletin, **4**, p. 342, equation (20); 1907.

and

$$A=\frac{\cos^2\gamma}{12d^2}\left(\alpha_1-\alpha_2-\alpha_3+(2\alpha_2-3\alpha_3)\cos^2\gamma+8\alpha_3\cos^4\gamma\right)$$

$$B=\frac{\sin^2\gamma}{12d^2}\left(\alpha_1+\frac{\alpha_2}{2}+2\alpha_3+(2\alpha_2+3\alpha_3)\cos^2\gamma+8\alpha_3\cos^4\gamma\right)$$

The values of α_1, α_2, and α_3 are as follows:

$$\alpha_1=b^2-c^2+\frac{c^4}{30a^2} \qquad \text{For square section: } \alpha_1=\frac{b^4}{30a^2}$$

$$\alpha_2=\frac{5b^2c^2-4c^4}{60a^2} \qquad \text{" " " } \alpha_2=\frac{b^4}{60a^2}$$

$$\alpha_3=\frac{2b^4+2c^4-5b^2c^2}{20d^2} \qquad \text{" " " } \alpha_3=-\frac{b^4}{20d^2}$$

Formula (33) is a very exact formula for all positions of the two coils, except when they are very close together.

Weinstein's original formula,[27] which is much less accurate than (33) for coils relatively near together, is not here given.

USE OF FORMULAS FOR SELF-INDUCTANCE IN CALCULATING MUTUAL INDUCTANCE

One can sometimes obtain the mutual inductance of adjacent coils, or of coils at a distance from one another, by means of a formula for the self-inductance of coils. Thus, suppose we have a coil of rectangular section, which we subdivide into three equal parts, 1, 2, 3, Fig. 9. Let L be the self-inductance of the whole coil, L_1 be the self-inductance of any one of the three equal smaller coils, and L_2 be the self-inductance of two adjacent coils taken together. Also let M_{12} be the mutual inductance of coil 1 on coil 2, or of coil 2 on coil 3, and M_{13} be the mutual inductance of coil 1 on coil 3. Then,

$$L=3L_1+4M_{12}+2M_{13}$$

Also, $L_2=2L_1+2M_{12}$

$$\therefore\ M_{12}=\frac{L_2-2L_1}{2} \qquad [34]$$

and $M_{13}=\dfrac{L+L_1-2L_2}{2}$

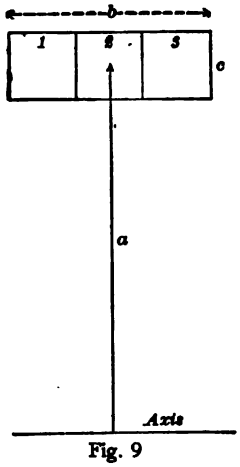

Fig. 9

[27] Wied. Annalen, **21**, p. 350; 1884.

Formula (34) will thus enable us to find the mutual inductance of two coils of equal radii adjacent or near each other by the calculation of self-inductances from such formulas as those of Weinstein (88) and Stefan (90). These latter formulas are not, however, exact enough when the section is large to permit us to apply them to coils at any considerable distance from one another.

GEOMETRIC MEAN DISTANCE FORMULA

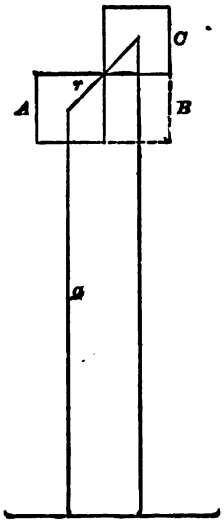

Fig. 10

The mutual inductance of two coaxial coils adjacent or very near can sometimes be obtained by means of the geometric mean distances. This method is accurate only when the sections are very small relatively to the radius. It can often be used to advantage in testing other formulas, but not often in determining the mutual inductance of actual coils.

Formula (10) gives the mutual inductance of two very near coaxial coils in terms of the geometric mean distance, if r be replaced by R, the geometric mean distance of the two sections. Formula (10) gives M_0 if r be used, where r is the distance between centers. Thus,

$$\Delta M = 4\pi a\left(1 + \frac{c}{2a}\right)\log\frac{r}{R} \qquad [35]$$

For coils A and C (Fig. 10), $R < r$ and ΔM is positive; $R = 0.99770\,r$
" " A " B, $R > r$ and ΔM is negative; $R = 1.00655\,r$

The same formula may also be used for squares not adjacent, but only when quite near.[28]

For illustrations and tests of the above formulas, see examples 20–33, pages 44–52.

[28] For other values of the geometric mean distances of squares in a plane see this Bulletin, 8, p. 1; 1907.

CHOICE OF FORMULAS

(*a*) For coils of equal radii and equal cross section (29) should be used if the coils are rather near together. If the cross section is square (29) takes the more simple form (30), and in some cases this may be used in its abbreviated form (31). For coils at all distances, except near together, (33) gives very good precision; (24) and (21) are not so accurate as this last, but give good results if the coils are far apart and their cross sections are not too large.

(*b*) For coils of equal radii but unequal section (32) is accurate for coils not too far away from one another. For coils farther separated (20), (23) or (28) may be used.

(*c*) For coils of unequal radii (23), (24), (27), and (28) apply, but unfortunately they are not as accurate as some of the others. except when the coils are relatively distant or have very small cross sections. The difficulty can be overcome by subdividing each of the two coils into two, four, or more equal parts, and taking the sum of the mutual inductances of all of the parts of one on all the parts of the other. This is a laborious operation, but in important cases it should be done. As the subdivision is carried further the results will approach a final value, and hence the results themselves show when the subdivision has been carried far enough.

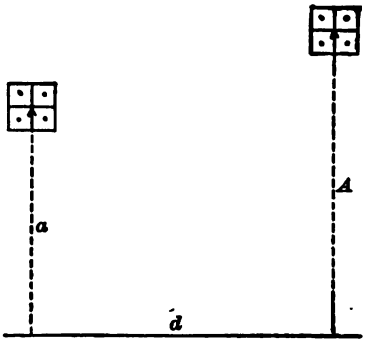

Fig. 11

Thus, suppose two coils A, B (Fig. 11) of square section are subdivided into four equal parts and by the method of Lyle, formula (27), the mutual inductance of the whole of B is computed on each of the four parts of A. If the sum differs appreciably from the result obtained by taking A and B as wholes in one calculation, then the four parts of B may be taken separately with respect to the separate parts of A. If one is doubtful whether this is sufficiently accurate, one of the sections of A may be subdivided further and calculated with respect to one section of B, to see whether there is any appreciable

difference due to this further subdivision. For coils of equal radii very accurate results for near coils can be obtained much more easily by using some of the other formulas.

EXAMPLES TO ILLUSTRATE THE FORMULAS FOR THE MUTUAL INDUCTANCE OF COILS OF RECTANGULAR SECTION

EXAMPLE 20. ROWLAND'S FORMULA (21). FOR COAXIAL COILS OF EQUAL RADII

$A=a=25 \qquad b=c=2 \text{ cm} \qquad d=10$ (Fig. 12.)

The mutual inductance of the two coils is $\frac{M}{n_1 n_2}=M_0+\Delta M$.

We find M_0 by formula 1, 8, or 13, and ΔM by 21 and 22.

$$M_0=107.4885\pi$$

$$k=\sin\gamma=\frac{50}{\sqrt{2600}}=0.9805807$$

$$k^2=0.9615383$$

$$\log_{10} F=0.4821754$$

$$\log_{10} E=0.0207625$$

By Table II, since $\tan\gamma=5$, $\log F=0.4821752$ and $\log E=0.0207626$. These slight differences in the logarithms obtained in the two different ways amount to scarcely one part in two million of F and E, respectively, and may usually be neglected. If more accurate values are required they may be obtained by carrying the interpolations further in Legendre's table, provided the angle γ is obtained with sufficient accuracy.

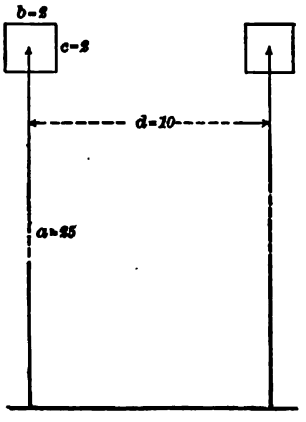

Fig. 12

Substituting these values in formula (22) we obtain

$$\frac{d^2M}{da^2}=-0.9081\pi$$

$$\frac{d^2M}{dx^2}=+1.0639\pi \qquad b^2=c^2=4$$

Substituting these values in formula (21) we obtain

$$\Delta M=.05194\pi$$

$$\therefore \frac{M}{n_1 n_2}=M_0+\Delta M=(107.4885+0.0519)\pi$$

$$=337.8481 \text{ cm.}$$

The correction ΔM thus amounts to about 1 part in 2000 of M. At a distance $d=20$ cm, the correction is over 1 part in 1000. For a coil of section 4×4 cm at $d=10$, ΔM would be four times as large as the value above, or about one part in five hundred, and at 20 cm one part in two hundred and fifty.

EXAMPLE 21. ROWLAND'S FORMULA (20). FOR COILS OF EQUAL RADII BUT UNEQUAL SECTION

Let us take $a=25$, $d=10$ as in the preceding example, but instead of supposing the sections of the coils to be equal let us take

$$b_1=4 \qquad b_2=2$$
$$c_1=1 \qquad c_2=2$$

The values of $\frac{d^2M_0}{dx^2}$ and $\frac{d^2M_0}{da^2}$ will be the same as in the preceding example. Substituting these in (20) we find $\Delta M=0.6974\pi$

$$M_0=107.4885\pi$$
$$\Delta M=\underline{\ \ 0.6974\pi}$$
$$\therefore\frac{M}{n_1n_2}=108.1859\pi=339.8761 \text{ cm.}$$

The correction here is fourteen times as great as in the previous example, although the areas of the cross sections of the two coils are the same as in the preceding case.

EXAMPLE 22. RAYLEIGH'S FORMULA (24). FOR COAXIAL COILS OF EQUAL RADII

$$A=a=25 \qquad b=4 \qquad c=1 \qquad d=10$$

We now find by formula (1) in accordance with formula (24) the mutual inductance of the following pairs of circles (Fig. 13):

O, 1 when $a=25$, $A=25.5$, $d=10$; O, 4 when $a=25$, $A=24.5$, $d=10$; O, 2 when $a=A=25$ and $d=8$; O, 3 when $A=a=25$, $d=12$ and finally O, O′ when $A=a=25$, $d=10$. Thus:

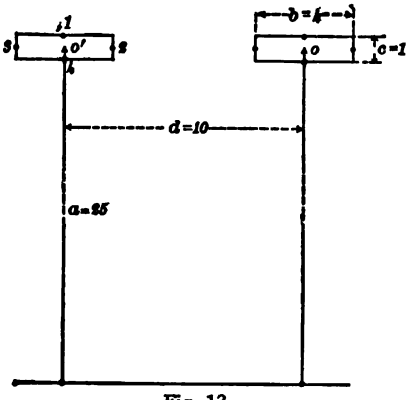

Fig. 13

$$M_1 = 109.3217\pi$$
$$M_4 = 105.4287\pi$$
$$M_2 = 127.3949\pi$$
$$M_3 = 91.9206\pi$$
$$434.0659\pi$$
$$M_0 = 107.4885\pi$$
$$326.5774\pi$$
$$\therefore M = 108.8591\pi$$
$$M_0 = 107.4885\pi$$
$$\Delta M = 1.3706\pi \text{ cm.}$$

EXAMPLE 23. RAYLEIGH'S FORMULA (23). COILS OF UNEQUAL RADII AND UNEQUAL SECTION

Let

$$A = 25 \quad b_1 = 4 \quad c_1 = 1 \quad d = 10$$
$$a = 20 \quad b_2 = 2 \quad c_2 = 3$$

We have then to calculate the mutual inductances of the following pairs of circles:

	A	a	d		A	a	d
M_1	24.5	20	10	M_6	25	20	11
M_2	25	20	8	M_7	25	21.5	10
M_3	25.5	20	10	M_8	25	20	9
M_4	25	20	12	M_0	25	20	10
M_5	25	18.5	10				

These have been calculated by means of Havelock's formula (16), with the following results:

$$M_1 = 248.41280$$
$$M_2 = 2[illegible].04027$$
$$M_3 = [illegible]8.77440 \qquad M_0 = 248.7873$$
$$M_4 = 214.75755$$
$$M_5 = 216.60185$$
$$M_6 = 231.04386$$
$$M_7 = 279.81417$$
$$M_8 = 268.09410$$
$$\text{Sum} = 1996.5390$$
$$2\,M_0 = 497.5746$$
$$\text{Diff} = 1498.9644$$
$$\frac{1}{6}\text{Diff.} = 249.8272 = \frac{M}{n_1 n_2}$$

EXAMPLE 24. LYLE'S FORMULA (27). FOR COILS OF SQUARE SECTION

$$A = a = 25 \text{ cm} \qquad b = c = 2 \text{ cm} \qquad d = 10 \text{ cm.}$$

The equivalent radius $r = a\left(1 + \frac{b^2}{24a^2}\right)$

$$r = 25\left(1 + \frac{4}{15000}\right) = 25.00667 \text{ cm.}$$

M is now found by using formula 1, 8, or 13, employing r in place of a as the radius.

The result is $M = 337.8475$, agreeing very closely with the result found under example 20.

$$M - M_0 = \Delta M = .0517\pi$$

EXAMPLE 25. LYLE'S FORMULA (28). FOR COILS OF RECTANGULAR SECTION

$$A = a = 25 \qquad b = 4 \qquad c = 1 \qquad d = 10$$

$$r = 25\left(1 + \frac{1}{15000}\right) = 25.00167$$

$\beta^2 = \frac{b^2 - c^2}{12} = \frac{15}{12} = 1.25$, $2\beta = 2.236$ cm, the distance apart of the two filaments which replace the coil (Fig. 14). We now find by formula (1), (8), or (13) the mutual inductances of two circles 1, 2 on the two circles 3, 4, where $a = 25.00167$ and d is 7.764, 10 and 12.236 cm, respectively. Thus:

$$\begin{aligned}
2\,M_{13} &= 215.00228\pi \\
M_{14} &= 90.31304\pi \\
M_{23} &= 130.14060\pi \\
4\,M &= 435.45592\pi \\
\therefore M &= 108.8640\ \pi \\
M_0 &= 107.4885\ \pi \\
\Delta M &= 1.3735\ \pi
\end{aligned}$$

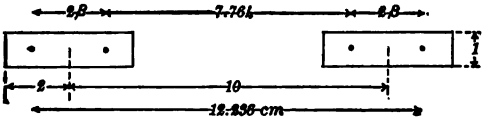

Fig. 14

ΔM = the correction for section of the coils whose dimensions are given above. These values of M and ΔM agree nearly with the results obtained in example 22 above.

EXAMPLE 26. LYLE'S FORMULA (28). FOR UNEQUAL COILS OF RECTANGULAR SECTION

Let us take the same coils as in example 23

$$A=25 \qquad b_1=4 \qquad c_1=1 \qquad d=10$$
$$a=20 \qquad b_2=2 \qquad c_2=3$$

For the first coil we find

$$r=25\left(1+\frac{c_1^2}{24A^2}\right)=25.001667 \text{ cm}$$
$$\beta=1.118034 \text{ cm}$$

For the second coil

$$r=20\left(1+\frac{b_1^2}{24a^2}\right)=20.008333 \text{ cm}$$
$$\delta=\ 0.645497$$
$$r+\delta=20.653830$$
$$r-\delta=19.362836$$

We then calculate the mutual inductance of the following pairs of circles:

	A	a	d
M_{13}	25.001667	20.653830	11.118034
M_{14}	"	19.362836	11.118034
M_{23}	"	20.653830	8.881966
M_{24}	"	19.362836	8.881966

The results by Havelock's formula (16) were

$$M_{13}=241.29369$$
$$M_{14}=216.91302$$
$$M_{23}=286.13490$$
$$M_{24}=255.03471$$
$$\text{Sum}=999.37632$$
$$\frac{1}{4}\text{Sum}=249.8441=\frac{M}{n_1n_2}$$

which differs from the value by Rayleigh's formula (23) by six or seven in a hundred thousand.

A more accurate value would, in each case, be found if each coil were subdivided and the formulas applied to each of the components as described on page 43. Such a proceeding is, however, rather tedious, although necessary in precise work.

EXAMPLE 27. ROSA'S FORMULA (29). FOR COILS OF EQUAL RADII

$$A=a=25 \qquad b=4 \qquad c=1 \qquad d=10$$

(same coils as examples 22, 25).

$$\log_e\frac{8a}{d}=\log_e 20=2.9957$$

$$\frac{3b^2+c^2}{96a^2}\cdot\log_e\frac{8a}{d}=\frac{49\times 2.9957}{60000}= \quad .0024465$$

$$\frac{b^2-c^2}{12d^2}=\frac{15}{1200} \quad = \quad .0125000$$

$$\frac{2b^4+2c-5b^2c^2}{120d^4}=\frac{434}{1200000} \quad = \quad .0003617$$

$$\frac{3b^6-3c^2+14b^2c^4-14b^4c^2}{504d^6}=\frac{8925}{504\times 10^6} \quad = \quad .0000177$$

$$\frac{6b^4+6c^4+5b^2c^2}{5760a^2d^2}=\frac{1622}{360\times 10^6} \quad = \quad .0000045$$

$$\frac{7c^2d^2}{1024a^4}\left(\log_e\frac{8a}{d}-\frac{163}{84}\right) \quad = \quad \underline{.0000018} \qquad .0153322$$

$$-\frac{11b^2-3c^2}{192a^2}= \quad -\frac{173}{120000}= -.0014417$$

$$-\frac{15b^2d^2}{1024a^4}\left(\log_e\frac{8a}{d}-\frac{97}{60}\right)= -\underline{.0000827} \qquad -\underline{.0015244}$$

$$.0138078$$

$$4a=100, \quad \therefore \quad \Delta M=1.3808\ \pi \text{ cm.}$$

This is a little larger value than found by formulas (24) and (28), and we shall see later that it is more nearly correct than either of the other values.

EXAMPLE 28. ROSA'S FORMULAS (30) AND (31). FOR COILS OF EQUAL RADII AND SQUARE SECTION

$$A=a=25 \qquad b=c=2 \qquad d=10$$

$$\log_e\frac{8a}{d}-1 \quad = \quad 2.9957-1 \quad = \quad 1.9957$$

$$\frac{17b^2}{240d^2} \quad = \quad \frac{68}{24000} \quad = \quad \underline{.0028} \qquad 1.9985$$

$$\frac{-a^2b^2}{5d^4} \quad = -\frac{2500}{50000} \quad = - \quad .0500$$

$$\frac{-3d^2}{16a^2}\left(\log_e\frac{8a}{d}-\frac{4}{3}\right) \quad = -\frac{300\times 1.6624}{10000}= - \quad \underline{.0499} \qquad -\underline{.0999}$$

$$1.8986$$

$$\frac{b^2}{6a}=\frac{4}{150} \quad \therefore \quad \Delta M=.05063\pi$$

The approximate formula (31) would have given .0519 (agreeing with formulas 21 and 27), which would be amply accurate for any experimental purpose. When the section is larger these small terms are, however, more important.

EXAMPLE 29. SECOND EXAMPLE BY FORMULA (30)

$$A=a=25 \qquad b=c=5 \qquad d=10$$

$$\log_e\frac{8a}{d}-1 = 1.9957$$

$$\frac{17b^2}{240d^2} = \underline{.0177} \qquad 2.0134$$

$$\frac{-a^2b^2}{5d^4} = -.3125$$

$$\frac{-3d^2}{16a^2}\left(\log_e\frac{8a}{d}-\frac{4}{3}\right) = -\underline{.0499} \qquad -\underline{.3624}$$

$$1.6510$$

$$\frac{b^2}{6a}=\frac{25}{150}$$

$$\therefore \Delta M=0.2752\pi$$

$$\underline{M_0=107.4885\pi} \text{ (see example 20)}$$

$$\therefore \frac{M}{n_1n_2}=107.7637\pi \text{ cm.}$$

This is a very simple formula for computing ΔM, and within a considerable range (i. e., d not larger than a and yet the coils not in contact) it is very accurate.

EXAMPLE 30. FORMULA (32). COILS OF EQUAL RADII, BUT UNEQUAL SECTION

For this we will take the coils of example 21

$$a=25 \qquad d=10$$
$$b_1=4 \qquad b_2=2$$
$$c_1=1 \qquad c_2=2$$

$$\begin{aligned}
\text{1st term} &= 0.0016227 \\
\text{2d} \quad \text{"} &= -0.0008542 \\
\text{3d} \quad \text{"} &= 0.0062500 \\
\text{4th} \quad \text{"} &= \underline{0.0000570} \\
\text{Sum} &= 0.0070755 \\
\therefore \Delta M &= 0.70755\pi \\
M_0 &= \underline{107.4885\pi} \\
\text{Sum} &= 108.1960\pi \\
\therefore \frac{M}{n_1 n_2} &= 339.9078 \text{ cm.}
\end{aligned}$$

This example shows that the fourth differentials neglected in (20) here amount to one part in ten thousand.

EXAMPLE 31. ROSA-WEINSTEIN FORMULA (33). FOR COILS OF EQUAL RADII AND EQUAL SECTION

$$a = 25 \qquad b = 4 \qquad c = 1 \qquad d = 10$$

$$\alpha_1 = 15.0000533 \qquad \sin^2\gamma = \frac{2500}{2600} = \frac{25}{26}$$

$$\alpha_2 = 0.0020267 \qquad \cos^2\gamma = \frac{100}{2600} = \frac{1}{26}$$

$$\alpha_3 = 0.2170000 \qquad \frac{c^2}{24a^2} = .0000667$$

$$\alpha_1 - \alpha_2 - \alpha_3 + (2\alpha_2 - 3\alpha_3)\cos^2\gamma + 8\alpha_3\cos^4\gamma = 14.7587120$$

$$\alpha_1 + \frac{\alpha_2}{2} + 2\alpha_3 + (2\alpha_2 + 3\alpha_3)\cos^2\gamma + 8\alpha_3\cos^4\gamma = 15.4628292$$

$$A = 0.0004730 \qquad \text{Also } F = 3.0351168$$

$$B = 0.0123901 \qquad E = 1.0489686$$

$$\left(F - E\right)\left(A + \frac{c^2}{24a^2}\right) = 0.0010719$$

$$EB = 0.0129968$$

$$\text{Sum} = 0.0140687$$

$$4\pi a \sin\gamma = 100\pi\sqrt{\frac{25}{16}} \quad \therefore \Delta M = 1.3795\pi \text{ cm.}$$

This is not as simple to calculate as (29) and when d is less than $a/2$ is less accurate than (29). But for $d = a$ or greater it is more accurate than (29), and indeed the most accurate of all the formulas.

EXAMPLE 32. FORMULA (34). MUTUAL INDUCTANCE IN TERMS OF SELF-INDUCTANCE. FOR COILS RELATIVELY NEAR

For $a=25$, $b=1$, $c=1$, we have, n being the number of turns in one of the two equal coils,

$$L_1 = 4\pi a n^2\,(4.103816)$$

For $b=2$, $c=1$,

$$L_2 = 4\pi a n^2\,(4\times 3.698695)$$

For $b=3$, $c=1$,

$$L = 4\pi a n^2\,(9\times 3.411766)$$

Then the mutual inductance of 1 on 3 is by formula (34)

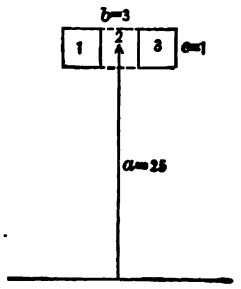

Fig. 15

$$M = 4\pi a n^2\left[\frac{L+L_1-2L_2}{2}\right]$$
$$= 4\pi a n^2\left[\frac{30.705894+4.103816-29.589560}{2}\right]$$
$$= 4\pi a n^2\times 2.610075$$
$$= 819.979\ n^2 \text{ cm.}$$

If $n=100$,

$$M = 8.19979 \text{ millihenrys,}$$

as the mutual inductance of coil 1 on coil 3, Fig. 15.

EXAMPLE 33. FORMULA (35). MUTUAL INDUCTANCE BY GEOMETRICAL MEAN DISTANCE

$$A = 25.1$$
$$a = 25.0$$
$$b = c = 0.1 \text{ cm}$$
$$d = 0.1 \text{ cm.}$$

The geometrical mean distance of two coils, corner to corner, as in Fig. 10, is 0.997701, and $\log \frac{r}{R} = 0.002302$

$$\therefore\ \Delta M = 100\times 0.002302\ (1.002)\pi$$
$$= 0.2307\pi \text{ cm.}$$

3. MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS

There are several formulas for the calculation of the mutual inductance of coaxial solenoids. Although few of these formulas

are exact, the approximate formulas often permit inductances to be calculated with very great accuracy by using a sufficient number of terms of the series by which they are expressed.

CONCENTRIC, COAXIAL SOLENOIDS OF EQUAL LENGTH

MAXWELL'S FORMULA [29]

The mutual inductance M of two coaxial solenoids of equal length (Fig. 16) is given by the following expression, due to Maxwell, where A and a are the radii of the outer and inner solenoids, respectively, l is the common length, and n_1 and n_2 the number of turns of wire per cm on the single layer winding of the outer and inner solenoids, respectively:

$$M = 4\pi^2 a^2 n_1 n_2 [l - 2A\alpha]$$

where

$$r = \sqrt{l^2 + A^2}$$

$$\begin{aligned}
\alpha = {} & \frac{A - r + l}{2A} - \frac{a^2}{16A^2}\left(1 - \frac{A^3}{r^3}\right) - \frac{a^4}{64A^4}\left(\frac{1}{2} + 2\frac{A^5}{r^5} - \frac{5}{2}\frac{A^7}{r^7}\right) \\
& - \frac{35}{2048}\frac{a^6}{A^6}\left(\frac{1}{7} - \frac{8}{7}\frac{A^7}{r^7} + 4\frac{A^9}{r^9} - 3\frac{A^{11}}{r^{11}}\right) \\
& - \frac{63}{2.128^2}\frac{a^8}{A^8}\left(\frac{5}{9} + \frac{64}{9}\frac{A^9}{r^9} - 48\frac{A^{11}}{r^{11}} + 88\frac{A^{13}}{r^{13}} - \frac{143}{3}\frac{A^{15}}{r^{15}}\right) \\
& - \frac{231}{512^2}\frac{a^{10}}{A^{10}}\left(\frac{7}{11} - \frac{128}{11}\frac{A^{11}}{r^{11}} + 128\frac{A^{13}}{r^{13}} - 416\frac{A^{15}}{r^{15}} + 520\frac{A^{17}}{r^{17}} - 221\frac{A^{19}}{r^{19}}\right) \\
& - \frac{429}{2.1024^2}\frac{a^{12}}{A^{12}}\left(\frac{21}{13} + \frac{512}{13}\frac{A^{13}}{r^{13}} - 640\frac{A^{15}}{r^{15}} + 3200\frac{A^{17}}{r^{17}} - 6800\frac{A^{19}}{r^{19}}\right. \\
& \qquad \left. + 6460\frac{A^{21}}{r^{21}} - 2261\frac{A^{23}}{r^{23}}\right) \qquad [36] \\
& - \frac{6435}{8192^2}\frac{a^{14}}{A^{14}}\left(\frac{11}{5} - \frac{1024}{15}\frac{A^{15}}{r^{15}} + 1536\frac{A^{17}}{r^{17}} - 10880\frac{A^{19}}{r^{19}} + \frac{103360}{3}\frac{A^{21}}{r^{21}}\right. \\
& \left. - 54264\frac{A^{23}}{r^{23}} + \frac{208012}{5}\frac{A^{25}}{r^{25}} - \frac{37145}{3}\frac{A^{27}}{r^{27}}\right) - \quad \cdots\cdots\cdots\cdots
\end{aligned}$$

[29] Electricity and Magnetism, Vol. II, § 678.

Putting

$$M = M_0 - \Delta M$$

$M_0 = 4\pi^3 a^2 n_1 n_2 l$ is the mutual inductance of an infinite outer solenoid and the finite inner solenoid, while ΔM is the correction due to the ends.

Equation (36) is Maxwell's expression, except that we have carried it out much further than Maxwell did. We would, however, emphasize that in the great majority of cases only three or four terms need be calculated in α, and in these only the first few terms in each parenthesis, to obtain a satisfactory accuracy.

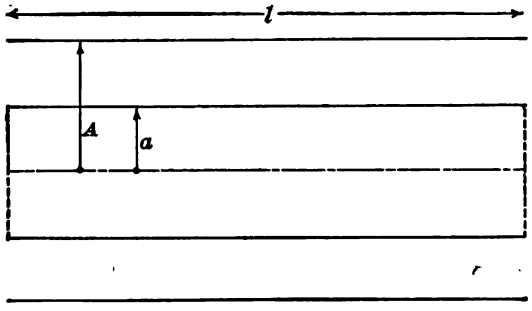

Fig. 16

Since, however, this formula is the most valuable single expression known for the case of solenoids of equal length, it has seemed advisable to extend the series far enough to take care of the most unfavorable cases, which may arise in practice. At the same time the extra terms found have proved of use in checking our extension of Ròiti's formula below.

It should be noticed that the algebraic sums of the coefficients in each of the parentheses is equal to zero. For very long coils $\left(\frac{A}{r} \text{ small}\right)$ the quantities in the parentheses are sensibly equal to the absolute term inside. For very short coils the parentheses are a little larger, reaching a maximum in the region $\frac{A}{r} = 0.9$, but falling abruptly to zero at the limit $\frac{A}{r} = 1$. The expression for α is

therefore rapidly convergent for coils of all lengths, even when the inner radius is nearly as great as the outer radius. In such cases the number of terms to be calculated in the above formula may become considerable, but even then it is simpler to use this series than to make the calculation with an absolute formula, such as those of Cohen or Nagaoka.

Equation (36) shows that the mutual inductance is proportional to $l-2A\alpha$; or the length l must be reduced by $A\alpha$ on each end. When a is small and l is large, α is $1/2$ approximately. That is, the length l is reduced by A, the radius of the outer solenoid.

For the case of two coils each of more than one layer the above formula may be used, A and a being the mean radii, and n_1 and n_2 the total number of turns per cm in all the layers. The result will be only approximate, but usually less in error than if one uses the formula of Maxwell § 679 quoted by Mascart and Joubert.[30]

When the solenoids are very long in comparison with the radii, formula (36) may be simplified by omitting the terms in A/l, A^2/r^2, A^4/r^4, etc. The expression for α then becomes

$$\alpha=\frac{1}{2}-\frac{a^2}{16A^2}-\frac{a^4}{128A^4}-\frac{5a^6}{2048A^6}-\cdots \qquad [37]$$

Heaviside[31] gives an extension of formula (37), but as it neglects $\frac{A}{l}$, $\frac{A^2}{r^2}$, etc., the additional terms are of no importance, being smaller than the terms already neglected in (37).

HAVELOCK'S FORMULA[32]

This formula for coaxial, concentric solenoids of equal length bears a close resemblance to the preceding, the main difference being that here l enters in place of the quantity $r=\sqrt{l^2+A^2}$ in

[30] Electricity and Magnetism, Vol. I, p. 533.

[31] There are some misprints in Heaviside, 2, p. 277. The radius of the *inner* solenoid should be c_2, of the *outer* c_1, and ρ is c_2^2/c_1^2.

[32] Phil. Mag., 15, p. 339; 1908. There is a misprint in Havelock's equation (25). In the factor outside the brackets, read a^2 instead of a.

equation (36). Using the same notation as in the latter this formula reads:

$$M = 4\pi^2 a^2 n_1 n_2 [l - 2A\beta]$$

where

$$\beta = \left[\frac{1}{2} - \frac{1}{16}\frac{a^2}{A^2} - \frac{1}{128}\frac{a^4}{A^4} - \frac{5}{2048}\frac{a^6}{A^6} - \frac{35}{32768}\frac{a^8}{A^8} - \cdots\right.$$
$$-\frac{1}{4}\frac{A}{l} + \frac{1}{16}\left(1 + \frac{a^2}{A^2}\right)\left(\frac{A}{l}\right)^3 - \frac{1}{32}\left(1 + \frac{3a^2}{A^2} + \frac{a^4}{A^4}\right)\left(\frac{A}{l}\right)^5$$
$$\left. + \frac{5}{256}\left(1 + 6\frac{a^2}{A^2} + 6\frac{a^4}{A^4} + \frac{a^6}{A^6}\right)\left(\frac{A}{l}\right)^7 - \cdots\right] \qquad [38]$$

Havelock gives the expressions for the general terms in $\frac{A}{a}$ and $\frac{A}{l}$, so that the computation of β may be carried out so as to include terms of higher order when necessary. These expressions are

$$-\frac{(2n-1)[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)]^2}{2^{2n+1}n!(n+1)!}\left(\frac{a}{A}\right)^{2n}$$

and

$$\frac{(-1)^s(2s)!\,F\left(-s-1, -s, 2, \frac{a^2}{A^2}\right)}{2^{2s+2}s!\,(s+1)!}\left(\frac{A}{l}\right)^{2s+1}$$

where F is a hypergeometric series in $\frac{a^2}{A^2}$, all of whose terms after that in $\left(\frac{a}{A}\right)^{2s}$ are zero.

$$F(\alpha,\beta,\gamma,z) = 1 + \frac{\alpha\beta}{1.\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}z^2$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}z^3 + \cdots$$

Formula (38) may be regarded as intermediate between (36) and (37), being applicable only to coils whose length is greater than the radius of the larger coil. In such cases, however, it furnishes a valuable check on Maxwell's formula.

CONCENTRIC COAXIAL SOLENOIDS, INNER COIL SHORTER THAN THE OUTER

RÒITP'S FORMULA

For a pair of concentric, coaxial solenoids of which the inner solenoid is shorter than the outer, we have the following:[23]

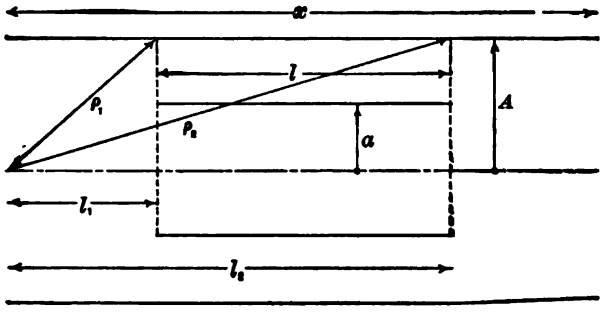

Fig. 17

$$
\begin{aligned}
M = 4\pi^2 a^2 n_1 n_2 \Bigg[& \rho_2 - \rho_1 + \frac{a^2A^2}{8}\left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right) - \frac{a^4A^2}{16}\left(\frac{1}{\rho_1^5} - \frac{1}{\rho_2^5}\right) \\
& + \frac{5}{64}a^4A^4\left(1 + \frac{1}{2}\frac{a^2}{A^2}\right)\left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7}\right) \\
& - \frac{35}{256}a^6A^4\left(1 + \frac{1}{5}\frac{a^2}{A^2}\right)\left(\frac{1}{\rho_1^9} - \frac{1}{\rho_2^9}\right) \\
& + \frac{105}{1024}a^6A^6\left(1 + \frac{9}{5}\frac{a^2}{A^2} + \frac{1}{5}\frac{a^4}{A^4}\right)\left(\frac{1}{\rho_1^{11}} - \frac{1}{\rho_2^{11}}\right) \\
& - \frac{693}{2048}a^8A^6\left(1 + \frac{2}{3}\frac{a^2}{A^2} + \frac{1}{21}\frac{a^4}{A^4}\right)\left(\frac{1}{\rho_1^{13}} - \frac{1}{\rho_2^{13}}\right) \\
& + \frac{3003}{16384}a^8A^8\left(1 + 4\frac{a^2}{A^2} + \frac{10}{7}\frac{a^4}{A^4} + \frac{1}{14}\frac{a^6}{A^6}\right)\left(\frac{1}{\rho_1^{15}} - \frac{1}{\rho_2^{15}}\right) - \cdots
\end{aligned}
\qquad [39]
$$

in which (see Fig. 17)

[23] For the derivation and method of extension of this formula see this Bulletin, 8, pp. 309-310. Recently we have carried it out still further to include the case of coils of moderate length. This formula was originally given (without proof and including the main term in $\left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7}\right)$ only) in this Bulletin, 2, p. 130; 1906.

$$\rho_1 = \sqrt{l_1^2 + A^2} \text{ where } l_1 = \frac{x - l}{2}$$

$$\rho_2 = \sqrt{l_2^2 + A^2} \quad \text{"} \quad l_2 = \frac{x + l}{2}$$

$l = l_2 - l_1 =$ length of inner solenoid.

$x =$ length of outer solenoid and A and a the radii.

When $\frac{l}{x}$ is small (case of short inner coil), $(\rho_2 - \rho_1)$ is most accurately calculated by the exact formula $(\rho_2 - \rho_1) = \frac{xl}{\rho_1 + \rho_2}$, the denominator being calculated from the above expressions for ρ_1 and ρ_2.

For long coils $\left(\frac{2A}{x} \text{ small}\right)$ the above formula is rapidly convergent, especially if the inner coil is considerably shorter than the outer. This formula may also be used for short coils $\left(\frac{x}{2A} \text{ small}\right)$, the convergence being most rapid when the radius of the inner coil is small in comparison with that of the outer. For very short coils, we have expanded formula (39) in a series in ascending powers of $\frac{a^2}{A^2}$. This formula is, however, not so accurate, nor so simple to use as that of Searle and Airey, and has not been included in this collection.

A peculiarity of Ròiti's formula is that the successive terms, especially in the case of short coils, are nearly equal in pairs. Thus the terms in $\left(\frac{1}{\rho_1^5} - \frac{1}{\rho_2^5}\right)$ and $\left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7}\right)$ are of the same order of magnitude, but of opposite sign; similarly for the terms involving the ninth and eleventh powers of ρ_1 and ρ_2, and so on. For the limiting case $x = l$, Ròiti's formula goes over into Maxwell's (36), as would be expected, since both are derived by integration of the same original expression between appropriate limits. To obtain, however, the same precision, twice as many terms have to be calculated in Ròiti's formula as in Maxwell's. We see from these considerations, that in using Ròiti's formula, the inner coil need not be very different in length from the outer coil, although in general the convergence is better with a relatively short inner solenoid.

GRAY'S FORMULA

Gray[34] gives a general expression for the mutual kinetic energy of two solenoidal coils which may or may not be concentric, and their axes may be at any angle ϕ. The most important case in practice is when the two coils are coaxial. In that case the zonal harmonic factors in each term reduce to unity, and half the terms become zero. Putting the current in each equal to unity, the mutual kinetic energy becomes the mutual inductance M.

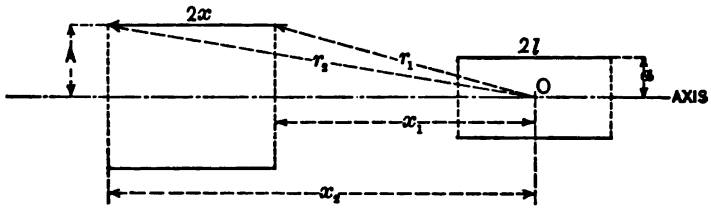

Fig. 18

Let $2x$, A, n_1 be respectively the length, radius, and number of turns per cm of one of the coils, and $2l$, a, n_2 be the corresponding quantities for the other solenoid. Let, further, x_1 and x_2 be the distances, along the axis, between the center of the coil with radius a and the nearer and further end planes, respectively, of the coil with radius A, and let r_1 and r_2 be the diagonals (Fig. 18).

$$r_1=\sqrt{x_1^2+A^2} \qquad r_2=\sqrt{x_2^2+A^2}$$

Gray's expression with these changes becomes

$$M=\pi^2a^2A^2n_1n_2[K_1k_1+K_3k_3+K_5k_5+\cdots] \qquad [40]$$

where K_1, K_3, etc., are functions of x and A, and k_1, k_3, etc., are functions of l and a.[35]

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[34] Absolute Measurements, 2, Part I, p. 274, equation 53.
[35] Rosa, this Bulletin, 8, p. 221; 1907.

$$K_1 = \frac{2}{A^2}\left(\frac{x_2}{r_2} - \frac{x_1}{r_1}\right)$$

$$K_3 = \frac{1}{2}\left(\frac{x_1}{r_1^5} - \frac{x_2}{r_2^5}\right)$$

$$K_5 = -\frac{A^2}{8}\left\{\frac{x_1}{r_1^9}\left(3 - 4\frac{x_1^2}{A^2}\right) - \frac{x_2}{r_2^9}\left(3 - 4\frac{x_2^2}{A^2}\right)\right\}$$

$$= -\frac{A^2}{8}\left\{\frac{x_1}{r_1^9}X_2' - \frac{x_2}{r_2^9}X_2''\right\} \qquad [40a]$$

$$K_7 = \frac{A^4}{8}\left\{\frac{x_1}{r_1^{13}}\left(\frac{5}{2} - 10\frac{x_1^2}{A^2} + 4\frac{x_1^4}{A^4}\right) - \frac{x_2}{r_2^{13}}\left(\frac{5}{2} - 10\frac{x_2^2}{A^2} + 4\frac{x_2^4}{A^4}\right)\right\}$$

$$= \frac{A^4}{8}\left\{\frac{x_1}{r_1^{13}}X_4' - \frac{x_2}{r_2^{13}}X_4''\right\}$$

. . . . . .

$$k_1 = 2l$$

$$k_3 = -a^2 l\left(3 - 4\frac{l^2}{a^2}\right) = -a^2 l L_2$$

$$k_5 = a^4 l\left(\frac{5}{2} - 10\frac{l^2}{a^2} + 4\frac{l^4}{a^4}\right) = a^4 l L_4$$

$$k_7 = -a^6 l\left(\frac{35}{16} - \frac{35}{2}\frac{l^2}{a^2} + 21\frac{l^4}{a^4} - 4\frac{l^6}{a^6}\right) = -a^6 l L_6$$

. . . . . .

This formula is simple and convenient for calculation, if only a few terms need be evaluated. This is the case when r_1 and r_2 are large (coils relatively far apart). The coefficients L_{2n}, X'_{2n} and X''_{2n} are derived from the same polynomial S_{2n} by substituting $\frac{l^2}{a^2}$ $\frac{x_1^2}{A^2}$, and $\frac{x_2^2}{A^2}$, respectively.

For short coils relatively far apart these polynomials are all small. Table XVIII gives values of the polynomial S_{2n} with varying argument, to aid in calculations where great accuracy is not desired, or to aid in making preliminary calculations to see whether the convergence will be satisfactory in any particular case.

If the coils are concentric, and the ratio of the length of the winding of the outer coil to the radius is $\sqrt{3}$ to 1, $K_5 = 0$, and if the same condition holds for the inner coil, $k_3 = 0$. If in addition a is considerably smaller than A, the terms of higher order become negligible and (40) reduces to

$$M = \frac{2\pi^2 a^2 N_1 N_2}{d} \qquad [41]$$

where d is half the diagonal of the outer coil, $=\sqrt{x^2+A^2}$. When the dimensions depart slightly from these theoretical ratios the small correction terms to (41) can be calculated.[36a] The general case for concentric coils is treated in the next section.

SEARLE AND AIREY'S FORMULA

The following expression for the mutual inductance of two concentric, coaxial solenoidal coils (Fig. 19) has been given by Searle and Airey:[36]

$$\begin{aligned} M &= g_1G_1+g_3G_3+g_5G_5+g_7G_7+\ \cdots \\ &= \frac{2\pi^2a^2N_1N_2}{d}\left[1-\frac{A^2}{2d^4}\,\frac{4l^2-3a^2}{4}-\frac{A^2(4x^2-3A^2)}{8d^8}\cdot\frac{8l^4-20l^2a^2+5a^4}{8}\right. \\ &\quad \left.-\frac{A^2(8x^4-20x^2A^2+5A^4)}{16d^{12}}\cdot\frac{(64l^6-336l^4a^2+280l^2a^4-35a^6)}{64}-\cdots\right] \end{aligned} \qquad [42]$$

The notation of (42) differs slightly from that used by Searle and Airey.

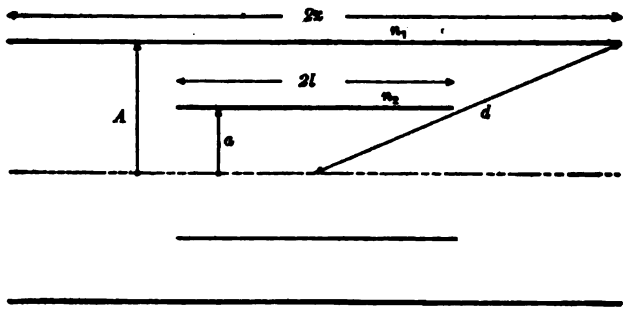

Fig. 19

Equation (42) has been extended and put for greater convenience in calculation into the form[37] shown on next page.

[36a] Rosa, this Bulletin, 8, p. 221; 1907.
[36] The Electrician (London), 56, p. 318; 1905.
[37] Rosa, this Bulletin, 8, p. 224; 1907.

$$M=\frac{2\pi^2a^2N_1N_2}{d}\Big[1+\frac{A^2a^2}{8d^4}L_2+\frac{A^4a^4}{32d^8}X_2L_4$$
$$+\frac{A^6a^6}{32d^{12}}X_4L_6+\frac{A^8a^8}{32d^{16}}X_6L_8+\cdots$$
$$+\frac{1}{32}\left(\frac{Aa}{d^2}\right)^{2n}X_{2n-2}L_{2n}+\cdots\Big] \qquad [43]$$

where

$$X_2=3-4\frac{x^2}{A^2} \qquad L_2=3-4\frac{l^2}{a^2} \qquad d=\sqrt{x^2+A^2}$$

$$X_4=\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4} \qquad L_4=\frac{5}{2}-10\frac{l^2}{a^2}+4\frac{l^4}{a^4}$$

$$X_6=\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6} \qquad L_6=\frac{35}{16}-\frac{35}{2}\frac{l^2}{a^2}+21\frac{l^4}{a^4}-4\frac{l^6}{a^6}$$

$$X_8=\frac{63}{32}-\frac{105}{4}\frac{x^2}{A^2}+63\frac{x^4}{A^4} \qquad L_8=\frac{63}{32}-\frac{105}{4}\frac{l^2}{a^2}+63\frac{l^4}{a^4}-36\frac{l^6}{a^6}+4\frac{l^8}{a^8}$$
$$-36\frac{x^6}{A^6}+4\frac{x^8}{A^8}$$

$$L_{10}=\frac{231}{128}-\frac{1155}{32}\frac{l^2}{a^2}+\frac{1155}{8}\frac{l^4}{a^4}-165\frac{l^6}{a^6}$$
$$+55\frac{l^8}{a^8}-4\frac{l^{10}}{a^{10}}$$

$N_1=2xn_1$ and $N_2=2ln_2$ are the total number of turns on the two solenoids. This formula reduces to (41) when the terms after the first are negligible, as they are when the conditions assumed for (41) are fulfilled. The above expressions for L_2, X_2 show what these conditions are in order to make the second and third terms zero. If l^2/a^2 is slightly more or less than ¾, (43) gives the value of the second term which is neglected in (41), etc.

The degree of convergence of Searle and Airey's formula depends primarily on the magnitude of the quantity $\frac{A^2a^2}{d^4}$; in certain cases, however, the values of the coefficients become of equal importance, making it necessary to examine carefully into the degree of convergence of the formula, since the terms of higher order are sometimes larger than those immediately preceding. Since the X and L

coefficients are polynomials in $\frac{l^2}{a^2}$ and $\frac{x^2}{A^2}$, each one will have a finite number of roots depending on the degree of the polynomial. The values of these coefficients will therefore, with increasing $\frac{l}{a}$ or $\frac{x}{A}$, oscillate between positive and negative values, each maximum or minimum being greater than that preceding, until, for values of the argument greater than the largest root, the values of the functions increase indefinitely without limit.

For short coils $\left(\frac{x}{A} \text{ and } \frac{l}{a} \text{ small}\right)$ the coefficients will evidently be confined to moderate values, and if, further, the inner radius is small relatively to the outer, the convergence will be very rapid. For longer coils the coefficients may attain very large values, and the convergence become very unsatisfactory, in spite of the fact that $\frac{A^2a^2}{d^4}$ is, for given radii, smaller with long coils than with short coils. The conditions are so complicated that we have calculated (Table XIX) certain values of the coefficients to aid in deciding whether, in any given case, the convergence will be satisfactory or not. The values given for $\frac{x}{A}$ and $\frac{l}{a}$ less than unity will also be found useful in calculations of the mutual inductance of short coils by Searle and Airey's formula, when the highest precision is not required. Coefficients of higher order than those given above are calculated by the formula

$$L_{2n} = \sum_{p=0}^{p=n} \frac{(-1)^{n-p}(2n+1)2n(2n-1)\cdot\cdot\cdot\cdot[2n-(2p-2)]}{\left(\frac{p+1}{4}\right)2^2\cdot 4^2\cdot 6^2\cdot\cdot\cdot\cdot\cdot(2p)^2}\left(\frac{l}{a}\right)^{2n-2p}$$

X_{2n} is calculated by the same expression in $\frac{x}{A}$ instead of $\frac{l}{a}$.

Table XVIII includes all the positive and negative maxima as well as the zero points of the coefficients up to and including L_{14} or X_{14}, together with the values at a number of intermediate points. Although, from the nature of the case, a table to serve as the basis of accurate calculations would be somewhat bulky, those given should suffice to simplify the use of this valuable formula.

COHEN'S FORMULA[38] FOR ANY TWO COAXIAL, CONCENTRIC SOLENOIDS

This is an absolute formula for two coaxial, concentric solenoids of lengths $2l_1$ and $2l_2$, Fig. 20.

$$M = 4\pi n_1 n_2 (V - V_1)$$

$$\begin{aligned} V = & -(A^2 - a^2)c[F\{F(k',\theta) - E(k',\theta)\} - EF(k',\theta)] \\ & + \frac{c^4 - (A^2 - 6Aa + a^2)c^2 - 2(A^2 - a^2)^2}{3\sqrt{(A+a)^2 + c^2}} . F \\ & + \frac{2(A^2 + a^2) - c^2}{3}\sqrt{(A+a)^2 + c^2} . E - c(A^2 - a^2)\frac{\pi}{2} \end{aligned} \qquad [44]$$

V_1 is obtained from V by replacing c by c_1,

$$c = l_1 + l_2 \qquad c_1 = l_1 - l_2,$$

F and E are the complete elliptic integrals of the first and second kind to modulus k, where $k^2 = \dfrac{4Aa}{(A+a)^2 + c^2}$

$F(k', \theta)$ and $E(k', \theta)$ are the incomplete elliptic integrals of modulus k' and amplitude θ,

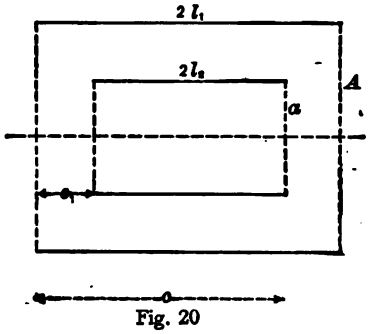

Fig. 20

$$k'^2 = 1 - k^2 = 1 - \frac{4Aa}{(A+a)^2 + c^2}$$

$$= \frac{(A-a)^2 + c^2}{(A+a)^2 + c^2}$$

$$\sin^2\theta = \frac{(A^2 - a^2)^2 + c^2(A-a)^2}{(A^2 - a^2)^2 + c^2(A+a)^2}$$

NAGAOKA'S FORMULA FOR ANY COAXIAL SOLENOIDS

Nagaoka has recently given[39] an absolute formula for the mutual inductance of two coaxial solenoids, whether concentric or not, and

[38] This Bulletin, 8, p. 301; 1907.

[39] Jour. Coll. Sci., Tokyo, 27, art. 6; 1909. There are a number of misprints in Nagaoka's article, which we have detected and corrected by a careful check on the derivation of the formulas.

has expanded this in q functions in a form suitable for calculation. In the notation of Fig. 18,

$$M = 4\pi n_1 n_2 Aa(I_1 - I_2 - I_3 + I_4) \qquad [45]$$

where I_1, I_2, I_3, and I_4 are the values of the integral I, given below with the arguments

$$c = d + (x + l),\ d + (x - l),\ d - (x - l) \text{ and } d - (x + l)$$

respectively, where d is the distance between centers.

The expression for I is, in the Weierstrassian notation,

$$I = 2\left\{\left(\frac{g_2}{6} - \mathrm{p}^2 v\right)\omega_1 + \mathrm{p}v.\eta_1 + \frac{\mathrm{p}'v}{2}\left(\eta_1 v - \omega_1 \frac{\sigma'}{\sigma}(v)\right)\right\}$$

where v is an auxiliary quantity, and ω_1 and g_2 are respectively the real semiperiod and invariant of the Weierstrassian function $\mathrm{p}u$.

To calculate I, Nagaoka divides it in two parts

$$I' = \left(\frac{g_2}{6} - \mathrm{p}^2 v\right)\omega_1 + \mathrm{p}v.\eta_1$$

$$I'' = \frac{\mathrm{p}'v}{2}\left(\eta_1 v - \omega_1 \frac{\sigma'}{\sigma}(v)\right)$$

We then calculate the following auxiliary quantities

$$\alpha = \left(\frac{2}{Aa}\right)^{\frac{1}{2}}$$

$$P_1 = (\mathrm{p}v - e_1) = -\frac{c^2}{2Aa\alpha}$$

$$P_2 = (\mathrm{p}v - e_2) = \frac{(A-a)^2}{2Aa\alpha}$$

$$P_3 = (\mathrm{p}v - e_3) = \frac{(A+a)^2}{2Aa\alpha}$$

and thence $(e_1 - e_2)$, $(e_1 - e_3)$, and $(e_2 - e_3)$, which with the relation $(e_1 + e_2 + e_3) = 0$ enable us to find $\mathrm{p}v$.

The very small quantity q is found, as in formula (8), by the relations (see also Table XV)

$$q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \cdots$$

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} \qquad k'^2 = \frac{e_1 - e_2}{e_1 - e_3} \qquad l = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} = \frac{k^2}{(1 + k')(1 + \sqrt{k'})^2}$$

and ω_1 may be calculated by any one of the following equations (the other two serving as checks):

$$\omega_1 = \frac{2\pi\sqrt{q}}{\sqrt{e_2 - e_3}}(1 + q^3 + q^6 + q^{12} + \cdots)^2$$

$$\omega_1 = \frac{\pi}{2\sqrt{e_1 - e_3}}(1 + 2q + 2q^4 + 2q^9 + \cdots)^2$$

$$\omega_1 = \frac{\pi}{2\sqrt{e_1 - e_2}}(1 - 2q + 2q^4 - 2q^9 + \cdots)^2$$

The term I' is now given by either of the following two formulas, which give a check on the calculation:

$$I' = -\left\{\frac{P_1(P_2 + P_3)}{2} + \frac{(P_1 + 2P_2)(e_2 - e_3)}{6}\right\}\omega_1 - \frac{pv}{4\omega_1}\frac{\theta''_3(0)}{\theta_3(0)}$$

$$I' = -\left\{\frac{P_1(P_2 + P_3)}{2} - \frac{(P_1 + 2P_3)(e_2 - e_3)}{6}\right\}\omega_1 - \frac{pv}{4\omega_1}\frac{\theta_0''(0)}{\theta_0(0)}$$

The quotients of the θ functions are easily calculated from the known value of q and the relations

$$\frac{\theta_3''(0)}{\theta_3(0)} = -\frac{8\pi^2(q + 4q^4 + 9q^9 + \cdots)}{1 + 2q + 2q^4 + 2q^9 + \cdots}$$

$$\frac{\theta_0''(0)}{\theta_0(0)} = \frac{8\pi^2(q - 4q^4 + 9q^9 - \cdots)}{1 - 2q + 2q^4 - 2q^9 + \cdots}$$

To calculate I'' we have

$$I'' = -\frac{\pi c}{8Aa}(A^2 - a^2)\left[\sqrt{\frac{b-1}{b+1}} + 4q^2\sqrt{b^2 - 1}\left\{1 - q^2(2b - 1)\right\}\right]$$

the expression in the brackets being nearly equal to unity. The quantity b is calculated from the equations

$$-b = \cos 2\pi w = \frac{s}{q}(1 + 2q^4 \cos 4\pi w + \cdots)$$

$$s = \frac{1}{2}\frac{\sqrt[4]{e_1 - e_3}\sqrt{P_2} - \sqrt[4]{e_1 - e_2}\sqrt{P_3}}{\sqrt[4]{e_1 - e_3}\sqrt{P_2} + \sqrt[4]{e_1 - e_2}\sqrt{P_3}}$$

first putting $\cos 2\pi w$ equal to its approximate value $\frac{s}{q}$, and then computing the small correction in $\cos 4\pi w$ from $\cos 2\pi w$, remembering that w is a pure imaginary. The correction to b thus found is often negligible.

The term I'' becomes less important as the difference of the radii of the solenoids becomes small, and vanishes for equal radii. If, further, the lengths of the solenoids be equal also, $I_2 = I_3$, and we have only three of the integrals to evaluate, and only the first term I' in each of these.

For concentric, coaxial solenoids $d = 0$, and consequently $I_1 - I_2 = I_4 - I_3$, so that only two integrals must be calculated.

On account of the number of auxiliary quantities involved, Nagaoka's formula should not be employed except when the various series formulas given in this section are all shown to be inadequate. It is, however, simpler to use Nagaoka's formula than the elliptic integral formula from which it is derived, or any other expression in incomplete integrals yet derived, even supposing Legendre's table of incomplete integrals to be available.

RUSSELL'S FORMULAS[40]

Russell's formula for coaxial solenoids in the notation of this paper is

$$M = 4\pi^2 a^2 n_1 n_2 \Big[R_1 \Big\{ 1 - \frac{1}{2} q_2 k_1^2 - \frac{1}{2} \cdot \frac{1}{4} q_3 k_1^4 - \frac{1.1.3}{2.4.6} q_4 k_1^6 - \frac{1.1.3.5}{2.4.6.8} q_5 k_1^8 - \frac{1.1.3.5.7}{2.4.6.8.10} q_6 k_1^{10} - \cdots \Big\}$$
$$- R_2 \Big\{ 1 - \frac{1}{2} q_2 k_2^2 - \frac{1}{2} \cdot \frac{1}{4} q_3 k_2^4 - \text{terms with above coefs.} \Big\} \Big] \quad [46]$$

where

$$R_1^2 = (A+a)^2 + (l_1 + l_2)^2 \qquad k_1^2 = \frac{4Aa}{R_1^2}$$

$$R_2^2 = (A+a)^2 + (l_1 - l_2)^2 \qquad k_2^2 = \frac{4Aa}{R_2^2}$$

[40] Alexander Russell, Phil. Mag., Apr. 1907, p. 420.

$$q_n = \frac{(A+a)^2}{4Aa} q_{n-1} - \frac{1}{n}\frac{1.3.5\ldots.2n-3}{2.4.6\ldots.2n-2}\frac{A}{a}$$

$$q_2 = \frac{(A+a)^2}{4Aa} - \frac{1}{2}\frac{1}{2}\frac{A}{a}$$

$$q_3 = \frac{(A+a)^2}{4Aa} q_2 - \frac{1}{3}\frac{1.3}{2.4}\frac{A}{a}$$

etc.

A and a are the radii of the outer and inner cylinders respectively, $2l_1$ and $2l_2$ their lengths, Fig. 20, and n_1, n_2 the number of turns of wire per cm in the two windings. This formula applies only when the inner coil is shorter than the outer. For two coils of equal length the second part of the above formula is not convergent, and hence it must be replaced by an expression in elliptic integrals. The formula thus becomes (equation 42 in Russell's paper)

$$M = 4\pi^2 a^2 n_1 n_2 \left[R_1 \left\{ 1 - \frac{1}{2} q_2 k_1^2 - \frac{1}{8} q_3 k_1^4 - \cdots \text{ as above} \right\} \right]$$

$$+ \frac{32\pi Aa}{3(A+a)} n_1 n_2 [(A^2+a^2)(F-E) - 2\,AaF] \qquad [47]$$

the modulus of the elliptic integrals being $k_2 = \frac{2\sqrt{Aa}}{A+a}$

This formula gives an accurate result for equal solenoids of considerable length, but Maxwell's formula (36) is just as accurate and much more convenient.

For short coils neither (46) nor (47) will apply, but for that case as well as other cases Russell's general formula may be used. As the latter is equivalent to (44) it is not here given.

MUTUAL INDUCTANCE OF A SHORT SECONDARY ON THE OUTSIDE OF A LONG PRIMARY

This is an important case in practice. Havelock[41] has shown that the mutual inductance of two such solenoids is the same as that of two coils with the same radii and lengths, but with the shorter coil inside. That is, the mutual inductance of a coil of length l and radius A outside of a coil of length x and radius a is the same as

[41] Phil. Mag., 15, p. 343; 1908.

the mutual inductance of a coil of length x and radius A outside of a coil of length l and radius a.

The series formulas already given for the latter case may therefore be applied to the present case directly if the quantities l and x, or l_1 and l_2, be interchanged.

In Ròiti's formula we put, therefore, $l_1 = \frac{l-x}{2}$ instead of $\frac{x-l}{2}$. The values of ρ_1 and ρ_2 are, however, unchanged and the formula may be used just as it stands.

Russell's formula being symmetrical in l_1 and l_2 requires no change whatever.

In Searle and Airey's formula we have to put

$$d = \sqrt{l^2 + A^2}$$

$$L_2 = 3 - 4\frac{x^2}{a^2}$$

$$X_2 = 3 - 4\frac{l^2}{A^2}$$

$$L_4 = \frac{5}{2} - 10\frac{x^2}{a^2} + 4\frac{x^4}{a^4}$$

etc.

Cohen's and Nagaoka's formulas apply without change as would be expected.

ROSA'S FORMULAS FOR SINGLE LAYER COILS OF EQUAL RADII

The mutual inductance of two coaxial single layer coils of equal radii is given by the following expression:

$$\frac{M}{N_1 N_2} = M_0 + \Delta M$$

where M_0 is the mutual inductance of the two parallel circles at the centers of the coils and ΔM is given by the following expression:[42]

[42] Rosa, this Bulletin, 2, p. 351; 1906.

$$\Delta M = 4\pi a\Bigg[\frac{1}{24}\frac{b_1^2+b_2^2}{d^2}\Big\{1+\frac{3}{8}\frac{d^2}{a^2}\Big(\log\frac{8a}{d}-\frac{11}{6}\Big)-\frac{45}{256}\frac{d^4}{a^4}\Big(\log\frac{8a}{d}-\frac{97}{60}\Big)$$
$$+\frac{1050}{128^2}\frac{d^6}{a^6}\Big(\log\frac{8a}{d}-\frac{54}{35}\Big)-\frac{44100}{128^3}\frac{d^8}{a^8}\Big(\log\frac{8a}{d}-\frac{3793}{2520}\Big)+\cdots\Big\}$$
$$+\frac{(b_1^4+b_2^4+\frac{10}{3}b_1^2b_2^2)}{320d^4}\Big\{1+\frac{1}{16}\frac{d^2}{a^2}-\frac{15}{256}\frac{d^4}{a^4}\Big(\log\frac{8a}{d}-\frac{187}{60}\Big) \qquad [48]$$
$$+\frac{2100}{128^2}\frac{d^6}{a^6}\Big(\log\frac{8a}{d}-\frac{893}{420}\Big)-\cdots\Big\}$$
$$+\frac{b_1^6+b_2^6+7(b_1^4b_2^2+b_1^2b_2^4)}{2688d^6}\Big\{1+\frac{3}{160}\frac{d^2}{a^2}-\frac{3}{1024}\frac{d^4}{a^4}+\cdots\Big\}$$
$$+\frac{(b_1^8+b_2^8)+12(b_1^6b_2^2+b_1^2b_2^6)+\frac{126}{5}b_1^4b_2^4}{18432d^8}\Big\{1+\frac{1}{112}\frac{d^2}{a^2}+\cdots\Big\}\Bigg]$$

For coils of equal breadth and equal radii (Fig. 21) $b_1 = b_2 = b$ and we may write the equation (48) as follows:

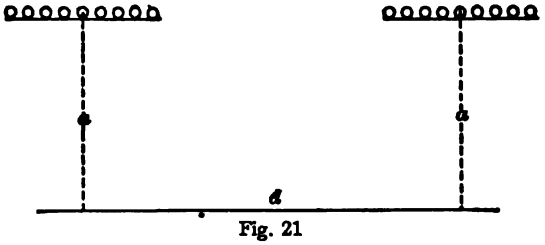

Fig. 21

$$\Delta M = 4\pi a\Bigg[\frac{b^2}{12d^2}\Big\{1+\frac{3}{8}\frac{d^2}{a^2}\Big(\log\frac{8a}{d}-\frac{11}{6}\Big)-\frac{45}{256}\frac{d^4}{a^4}\Big(\log\frac{8a}{d}-\frac{97}{60}\Big)$$
$$+\frac{1050}{128^2}\frac{d^6}{a^6}\Big(\log\frac{8a}{d}-\frac{54}{35}\Big)-\frac{44100}{128^3}\frac{d^8}{a^8}\Big(\log\frac{8a}{d}-\frac{3793}{2520}\Big)+\cdots\Big\}$$
$$+\frac{1}{60}\frac{b^4}{d^4}\Big\{1+\frac{1}{16}\frac{d^2}{a^2}-\frac{15}{256}\frac{d^4}{a^4}\Big(\log\frac{8a}{d}-\frac{187}{60}\Big) \qquad [49]$$
$$+\frac{2100}{128^2}\frac{d^6}{a^6}\Big(\log\frac{8a}{d}-\frac{893}{420}\Big)-\cdots\Big\}$$
$$+\frac{1}{168}\frac{b^6}{d^6}\Big\{1+\frac{3}{160}\frac{d^2}{a^2}-\frac{3}{1024}\frac{d^4}{a^4}+\cdots\Big\}+\frac{1}{360}\frac{b^8}{d^8}\Big\{1+\frac{1}{112}\frac{d^2}{a^2}-\cdots\Big\}\Bigg]$$

This expression will give a very accurate value of ΔM for two coils not nearer together than their breadth if a is considerably greater than b, the breadth of the coil.

For coils which are not so near together the Rosa-Weinstein formula[43] may be used.

$$\Delta M = 4\pi a \sin\gamma \, [(F-E)\, P + EQ] \qquad [50]$$

where

$$P = \frac{\cos^2\gamma}{24d^2}[\alpha_1 - \alpha_2 - 3\alpha_3 \cos^2\gamma + 8\alpha_3 \cos^4\gamma]$$

$$Q = \frac{\sin^2\gamma}{24d^2}[\alpha_1 + 2\alpha_2 + 3\alpha_3 \cos^2\gamma + 8\alpha_3 \cos^4\gamma]$$

$$\alpha_1 = (b_1^2 + b_2^2) \qquad \alpha_3 = \frac{3(b_1^4 + b_2^4) + 10 b_1^2 b_2^2}{80d^2}$$

$$\sin^2\gamma = \frac{4a^2}{4a^2 + d^2} \qquad \cos^2\gamma = \frac{d^2}{4a^2 + d^2}$$

and F and E are the complete elliptic integrals of the first and second kinds with modulus $k = \sin\gamma$.

When the coils have equal breadth $b_1 = b_2 = b$ and $\alpha_1 = b^2$, $\alpha_3 = \frac{b^4}{5d^2}$.

If the lengths of the coils are not very small in comparison with d a greater precision may be attained by adding to (50) the last two terms of (48) or (49) which depend on differentials of the sixth and eighth order.

MUTUAL INDUCTANCE BY MEANS OF SELF-INDUCTANCE FORMULA

The mutual inductance of two coils having the same radii and the same number of turns per unit of length may be calculated with great accuracy from a knowledge of several self-inductances.

If the two coils be designated as A and B and a coil C having the same radius and number of turns per unit length be imagined to exactly fill up the space between A and B, the self-inductance of coils A, B and C in series will be

$$L_{ABC} = L_A + L_B + L_C + 2M_{AC} + 2M_{BC} + 2M_{AB}$$

[43] This Bulletin, 2, p. 342; 1906.

Similarly the self-inductances of the coils A and C in series, and of B and C in series are given by the equations.

$$L_{AC} = L_A + L_C + 2M_{AC}$$

$$L_{BC} = L_B + L_C + 2M_{BC}$$

Eliminating M_{AC} and M_{BC} in the equation above we find

$$2M_{AB} = (L_{ABC} + L_C) - (L_{AC} + L_{BC}) \qquad [51]$$

The self-inductances may be calculated with all the accuracy desired by Lorenz's or Nagaoka's formulas. Formula (51) is of especial value in testing new formulas and in the case where the two coils are in contact. In the latter case the formula becomes

$$2M_{AB} = L_{AB} - (L_A + L_B) \qquad [52]$$

OTHER FORMULAS

Himstedt has given several formulas for different cases of coaxial solenoids. The first[44] is for the case of a short secondary on the outside of a long primary. The formula is very complicated, and the calculation tedious. The formulas of Ròiti and Searle and Airey may be used to much better advantage.

Himstedt's second expression is for the case of two coaxial solenoids not concentric, the distance between their mean planes having any value; the radius of one is supposed to be considerably smaller than the other. This also is a very complicated formula, involving second and fourth derivatives of expressions containing the elliptic integrals F and E. Gray's general equation is much simpler to calculate. This is not, however, an important case in practice, and we do not therefore give Himstedt's equation. Himstedt's third equation is general and applies to two coaxial solenoids of nearly equal or very different radii, which may or may not be concentric. This expression of Himstedt's consists of four terms, each of which is a somewhat complicated expression involving both complete and incomplete elliptic integrals. A less inconvenient general expression for M in elliptic integrals is given above (44).

Havelock[45] gave a formula for the mutual inductance of two coaxial, concentric solenoids, which resembles somewhat the formula

[44] Wied. Annalen, **26**, p. 551; 1885.
[45] Phil. Mag., **15**, p. 342; 1908.

of Ròiti. It is, however, not so convergent as the latter, except when one coil is very short in comparison with the other.

After the present work had gone to press a valuable article appeared[45a] by Olshausen, in which the author derived a general absolute expression for the mutual inductance of two coaxial solenoids. Adopting the same nomenclature as in Nagaoka's formula (45), the integral I is in this case given by

$$I = m^{\frac{1}{2}}\sqrt{2Aa}\left[\left(\frac{g_2}{6} - \mathrm{p}^2 v\right)\omega_1 + \mathrm{p}v \cdot \eta_1 + \frac{\mathrm{p}'v}{2}\left\{\eta_1 v - \omega_1 \frac{\sigma_1}{\sigma}(v) + n\pi i\right\}\right] \quad [52a]$$

Here m is a parameter, which is to be arbitrarily assigned, and consequently (52a) may be put into various special forms depending on the value assumed for m. The integer n, which may be positive or negative, enters because of the many-valuedness of a logarithm, and is to be found from the equation defining v.

If we place $m = \left(\frac{2}{Aa}\right)^{\frac{1}{2}}$ and let $n = 0$, the result is Nagaoka's equation (45). The author shows further that by expressing the quantities in (52a) in terms of the elliptic integrals of Legendre, Cohen's absolute formula (44) may be shown to be a special case of the general equation (52a).

As a third example, the author shows that the absolute formula of Kirchhoff, published for the first time by Coffin[45b] in a form subsequently shown by Cohen[45c] to be in error, is included in (52a), and the correct expression is given in Olshausen's equation (38).

Olshausen showed further that if the value $\frac{(A+a)^2 + c^2}{2Aa}$ be assigned to m, the expressions for some of the auxiliary quantities become very simple. For the details of calculation as arranged by him we refer to the original article.

CHOICE OF FORMULAS

1. *Coaxial solenoids, not concentric.*—(a) For the general case, if the greatest precision is required, Nagaoka's absolute formula (45) should be used. Since, however, the mutual inductance of such

[45a] Phys. Rev., **31**, p. 617; 1910.
[45b] This Bulletin, **2**, p. 125; 1906.
[45c] This Bulletin, **3**, p. 301; 1907.

coils will not in general be needed with extreme accuracy, it will usually be found sufficient to apply Gray's formula (40), taking the precaution to determine by a rough preliminary calculation, whether or not the terms of higher order will have an appreciable effect in the case at hand. For this purpose Table XVIII will be found of material aid.

If the convergence is not satisfactory, or if more than three or four terms must be calculated, it will be found advantageous to subdivide one or both of the coils, and to apply Gray's formula to the calculation of the mutual inductance of the several pairs of sections; for these the convergence will be more rapid. For example, if coil A be divided into two parts, C and D, and the coil B into sections E and F, then the mutual inductance of A or B will be given by the relation

$$M_{AB} = M_{CE} + M_{CF} + M_{DE} + M_{DF}$$

It may be stated as a general criterion for the rapid convergence of Gray's formula, that the distance between the coils should be great relatively to the radii, and that the coils should not be very long. With long coils it is necessary to carry the subdivision further than with short coils, with a corresponding increase in the number of terms to be calculated, but even then the labor will generally be much less than in using Nagaoka's formula.

If the coils be relatively far apart, and great precision is not desired, the formula of quadratures (23) may be adapted to this case, by making the radial dimension of the cross section of the coils in Fig. 4 equal to zero. We have then

$$M_1 = M_3 = M_5 = M_7 = M_0$$

and the formula of quadratures becomes

$$M = \frac{1}{6}(2M_0 + M_2 + M_4 + M_6 + M_8)$$

It is, therefore, only necessary to calculate, by an appropriate formula or formulas, the mutual inductances of the five pairs of circles, and to take the weighted mean indicated. This formula is more accurate, the shorter the axial lengths of the solenoids in comparison with their distance apart, and the process of subdivision above described will be, in general, necessary. Gray's formula is, however, to be preferred.

(*b*) An important case in practice is that of solenoids of *equal radii.* If the coils be in contact or very near together the formulas (52) or (51), respectively, should be employed.

If the solenoids be separated so that the distance between their medial planes is greater than the axial length of either, the mutual inductance may be calculated from the mutual inductance of the two circles at their centers, a correction being applied to take account of the lengths of the coils. For this purpose formula (48) should be used for coils relatively near together and (50) for coils farther apart. The corresponding formulas, for coils of *equal radii* and *equal length* are (49) and (50).

2. *Coaxial, concentric solenoids of equal length.*—If the solenoids be long relatively to their radii, Havelock's formula (38) will be found to be very accurate. Maxwell's formula (36), however, is applicable to both long and short solenoids, provided the radii are not too nearly equal, and should be given the preference, using Havelock's, when desired, as a check on the result. It may be necessary in rare cases to use the absolute formulas of Nagaoka or Cohen. One should also bear in mind that Ròiti's and Searle and Airey's formulas also hold for equal length solenoids, and may be used in checking the results.

3. *Coaxial, concentric solenoids—Inner coil the shorter.*—For relatively long coils Ròiti's formula (39) will give very accurate values, whatever the length of the inner solenoid, provided the radius of the inner coil is not closely equal to that of the outer. Ròiti's formula is also applicable to short solenoids in case the inner radius is considerably smaller than the outer. For short solenoids, however, Searle and Airey's formula (43) is preferable, and gives a very rapidly converging value unless the inner radius be nearly equal to the outer. Russell's formula (46) is most convergent for long solenoids, of which the inner one is a good deal shorter than the outer one.

In those cases for which none of the above formulas converge rapidly, and great precision is desired, Nagaoka's or Cohen's absolute formula should be used.

4. *Coaxial, concentric solenoids—Outer coil the shorter.*—The formulas of the preceding section are to be used interchanging x and l, or l_1 and l_2 as the case may be. The formulas of Ròiti,

Russell, Cohen, and Nagaoka are unchanged in form; Searle and Airey's is slightly changed as regards the coefficients L and X.

Usually, it will be found that more than one formula will apply to a given case. The advantage of such a check can not be overestimated.

For illustrations and tests of the above formulas, see examples 34–47.

In taking the dimensions of coils where an accurate value of the mutual inductance is sought it should be borne in mind that the above formulas have been derived on the supposition that the current is uniformly distributed over the length of the coaxial solenoids; in other words, these formulas are all current-sheet formulas. Hence, for coils made up of many turns of wire we must meet the conditions imposed by current-sheet formulas. In calculating self-inductances, this requires an accurate determination of the size of the wire and of the distance between the axes of successive wires, from which we can calculate two correction terms to be combined with the value of L given by the current-sheet formulas.[46]

In the case of mutual inductances, however, there are no correction terms to calculate; but we must take the dimensions of the current sheets that are equivalent to the coils of wire; that is, the radius of each coil will be the mean distance to the center of the wire, and the length of each will be the over-all length, including the insulation, when the coil is wound of insulated wire in contact, or the length from center to center of a winding of $n+1$ turns, where n is the whole number of turns used.[47] It is also very important that the winding on both coils shall be uniform,[48] and that the leads shall be brought out so that there shall be no mutual inductance due to them.

The mutual inductance will of course be different at high frequencies from its value at low frequencies. We assume, however, that for all purposes for which an extremely acturate mutual inductance is desired the frequency of the current would be low, say,

[46] Rosa, this Bulletin, **2**, p. 181; 1906.

[47] Rosa, this Bulletin, **2**, p. 161, 1906; and **8**, p. 1; 1907.

[48] Searle and Airey, Electrician (London), **56**, p. 318; 1905.

not more than a few hundred per second. If the value at very high frequency is desired the coil should be wound with stranded wire, each strand of which is separately insulated.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS

EXAMPLE 34. MAXWELL'S FORMULA (36) AND COHEN'S (44)

Two solenoids, Fig. 22, of equal length, 200 cm, each wound with a single layer coil.

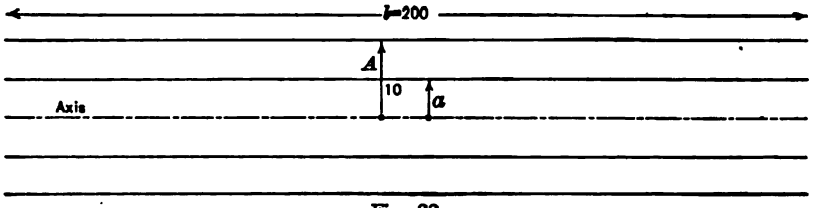

Fig. 22

$$A = 10 = \text{radius of outer.}$$
$$a = 5 = \text{radius of inner.}$$

Substituting in (36) for α we have the following:

$$\alpha = 0.487508 - \frac{1}{16}\frac{a^2}{A^2}(0.999875) - \frac{1}{64}\frac{a^4}{A^4}(0.500001) - \frac{35}{2048}\frac{a^6}{A^6}\left(\frac{1}{7}\right)$$
$$= 0.487508 - 0.015623 - 0.000488 - 0.000038$$
$$= 0.471359$$
$$\therefore M = 4\pi^2 a^2 n^2 (200 - 9.42718)$$
$$M = 19057.28\pi^2 n^2$$

If $n = 10$ turns per cm, $M = \dfrac{100\,\pi^2 \times 19057.28}{10^9}$ henry

$$= 0.01880878 \text{ henry.}$$

The effect of the next term in the series for α beyond those calculated is to raise the value of M by only one part in five million.

By the approximate formula of Maxwell (37)

$$2\alpha = 1 - \frac{1}{8\times 4} - \frac{1}{64\times 16} - \frac{1}{1024\times 64} - \cdots$$
$$= 0.96773$$
$$\therefore M = 0.018784 \text{ henry.}$$

This example by Heaviside's extension of Maxwell's formula (see p. 55) has exactly the same value of M; that is, the added terms do not amount to as much as a millionth of a henry in this particular case.

To show that the result by Maxwell's formula (36) is very accurate for this case we may now calculate M by Cohen's absolute formula:

$$M = 4\pi n^2(V - V_1)$$

Substituting in (44) for V we have the following terms:

$$\begin{aligned} V &= 7863.79 + 4200532.04 - 4169106.25 - 23561.95 \\ &= 15727.63 \\ V_1 &= 1392.18 - 632.16 = 760.02 \\ \therefore M &= 4\pi n^2 (15727.63 - 760.02) = 4\pi n^2 (14967.61) \\ &= 19057.36\pi^2 n^2 \\ M &= 0.01880886 \text{ henry.} \end{aligned}$$

This agrees with the result by Maxwell's formula to within five parts in a million, the value by Maxwell's formula being more nearly correct, as is shown in the next example.

The example by Cohen's formula illustrates the disadvantage of that formula for numerical calculations. Aside from the fact that it is complicated, and involves the use of both complete and incomplete elliptic integrals, the value of M depends on the difference between very large positive and negative terms, so that in order to get a value of M correct to one part in one hundred thousand it is necessary in the above example to calculate the large terms to one part in twenty-five million. As a means of testing other formulas, however, this absolute formula is of great value.

EXAMPLE 35. HAVELOCK'S FORMULA (38)

We will take the same problem as in the preceding example:

$$a = 5 \qquad A = 10 \qquad l = 200$$

$$\frac{1}{2} - \frac{1}{16}\frac{a^2}{A^2} = 0.484375$$

$$- \frac{1}{128}\frac{a^4}{A^4} = -0.000488$$

$$-\frac{5}{2048}\frac{a^6}{A^6} = -0.000038$$

$$-\frac{1}{4}\frac{A}{l} = -0.012500$$

$$\frac{1}{16}\left(1+\frac{a^2}{A^2}\right)\frac{A^2}{l^2} = +0.000010$$

$$\text{Sum} = \beta = \quad 0.471359$$

which is exactly the same as the value of α found by Maxwell's formula in the preceding example. The value of the mutual inductance agrees, therefore, exactly to seven significant figures with the value given by Maxwell's formula. For this example, accordingly, we see that Maxwell's and Havelock's formulas give a more accurate value than Cohen's formula, unless the quantities in the latter are carried out to a greater number of places of decimals. This was pointed out by Havelock.[40]

EXAMPLE 36. MAXWELL'S FORMULA (36). FOR EQUAL SHORT SOLENOIDS

$$a=5 \qquad A=10 \qquad l=2$$

$$r=\sqrt{104} \qquad \frac{A}{r}=0.9805808 \qquad \frac{a}{A}=\frac{1}{2}$$

$$\frac{A-r+l}{2A} = 0.09009805$$

$$-\frac{1}{16}\frac{a^2}{A^2}\left(1-\frac{A^2}{r^2}\right) = -\ 0.00089271$$

$$-\frac{1}{64}\frac{a^4}{A^4}\left(\frac{1}{2}+2\frac{A^5}{r^5}-\frac{5}{2}\frac{A^7}{r^7}\right) = -\ 0.00013073$$

$$-\frac{35}{2048}(0.080378)\frac{a^6}{A^6} = -\ 0.00002146$$

$$-\frac{63}{2.128^2}(0.5079)\frac{a^8}{A^8} = -\ 0.00000381$$

$$-\frac{231}{512^2}(0.788)\frac{a^{10}}{A^{10}} = -\ 0.00000068$$

$$-\frac{429}{2.1024^2}(2.43)\frac{a^{12}}{A^{12}} = -\ 0.00000012$$

$$\text{Sum} = \alpha = \quad 0.08904854$$

[40] Phil. Mag., 15, p. 341.

$$2A\alpha = \quad 1.7809708$$
$$l - 2A\alpha = \quad 0.2190292$$
$$\therefore \frac{M}{4\pi n_1 n_2} = \quad 17.20251$$

The formula is not so favorable in this case as for long coils, since the quantity $2A\alpha$ is nearly equal to l. Further, the quantities involved in the parentheses are rather large, although their sum is in only one case greater than unity. There is, however, no difficulty in obtaining these factors with all the accuracy required. We have carried out the calculation with this formula further than would in practice be desired, in order to test the formula. We find that to get the same order of accuracy by Searle and Airey's formula terms including the product $X_{12}L_{14}$ must be calculated. The result found was

$$\frac{M}{4\pi n_1 n_2} = 17.20252$$

or only one part in two million different. We have also calculated the mutual inductance of these coils by Nagaoka's formula and obtained a value not very different, but this is a very unfavorable case for this formula, no great accuracy being obtainable using seven-place logarithms.

EXAMPLE 37. RÒITI'S FORMULA (39) COMPARED WITH SEARLE AND AIREY'S (43)

We will now calculate the example, Fig. 23 (originally given by Searle and Airey[50]), by Ròiti's formula, and also by the formula of Searle and Airey.

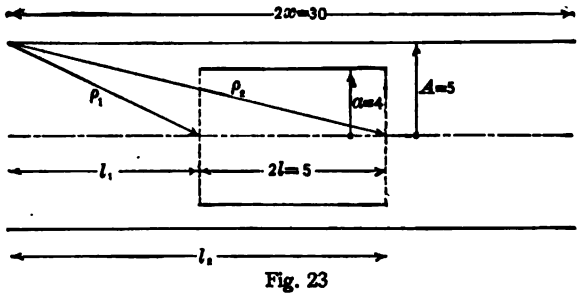

Fig. 23

[50] Electrician (London), 56, p. 319; 1905.

30 cm = length of outer solenoid.
5 " = " " inner "
$A =$ 5 " = radius " outer "
$a =$ 4 " = " " inner "

$N_1 = 300$ turns $\therefore n_1 = \frac{300}{30} = 10$ per cm

$N_2 = 200$ " $n_2 = \frac{200}{5} = 40$ per cm

$$l_1 = 12.5 \qquad \rho_1 = \sqrt{12.5^2 + 25} = 13.462912$$
$$l_2 = 17.5 \qquad \rho_2 = \sqrt{17.5^2 + 25} = 18.200275$$
$$\therefore \rho_2 - \rho_1 = 4.737363$$
$$\rho_1 + \rho_2 = 31.663187$$

It is more accurate to calculate $(\rho_2 - \rho_1)$ by the formula

$$\rho_2 - \rho_1 = \frac{xl}{\rho_1 + \rho_2}.$$

This gives $(\rho_2 - \rho_1) = 4.7373620$, which value will be used in the calculation of M.

$$\rho_2 - \rho_1 = 4.7373620$$
$$\frac{A^2a^2}{8}\left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right) = + .0121975$$
$$-\frac{A^4a^2}{16}\left(\frac{1}{\rho_1^5} - \frac{1}{\rho_2^5}\right) = - .0007041$$
$$\frac{5}{64}A^4a^4\left(1 + \frac{1}{2}\frac{a^2}{A^2}\right)\left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7}\right) = + .0001808$$
$$-\frac{35A^4a^6}{256}\left(1 + \frac{1}{5}\frac{a^2}{A^2}\right)\left(\frac{1}{\rho_1^9} - \frac{1}{\rho_2^9}\right) = - .0000254$$
$$+\frac{105A^6a^6}{1024}\left(1 + \frac{9}{5}\frac{a^2}{A^2} + \frac{1}{5}\frac{a^4}{A^4}\right)\left(\frac{1}{\rho_1^{11}} - \frac{1}{\rho_2^{11}}\right) = + .0000054$$
$$-\frac{693A^6a^8}{2048}\left(1 + \frac{2}{3}\frac{a^2}{A^2} + \frac{1}{21}\frac{a^4}{A^4}\right)\left(\frac{1}{\rho_1^{13}} - \frac{1}{\rho_2^{13}}\right) = - .0000010$$
$$\frac{3003A^8a^8}{16384}\left(1 + 4\frac{a^2}{A^2} + \frac{10}{7}\frac{a^4}{A^4} + \frac{1}{14}\frac{a^6}{A^6}\right)\left(\frac{1}{\rho_1^{15}} - \frac{1}{\rho_2^{15}}\right) = + .0000002$$
$$\text{Sum} = 4.7490149$$

$$4\pi^2 a^2 n_1 n_2 = 25600\,\pi^2$$

$$\therefore M = \frac{25600\,\pi^2 \times 4.7490149}{10^9} \text{ henry}$$

$$\text{or } M = 0.0011998950 \text{ henry.}$$

The sum of the next two terms in the series is equal to about one part in ten million. The value of the mutual inductance is therefore given with great precision by this formula. If the inner radius had been relatively smaller, the convergence would have been more rapid. We have, however, carried the computation much further than would in practice be necessary.

Calculating the same problem by Searle and Airey's formula we have

$$2x = 30 \qquad 2l = 5 \qquad A = 5 \qquad a = 4$$
$$N_1 = 300 \qquad N_2 = 200$$

$$d = \sqrt{250} \qquad \frac{A^2a^2}{d^4} = \frac{4}{625}$$

$$X_2 = -\ 33.00 \qquad L_2 = 1.43750$$
$$X_4 = 236.5 \qquad L_4 = -0.7959$$
$$X_6 = -1370 \qquad L_6 = -1.71$$
$$X_8 = 4869 \qquad L_8 = -0.72$$

$$1 + \frac{A^2a^2}{d^4}\cdot\frac{L_2}{8} = 1.0011500$$
$$\frac{A^4a^4}{d^8}\,\frac{L_4X_2}{32} = 0.0000344$$
$$\frac{A^6a^6}{d^{12}}\,\frac{L_6X_4}{32} = -0.0000033$$
$$\text{Sum} = 1.0011811$$
$$\frac{2\pi^2a^2N_1N_2}{d} = 1198480.5$$
$$\therefore M = 0.0011998957 \text{ henry.}$$

The terms neglected are less than one part in ten million. The value of the mutual inductance found is only six parts in ten million greater than that found by Ròiti's formula, and for this problem the convergence of Searle and Airey's formula is the more satisfactory.

The same problem by Russell's formula (46) (extended to include six terms in each part of the formula) gives

$$M = 0.00119989 \text{ henry.}$$

Of the three formulas Searle and Airey's is for this case the most convergent, and Russell's the least convergent. If the ratio of the radii was still more nearly equal to unity, Searle and Airey's formula would still be satisfactory; the convergence of Ròiti's formula would, however, become poorer.

If in the above problem the length $2l$ of the inner coil be increased without changing the radii, the quantities L_{nn} in Searle and Airey's formula would become rapidly larger, and the convergence would become poorer. Ròiti's formula also becomes less satisfactory as l is increased. For $2l = 24$, however, Searle and Airey's formula will still give the correct result to about one part in 100000, but Ròiti's formula in this case converges very slowly. On the other hand, if the radius of the inner coil were smaller in the latter case, Ròiti's formula could be used, but Searle and Airey's would not converge rapidly enough. This is shown in the next example.

EXAMPLE 38. RÒITI'S FORMULA COMPARED WITH SEARLE AND AIREY'S FORMULA. COILS OF NEARLY EQUAL LENGTH

Length of outer solenoid = 30 cm
" inner " = 24

$a = 2 \qquad A = 5 \qquad n_1 = 10 \qquad n_2 = 40$

In Ròiti's formula:

$\rho_2 - \rho_1 =$		21.628108
2d term	= +	0.062447
3d "	= −	0.003707
4th "	= +	0.003682
5th "	= −	0.000724
6th "	= +	0.000501
7th "	= −	0.000170
8th "	= +	0.000159
Sum =		21.690296
$4\pi^2 a^2 n_1 n_2 =$		$6400\pi^2$
$\therefore M =$		0.00137008 henry.

In Searle and Airey's formula

$$X_2 = -\ 33 \qquad L_2 = -\ 141$$
$$X_4 = 236.5 \qquad L_4 = 4826.5$$
$$X_6 = -1370.3 \qquad L_6 = -160036$$
$$X_8 = 4869 \qquad L_8 = 5.019 \times 10^7$$
$$X_{10} = 15746 \qquad L_{10} = -1.571 \times 10^8$$
$$L_{12} = 4.483 \times 10^9$$

$$d^2 = 250$$

$$\begin{aligned} & 1.000000 \\ \text{2d term} &= -0.028200 \\ \text{3d} \quad \text{"} &= -0.012742 \\ \text{4th} \quad \text{"} &= -0.004845 \\ \text{5th} \quad \text{"} &= -0.001409 \\ \text{6th} \quad \text{"} &= -0.000251 \\ \text{7th} \quad \text{"} &= +0.000037 \\ \hline \text{Sum} &= \ \ 0.952590 \end{aligned}$$

$$\frac{2\pi^3 a^2 N_1 N_2}{d} = \frac{2304000\pi^2}{d}$$

$$\therefore M = 0.00136999 \text{ henry.}$$

In this case we see that the higher order terms in Ròiti's formula arrange themselves in pairs of nearly equal values with opposite signs. The convergence is, therefore, better than appears at first sight, and the terms here neglected do not amount to more than one part in a million in M. Searle and Airey's formula does not converge so rapidly, the eighth and still higher order terms being appreciable. If the length of the inner solenoid were made still greater the L coefficients would become even larger than they are here, and the convergence would become unsatisfactory.

EXAMPLE 39. RÒITI'S AND SEARLE AND AIREY'S FORMULAS IN THE CASE OF SHORT COILS

Length of the outer solenoid = 5 cm
" " " inner " = 2

$$a = 2 \qquad A = 10$$

In Ròiti's formula:

$$\rho_1 = 10.111873$$
$$\rho_2 = 10.594808$$

which used in the formula $\frac{xl}{\rho_1+\rho_2}$ give a more accurate value of $\rho_2-\rho_1$, viz:

$$\begin{aligned}
\rho_2-\rho_1 &= 0.4829359 \\
\text{2d term} &= 0.0063160 \\
\text{3d } \text{"} &= -0.0001968 \\
\text{4th } \text{"} &= 0.0003286 \\
\text{5th } \text{"} &= -0.0000274 \\
\text{6th } \text{"} &= 0.0000250 \\
\text{7th } \text{"} &= -0.0000035 \\
\text{8th } \text{"} &= \underline{0.0000023} \\
\text{Sum} &= 0.4893801 \\
4\pi^2a^2 &= 16\pi^2 \\
\therefore \frac{M}{n_1n_2} &= 77.27981
\end{aligned}$$

In Searle and Airey's formula

$$\begin{aligned}
X_2 &= 2.75 & L_2 &= 2 \\
X_4 &= \frac{121}{64} & L_4 &= \frac{1}{4} \\
X_6 &= \frac{1203}{1024} & L_6 &= -\frac{15}{16} \\
X_8 &= \frac{9265}{16384} & L_8 &= -\frac{77}{64} \\
d^2 &= 106.25 & L_{10} &= -\frac{9}{16}
\end{aligned}$$

$$\begin{aligned}
& & & 1.0000000 \\
\text{1st term} &= & & 0.0088581 \\
\text{2d } \text{"} &= & & 0.0000270 \\
\text{3d } \text{"} &= & & -0.0000025 \\
\text{4th } \text{"} &= & & \underline{-0.0000001} \\
\text{Sum} &= & & 1.0088825 \\
\frac{2\pi^2a^2N_1N_2}{d} &= & & \frac{80\pi^2n_1n_2}{d} \\
\therefore \frac{M}{n_1n_2} &= & & 77.27980
\end{aligned}$$

The neglected terms in Searle and Airey's forumula are entirely inappreciable. The convergence of Ròiti's formula is not quite so

good. The sum of the next two terms is such as to reduce M by three units in the last place, but the following terms do not decrease very rapidly. We may evidently regard the use of either formula as entirely justified in the problem. If the radius of the outer coil had been only one-half as great, the lengths of the two coils and the radius of the inner remaining unchanged, the value found by Ròiti's formula would be in error by more than one part in ten thousand; the convergence of Searle and Airey's would, however, be satisfactory in this case. This formula would, on the other hand, be less convergent when the length of the inner coil is nearly as great as that of the outer coil. In general, it will be found that these two formulas between them cover sufficiently well the cases which may arise in practice.

EXAMPLE 40. GRAY'S FORMULA (41) COMPARED WITH RÒITI'S (39)

Let the winding be 20 turns per cm on each coil (Fig. 24), $n_1 = n_2 = 20$.

$A = 25$ cm $\qquad N_1 = n_1 A\sqrt{3}$

$a = 10$ cm $\qquad N_2 = n_2 a\sqrt{3}$

$$\therefore N_1N_2 = 3n_1n_2Aa$$

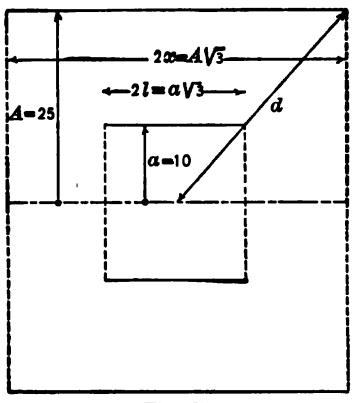

Fig. 24

$$d = \sqrt{x^2 + A^2} = \frac{A}{2}\sqrt{7}$$

$$\therefore \quad M = \frac{2\pi^2 a^2 N_1 N_2}{d} = 4\pi^2 a^2 n_1 n_2 \left[\frac{3a}{\sqrt{7}}\right]$$

$$M = .0179057 \text{ henry.}$$

We have also calculated the mutual inductance for these coils by Ròiti's formula (39).

The value is, $M = .0179058$, which is practically identical with the value by Gray's formula.

When $A = 25$ cm and $a = 10$ cm, $N_1 = 20A\sqrt{3} = 866.025$ and $N_2 = 20a\sqrt{3} = 346.4$. As there must be an integral number of turns, let us suppose the winding is exactly 20 turns per cm on each coil and the lengths therefore 43.3 cm and 17.3 cm, respectively. Then $d = \sqrt{x^2 + A^2} = \sqrt{625 + \left(\frac{43.3}{2}\right)^2} = 33.0715$ cm. This does not exactly

conform to the condition imposed in deriving the simple formula (41) of Gray used above. Hence (41) will not be as exact with these slightly altered dimensions, and we must calculate at least one correction term to get an accurate value of M.

By Gray's formula (41), $M = \frac{2\pi^2 100 \times 866 \times 346}{33.0715 \times 10^9} = .0178842$ henry.

The first correction term in (43) increases this value to .0178854 henry.

We will now calculate the mutual inductance of these coils by Rôiti's formula (39):

$A = 25$	$2x = 43.3$	$l_1 = 13.0$ cm	$\rho_1 =$	28.17800
$a = 10$	$2l = 17.3$	$l_2 = 30.3$ cm	$\rho_2 =$	39.28218
			$\rho_2 - \rho_1 =$	11.10418
			2nd term = +	.22030
			3rd = −	.01781
$M = \frac{4\pi^2 a^2 n_1 n_2 \times 11.32596}{10^9}$ henry,			4th = +	.01952
			5th = +	.00156
$= .0178853$ henry.			6th = −	.00453
			7th = +	.00274
			Sum =	11.32596

This differs from the result by Gray's formula by only 1 part in 178000.

EXAMPLE 41. GRAY'S FORMULA (40) COMPARED WITH NAGAOKA'S FORMULA (45)

We will next consider a practical problem suggested by Prof. Nasmyth.

$2x = 20.55$	$A = 6.44$	$N_1 = 15$ turns.
$2l = 27.38$	$a = 4.435$	$N_2 = 75$ "

The distance between the adjacent ends of the two solenoids was 7.2 cm. From this we find

$n_1 = 0.7296$ turns per cm	$k_1K_1 = 0.042937$
$n_2 = 2.737$ " " "	$k_3K_3 = 0.018274$
$x_1 = 20.89$	$k_5K_5 = 0.005193$
$x_2 = 41.44$	$k_7K_7 = 0.001423$
	$k_9K_9 = 0.000116$
	Sum = 0.067943
	$\therefore$ M = 1092.3 cm.

It is evident that the convergence is not rapid enough to give a very precise value for the mutual inductance. We next divide the longer coil S into two sections C and D, such that C has 37 turns and D has 38 turns, C being the section nearer the other coil R. The axial lengths of these sections are, respectively, 13.51 and 13.87 cm. It would be just as well, if not better, to divide the coil into two equal sections of 37½ turns each. The division chosen was for greater convenience in the solution of the same problem by the method of quadratures. (Example 42.)

We now proceed to find M_{RC} and M_{RD}. The L coefficients are much smaller than before on account of the ratio $\frac{l}{a}$ being now smaller than was previously the case, and the convergence is much more satisfactory. These coefficients would be still smaller if we had divided coil R instead of S into two sections, measuring the x's from the center of R instead of using S for the reference coil as is done here. This advantage would, nevertheless, be in large measure offset by the smaller values of the distances r_1 and r_3.

We find for M_{RC}

$$\begin{aligned} k_1K_1 &= 0.048894 \\ k_3K_3 &= 0.006520 \\ k_5K_5 &= 0.000051 \\ \hline \text{Sum} &= 0.055465 \\ \therefore M_{RC} &= \quad 891.7 \text{ cm} \end{aligned}$$

and for M_{RD}

$$\begin{aligned} k_1K_1 &= 0.011549 \\ k_3K_3 &= 0.000613 \\ k_5K_5 &= 0.000004 \\ \hline \text{Sum} &= 0.012166 \\ \therefore M_{RD} &= \quad 195.6 \text{ cm.} \end{aligned}$$

Consequently $M = M_{RC} + M_{RD} = 1087.3$ cm.

To test the correctness of this value, the coil R was divided into two sections A and B (B being the section nearer to S), and the four mutual inductances between these sections and the two sections C and D of the coil S were calculated.

	M_{AC}	M_{BC}	M_{AD}	M_{BD}
$k_1K_1=$	0.012024	0.036869	0.003893	0.007657
$k_2K_2=$	0.000863	0.005654	0.000141	0.000471
$k_3K_3=$	0.000004	0.000047	0.000001	0.000004
	0.012891	0.042570	0.004035	0.008132
	$M_{AC}=207.2$	$M_{BC}=684.4$	$M_{AD}=64.9$	$M_{BD}=130.7$

$$M=\text{sum}=1087.2 \text{ cm.}$$

The only component for which the convergence was not entirely satisfactory was M_{BC}. Here the sections are relatively near together and the coefficients L and X are not very favorable. Accordingly M_{BC} was calculated by two other methods (*a*) by dividing B into two sections, H and J, and by calculating M_{HC} and M_{JC}, (*b*) by dividing C into two sections, F and G, and by calculating M_{BF} and M_{BG}. The first procedure, on account of the relatively smaller values of r_1 and r_2, did not give a satisfactory degree of convergence. The latter, however, is better, the values found being

$$M_{BC}=463.8+220.0=683.8$$

Using this value instead of the above the value of M is 1086.6 cm.

As the final check we have calculated the mutual inductance by Nagaoka's formula (45). The entire calculation has to be carried through for four different values of c, viz, 55.13, 34.58, 27.75, and 7.20. The corresponding values of I are

$$I_1=60.041802$$
$$I_2=38.047638$$
$$I_3=30.811676$$
$$I_4=10.333503$$

and $(I_1+I_4)-(I_2+I_3)=70.375305-68.859314$

$$=1.515991$$

$$\therefore M=4\pi n_1n_2Aa(1.515991)=1086.55 \text{ cm.}$$

An inspection of the various details of the calculation shows that the last figure may be in error by several units, although the utmost precision of which the seven-place logarithms are capable was striven

for. Of course by carrying the computation of the various quantities to a still greater number of decimal places, the accuracy of the result would be enhanced. Similarly the component values of the mutual inductance by Gray's formula would have been more accurate if we had not stopped with k_s K_s. Since the dimensions of such coils would not ordinarily be obtained with greater precision than the accuracy here attained by Gray's formula, it is evident that the latter is for such cases much to be preferred to Nagaoka's formula, and the same would be true if the number of components were considerably increased. Nagaoka's formula has nevertheless the advantage in checking other formulas.

EXAMPLE 42. FORMULA OF QUADRATURES

The problem treated in the preceding example may also be solved by the formula of quadratures, using formula (8) for the calculation of the mutual inductance of the various pairs of circles. In general the method is not so accurate as that in the preceding example, and no time is saved. Only the results are here given, together with those by Gray's formula for comparison.

Single coils	Two sections	Four sections		
$M = 964.1$	$M_{RC} = 848.1$	$M_{AC} = 205.7$		$M_{BF} = 504.5$
	$M_{RD} = 193.7$	$M_{BC} = 669.5$		$M_{BG} = 182.2$
	$M = 1041.8$	$M_{RC} =$	875.2	$M_{BC} = 686.7$
		$M_{AD} = 66.5$		
		$M_{BD} = 130.4$		
		$M_{RD} =$	196.9	
			$M = 1072.1$	

Using the value of M_{BC} in the last column, $M = 1089.3$.

By Gray's formula:

Single coils	Two sections	Four sections		
$M = 1092.3$	$M_{RC} = 891.7$	$M_{AC} = 207.2$		$M_{CH} = 463.8$
	$M_{RD} = 195.6$	$M_{BC} = 684.4$		$M_{CJ} = 220.0$
	$M = 1087.3$	$M_{RC} =$	891.6	$M_{BC} = 683.8$
		$M_{AD} = 64.9$		
		$M_{BD} = 130.7$		
		$M_{RD} =$	195.6	
			$M = 1087.2$	

Using the value of M_{BC} in the last column, $M = 1086.6$.

EXAMPLE 43. MUTUAL INDUCTANCE OF CONCENTRIC COAXIAL SOLENOIDS BY NAGAOKA'S FORMULA (45)

$$2x = 200 \text{ cm}$$
$$2l = 20$$
$$A = 15$$
$$a = 10$$

This pair of coils was used by Cohen[51] in testing his absolute formula. He gave as the result $M = 4\pi n_1 n_2$ (6213.4). The same problem was worked out by Nagaoka[52] as an illustration of the use of his formula, the value of M found by him being $M = 4\pi n_1 n_2$ (6213.51). We have repeated his calculation, which was given only in abbreviated form, and agree substantially with his result, the value found being $M = 4\pi n_1 n_2$ (6213.52). Using seven-place logarithms it is very difficult to be sure of the last place of decimals given here. On the other hand, we find with Ròiti's formula, only three terms being necessary $M = 4\pi n_1 n_2$ (6213.509), and the same number of terms in Searle and Airey's formula give $M = 4\pi n_1 n_2$ (6213.510) with no uncertainty greater than one unit in the last place given. Olshausen found for the same coils the values $4\pi n_1 n_2$ (6213.77), and $4\pi n_1 n_2$(6213.63), using two methods of calculation in his formulas (21) and (61) and with $m = \frac{(A+a)^2 + c^2}{2aA}$. By the Kirchhoff formula he found $4\pi n_1 n_2$(6212.9). This is, however, an unfavorable case for this formula, since the angles φ and θ, on which the incomplete integrals $E(\varphi, k')$ and $F(\varphi, k')$ depend, are too near 90° to allow of accurate interpolation in Legendre's tables.

These differing results by the various absolute formulas, which arise from the fact, that in all of them the auxiliary quantities must be calculated with a considerably greater degree of accuracy than that desired in the result, serve to emphasize the advantage of the series formulas. In the great majority of practical cases the values, found by the use of series formulas, are not only obtained with a much smaller expenditure of labor, but are more accurate than when an absolute formula is used.

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[51] This Bulletin, 3, p. 8; 1907.

[52] Jour. Tokyo, Math. Phys. Soc. (2), 4, p. 284; 1908.

We reproduce below the principal results in the calculation of this problem by Nagaoka's formula.

$$A+a=25 \qquad A-a=5 \qquad A^2-a^2=125$$
$$(A+a)^2=625 \qquad (A-a)^2=25$$

The argument of $I_1=-I_3$ is $c=110$; that of $I_2=-I_4$ is $c=90$.

	C=110	C=90
$\log \alpha =$	$\bar{1}.3749796$	$\bar{1}.3749796$
$\log 2Aa\alpha =$	1.8521009	1.8521009
$P_1=\mathrm{p}v-e_1=$	-170.09225	-113.86339
$P_2=\mathrm{p}v-e_2=$	$+\ 0.3514302$	$+\ 0.3514302$
$P_3=\mathrm{p}v-e_3=$	$+\ 8.7857551$	$+\ 8.7857551$
$\mathrm{p}v=\frac{1}{3}(P_1+P_2+P_3)=$	$-\ 53.651687$	$-\ 34.908733$
$e_2-e_3=$	8.4343249	8.4343249
$e_1-e_3=$	178.87801	122.64916
$e_1-e_2=$	170.44368	114.21483
$k'=\sqrt{\frac{e_1-e_2}{e_1-e_3}}=$	0.9761398	0.9650037
$1+\sqrt{k'}=$	1.9879979	1.9823461
$\log k^2=$	$\bar{2}.6734934$	$\bar{2}.8373857$
$q=$	0.0030186570	0.0044528010
$\log \omega_1$ (1st equat.)$=$	$\bar{1}.0750696$	$\bar{1}.1594886$
" (2d ")$=$	$\bar{1}.0750696$	$\bar{1}.1594886$
" (3d ")$=$	$\bar{1}.0750696$	$\bar{1}.1594887$
$P_2+\ P_3=$	9.1371853	9.1371853
$P_1+\ 2P_2=$	-169.38939	-113.16053
$P_1+\ 2P_3=$	-152.52074	$-\ 96.29188$
$\frac{P_1}{2}(P_2+\ P_3)=$	$-\ 777.08214$	-520.19542
$\frac{(e_2-e_3)}{6}(P_1+2P_2)=$	$-\ 238.11479$	-159.07205
$-\mathrm{Sum}=$	$+1015.19693$	679.26747
$\times\omega_1=$	120.67564	98.068477
$\frac{\mathrm{p}v}{4\omega_1}\frac{\theta_2''(0)}{\theta_2(0)}=$	26.73272	21.064838
Diff. $=I'=$	93.94292	77.003639
(other equat.)$=$	93.942921	77.003659

$$s = -\ 0.33164947 \qquad -\ 0.33084473$$

$$-\frac{s}{q} = 109.86656 \qquad 74.300356$$

$$1 + 2q^4 \cos 4\pi w = 1.0000042 \qquad 1.0000087$$

$$b = 109.86702 \qquad 74.301000$$

$$\sqrt{\frac{b-1}{b+1}} = 0.99093909 \qquad 0.98663068$$

$$4q^2\sqrt{b^2-1}\,[1-q^2(2b-1)] = 0.00399641 \qquad 0.00587502$$

$$\text{Sum} = 0.99493550 \qquad 0.99250570$$

$$I'' = -\ 35.815107 \qquad -\ 29.231710$$

$$\frac{I}{2} = I' + I'' = 58.127814 \qquad 47.771939$$

$$I_1 - I_2 = 2(58.127814 - 47.771939)$$

$$= 2(10.355875)$$

$$M = 4\pi n_1 n_2 Aa.2(I_1 - I_2)$$

$$= 4\pi n_1 n_2 (6213.52)$$

In the calculation of I we have used the second value found for I' with $c = 110$ and the mean of the two values for I' with $c = 90$.

EXAMPLE 44. SHORT SECONDARY ON THE OUTSIDE OF A LONG PRIMARY

Length of primary $= 200$ cm
" " secondary $= 5$
Radius of primary $= 4 = a$
" " secondary $= 5 = A$

$$n_1 = 10 \qquad n_2 = 40$$

In Ròiti's formula:

$$\rho_1 = 97.62811$$

$$\rho_2 = 102.62188$$

$$\rho_2 - \rho_1 = \frac{xl}{\rho_1 + \rho_2} = 4.9937586$$

$$\frac{A^2a^2}{8}\left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right) = 0.0000075$$

$$\text{Sum} = 4.9937661$$

$$4\pi^2 a^2 n_1 n_2 = 25600\pi^2$$

$$\therefore M = 0.0012617342 \text{ henry.}$$

In Searle and Airey's formula:

$$d^2 = 10025 \qquad N_1N_2 = 400000$$
$$\frac{l}{A} = 20 \qquad \frac{x}{a} = \frac{5}{8}$$
$$X_2 = -1597 \qquad L_2 = 1.4375$$
$$L_4 = -0.796$$

	1.00000000
1st term =	0.00000071
2d " =	-1.3×10^{-10}
Sum =	1.00000071
$\therefore M =$	0.0012617342 henry

in exact agreement with the above. Both formulas are very rapidly convergent, and give as nearly the same value for M as can be calculated with seven-place logarithms.

EXAMPLE 45. COILS OF EQUAL RADII NEAR TOGETHER, BY FORMULA (48)

Lengths of the coils 4 cm and 6 cm $= b_1$ and b_2
Radius " " " 20 cm $= a$
Distance between centers 10 cm $= d$
The calculation of the quantities in the parentheses is as follows:

	First	Second	Third	Fourth
1st term =	0.088055	0.01562	0.0047	.0022
2d term =	−0.012694	0.00126	−0.0002	
3d term =	0.001232	0.00130		
4th term =	−0.000104			
Sum =	0.076489	0.01818	0.0045	.0022

The expression for ΔM then gives:

First term	= 0.0233240
Second "	= 0.0011047
Third "	= 0.0000973
Fourth "	= 0.0000113
Sum	= 0.0245373

The value of M_0 calculated for two circles of radius 20 cm and at a distance apart = 10 cm was found by (4) and checked by (1) to be

$$\frac{M_0}{N_1 N_2} = 4\pi a\,(0.8853877)$$

$$\frac{\Delta M_0}{N_1 N_2} = 4\pi a\,(0.0245373)$$

$$\therefore\ \frac{M}{N_1 N_2} = 4\pi a\,(0.9099250)$$

This was checked by means of (51) with the result, (assuming one turn per cm of the length of the coils)

$$\begin{aligned} L_{ABC} &= 4\pi a\,(430.339736) \\ L_0 &= 4\pi a\,(74.324564) \\ \hline \text{Sum} &= 4\pi a\,(504.664300) \end{aligned}$$

$$\begin{aligned} L_{AO} &= 4\pi a\,(194.210135) \\ L_{BO} &= 4\pi a\,(266.777705) \\ \hline \text{Sum} &= 4\pi a\,(460.987840) \\ \tfrac{1}{2}\,\text{Diff.} &= 4\pi a\,(21.838230) \end{aligned}$$

Dividing by 24, the product of the number of turns assumed in calculating the self-inductances,

$$\frac{M}{N_1 N_2} = 4\pi a\,(0.9099264)$$

which agrees with the value by (48) to about one and a half in a million.

For these coils, therefore, (48) is adequate to give a high degree of precision. If the distance between the same coils were, however, smaller, or if the lengths of the coils were greater the accuracy would not be so great, and it might be necessary to use (51). The latter, should, however, not be used when (48) converges well, since to get the same accuracy the calculation of the four self-inductances must be carried out to a greater number of decimal places than appear in the value of M. For the rapid convergence of (48) the ratios $\frac{b_1}{d}$, $\frac{b_2}{d}$ and $\frac{d}{a}$ should all be small.

For the more unfavorable case $b_1 = 6$, $b_2 = 10$, $d = 10$, $a = 20$ the value of ΔM comes out too small by three parts in ten thousand.

EXAMPLE 46. ROSA-WEINSTEIN FORMULA (50). FOR COILS FARTHER APART

As a rather unfavorable case we may take

$$b_1 = 10 \qquad b_2 = 20 \qquad d = 50 \qquad a = 25$$

$$k = \sin\gamma = \frac{50}{\sqrt{5000}} = \frac{1}{\sqrt{2}} = \cos\gamma$$

$$\frac{\cos^2\gamma}{24d^2} = \frac{\sin^2\gamma}{24d^2} = \frac{1}{120000}$$

$\alpha_1 =$	500	$\alpha_1 =$	500
$-\alpha_2 =$	$-$ 4.55	$2\alpha_2 =$	9.10
$-3\alpha_2\cos^2\gamma =$	$-$ 6.825	$+3\alpha_2\cos^2\gamma =$	$+$ 6.825
$8\alpha_2\cos^4\gamma =$	$+$ 9.10	$8\alpha_2\cos^4\gamma =$	9.10
Sum =	497.725	Sum =	525.025
$P =$	.0041477	$Q =$	.0043752
$F =$	1.854075		
$E =$	1.350644		
$F - E =$	0.503431		

$$(F-E)P + EQ = 0.0079974$$

$$\therefore \Delta M = 1.7766$$

Terms in the 6th and 8th differentials = 0.0016

Sum = 1.7782

From formula (19) which applies to the two circles at the centers of these coils

$$M_0 = 1.418599 \times 25 \qquad = 35.4650$$

$$\therefore \frac{M}{N_1N_2} = 35.4650 + 1.7782 = 37.2432$$

If we calculate the mutual inductance by formula (51) we find

$L_{ABC} =$	3792.2261	$L_{AC} =$	2350.4870
$L_C =$	1667.7268	$L_{BC} =$	3062.0405
	5459.9529		5412.5275

$\frac{1}{2}$ Diff. = 23.7127

Dividing by 200 = 0.1185635 $\qquad = \dfrac{M}{4\pi a N_1 N_2}$

$$\therefore \frac{M}{N_1N_2} = 37.2478$$

which is more than one in ten thousand greater than the value by (50). If the coils had been shorter and their diameter had been greater than the distance between their medial planes, the quantities P and Q in (50) would have been more convergent and the value of ΔM would have been more nearly correct. The accuracy here obtained would, however, suffice in many cases.

This formula when applied to the coils in the preceding problem gives a very accurate result viz, $\frac{M}{N_1N_2} = 4\pi a(0.909932)$, or about six in a million too large. (The terms in the sixth and eight order differentials as calculated by (48) are taken into account in this result.)

The mutual inductance of the coils in this example could also be calculated with a good degree of accuracy by Gray's formula.

EXAMPLE 47. METHOD OF OBTAINING THE DIMENSIONS OF THE EQUIVALENT CURRENT SHEETS

Suppose it is desired to obtain the mutual inductance of two solenoids, whose measured dimensions are as follows:

Coil I is wound with 100 turns of insulated wire of 0.15 cm covered diameter, the successive turns being in contact. The measured external diameter of the coil is 50.4 cm.

Coil II is wound with 50 turns of bare wire, 0.1 cm in diameter, in a thread of 2 mm pitch. The diameter measured over the wire is 10.25 cm.

Then the mean radius of coil I, to the center of the wire, is equal to $\frac{1}{2}(50.4-0.15)$ or 25.125 cm. The length of the equivalent current sheet will be the distance between the center of the first and the one hundred and first wire, or one hundred times the covered diameter of the wire; that is, 15 cm. Since the turns are in contact, the equivalent length may, in this case, also be found by measuring the over-all length of the winding, including the insulation. Both these methods are equivalent to taking one hundred times the pitch of the winding, which, in this case, is equal to the covered diameter of the wire.

For coil II, the equivalent radius is $\frac{1}{2}(10.25-0.1)=5.075$ cm. The equivalent length is fifty times $0.2=10$ cm.

The dimensions found in this way for coils I and II are to be used in the appropriate current sheet formula. (See p. 73.)

4. THE MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SINGLE-LAYER COIL

LORENZ'S FORMULA

The problem of finding the mutual inductance of a circle and a coaxial single layer winding was first solved by Lorenz.[53] Assuming that the current was uniformly distributed over the surface of the cylinder, forming a current sheet, he integrated over the length of the cylinder the expression for the mutual inductance of a circular element and the given circle. This expression is an elliptic integral. Lorenz reduced the integrated form to a series and gave the following formula for the mutual inductance of the disk and solenoid of what is now called the Lorenz apparatus. He called it, however, the constant of the apparatus instead of mutual inductance, and denoted it by C. It is of course the whole number of lines of magnetic force passing through the disk due to unit current in the surrounding solenoid.

$$M=\frac{\pi q r^2}{d}\left[Q(\alpha_1)+Q(\alpha_2)\right]$$

$$Q(\alpha)=2\pi q\sqrt{\frac{\alpha-1}{\alpha}}\left[1+\frac{3}{8}\frac{q^2}{\alpha^2}+\frac{5}{16}\frac{q^4}{\alpha^4}\left(\frac{7}{4}-\alpha\right)\right. \qquad [53]$$

$$\left.+\frac{35}{128}\frac{q^6}{\alpha^6}\left(\frac{33}{8}-\frac{9}{2}\alpha+\alpha\right)+\cdots\right.$$

ρ = radius of the disk, Fig. 25.

r = radius of the solenoid.

$2x$ = length of winding of solenoid.

$q=\rho/r$ = ratio of the two radii.

$d=\frac{2x}{n}$ = distance between centers of successive turns of wire.

$\alpha=\frac{x^2+r^2}{r^2}$

[53] Wied. Annalen, 25, p. 1; 1885. Oeuvres Scientifiques, 2, p. 162.

If the disk be not exactly in the mean plane of the solenoid, and x_1 be the distance from the plane of the disk to one end of the solenoid and x_2 to the other,

$$\alpha_1 = \frac{x_1^2 + r^2}{r^2} \qquad \alpha_2 = \frac{x_2^2 + r^2}{r^2}$$

Then $Q(\alpha_1)$ is found by substituting the values of α_1 in equation (53) above, and $Q(\alpha_2)$ by using the value of α_2 for α in the same equation. The sum of these two quantities multiplied by $\frac{\pi q r^2}{d}$ gives the constant of the instrument; that is, the mutual inductance sought.

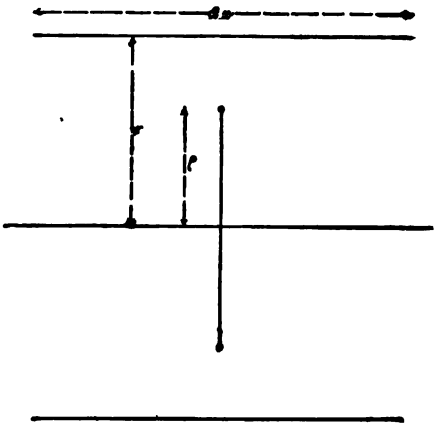


Fig. 25

As Lorenz gave the expression for the general term of (53), his equation can be extended. The following is the general term:

$$Q(\alpha) = 2\pi \sum_{m=0}^{m=\infty} q^{2m+1} \frac{1.3 \cdot \cdots \cdot 2m - 1}{2.4 \cdot \cdots \cdot 2m} \cdot \frac{1}{1.2 \cdot \cdots \cdot (m+1)} \cdot \frac{a^m}{da^m} \left(\frac{\alpha - 1}{\alpha} \right)^{m+\frac{1}{2}}$$

JONES'S FORMULAS

Two solutions of the above problem were given by Jones,[54] both in terms of elliptic integrals. The current was considered to flow not in a current sheet, but along a spiral winding or helix. The first solution was in the form of a series, convergent only when O_1A, Fig. 26, is less than the difference in the radii of inner and outer coils; that is, when O_1A is less than $A - a$. As this is a serious limitation, and the formula is a laborious one to use, it is not here given. The second solution is exact and general, and is in terms of elliptic integrals of all three kinds. The second formula is as follows:

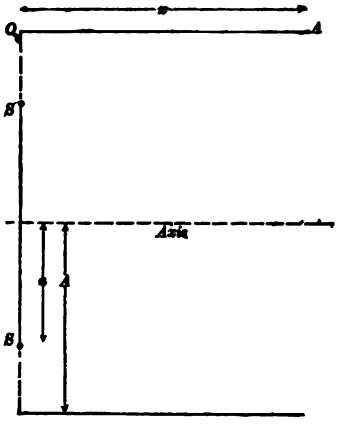


Fig. 26

$$M_0 = \Theta(A + a)ck\left\{\frac{F - E}{k^2} + \frac{c'^2}{c^2}(F - \Pi)\right\} \qquad [54]$$

[54] J. V. Jones, Proc. Roy. Soc., **63**, p. 198; 1898. Also, Trans. Roy. Soc., **182**, A; 1891. Jones's first formula was given in Phil. Mag., **27**, p. 61; 1889.

M_θ = mutual inductance of helix O_1A, Fig. 26, with respect to the disk S in the plane of one end.

$\Theta = 2\pi n$, $1/n$ = pitch of winding, Θ = whole angle of winding.

F, E, and Π are the complete elliptic integrals to modulus k, where

$$k^2 = \frac{4Aa}{(A+a)^2+x^2} = \sin^2\gamma,\ c^2 = \frac{4Aa}{(A+a)^2},\ c'^2 = 1 - c^2.$$

Π, the complete elliptic integral of the third kind, can be expressed in terms of incomplete integrals of the first and second kinds, and the value of M_θ can then be calculated by the help of Legendre's tables. (See example 50.) The calculation is, however, extremely tedious, especially when the value is to be determined with high precision.

Campbell has given Jones's formula (54) a slightly different form,[55] somewhat more convenient in calculation, as follows:

$$M = 2\,\pi n_1 n_2 (A+a)\left\{\frac{c}{k}(F-E) + \frac{A-a}{b}\psi\right\} \qquad [55]$$

where n_1 is the same as n above, the number of turns per cm on the solenoid, n_2 is the number of turns in the secondary coil (in the above case it was taken as one), A is the greater and a the less of the two radii (in the above case A was the radius of the solenoid and a of the circle within), and

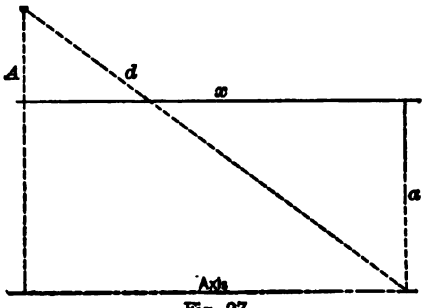


Fig. 27

$$\psi = F(k)E(k',\beta) - [F(k) - E(k)]F(k',\beta) - \frac{\pi}{2}$$

where $F(k)$ and $E(k)$ are the complete elliptic integrals to modulus k, and $F(k',\beta)$ and $E(k',\beta)$ are the incomplete elliptic integrals to modulus k' and amplitude β; $k' = \cos\gamma$, $\sin\beta = c'/k'$; k, c, and c' are given above. If a second-

[55] A. Campbell, Proc. Roy. Soc, A, 79, p. 428; 1907. There is a misprint in the formula as given in Campbell's paper. It was, however, used correctly in the numerical calculations given in the paper.

ary circle or coil has a radius greater than that of the solenoid, the same formula can be used if A is taken for the radius of the larger secondary and a is the radius of the solenoid (Fig. 27).

ROSA'S FORMULA[56]

The following formula gives the mutual inductance of a single layer coil of length x and a coaxial circle of radius a in the plane of one end of the coil, as shown in Fig. 26. It is the same quantity represented by M of equations (53) and (55) and M_0 of (54).

$$M_{0A}=\frac{2\pi^2a^2N}{d}\left[1+\frac{3}{8}\frac{a^2A^2}{d^4}+\frac{5}{64}\frac{a^4A^4}{d^8}X_2+\frac{35}{512}\frac{a^6A^6}{d^{12}}X_4+\frac{63}{1024}\frac{a^8A^8}{d^{16}}X_6\right.$$
$$\left.+\frac{231}{4096}\frac{a^{10}A^{10}}{d^{20}}X_8+\frac{429}{8192}\frac{a^{12}A^{12}}{d^{24}}X_{10}+\cdots\right] \quad [56]$$

$$X_2=3-4\frac{x^2}{A^2}$$

$$X_4=\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4}$$

$$X_6=\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6}$$

$$X_8=\frac{63}{32}-\frac{105}{4}\frac{x^2}{A^2}+63\frac{x^4}{A^4}-36\frac{x^6}{A^6}+4\frac{x^8}{A^8}$$

$$X_{10}=\frac{231}{128}-\frac{1155}{32}\frac{x^2}{A^2}+\frac{1155}{8}\frac{x^4}{A^4}-165\frac{x^6}{A^6}+55\frac{x^8}{A^8}-4\frac{x^{10}}{A^{10}}$$

(For general coefficient, see p. 63.)

a = radius of disk or circle S, Fig. 26.

A = radius of the solenoid.

x = length O_1A of one end of the solenoid.

$d=\sqrt{x^2+A^2}$ = half the diagonal of the solenoid.

N is the whole number of turns of wire in the length x.

This formula is very easy to use in numerical calculation, notwithstanding it looks somewhat elaborate. The logarithm of $\frac{a^2A^2}{d^4}$, multiplied by 2, 3, 4, etc., gives the logarithm of the corresponding factor in each of the other terms. Similarly, the various terms X_2, X_4, etc., contain only powers of $\frac{x^2}{A^2}$ besides the numerical

[56] This Bulletin, 8, p. 209; 1907.

coefficients. It is hence a far simpler matter to compute M with high precision by this formula than by Jones's formula, the latter containing as it does elliptic integrals of all three kinds and involving the tedius interpolations for incomplete elliptic integrals.

If the secondary circle has a larger radius than the solenoid, A will be the radius of the circle and a the radius of solenoid. In every case A is the greater and a the less of the two radii, and d is $\sqrt{A^2+x^2}$.

Equation (56) may be written

$$M=\frac{2\pi^2a^2n_1x}{d}S$$

where n_1 is the number of turns of wire per cm, x is the length of the coil, Fig. 26, and S is the value of the quantity in brackets in (56), which is always somewhat greater than unity. This may also be put as follows:

$$M=a^2n_1\left(\frac{2\pi^2x}{d}\right)S=a^2n_1RS$$

or,

$$M=a^2n_1K \qquad [57]$$

The quantity R depends on x/d; that is, only upon the shape of the solenoid. S depends upon x/A, a/A, and A/d; that is, upon the relative sizes of the inner circle and the solenoid and the shape of the solenoid. If we have the value of RS, or K of equation (57) for a given solenoid and circle, we can get M by multiplying by a^2n_1, and for any other system of similar shape but different size by multiplying the same value of K by a^2n_1. The values of the constant K for various values of a/A and x/A are given in Table III, page 193.

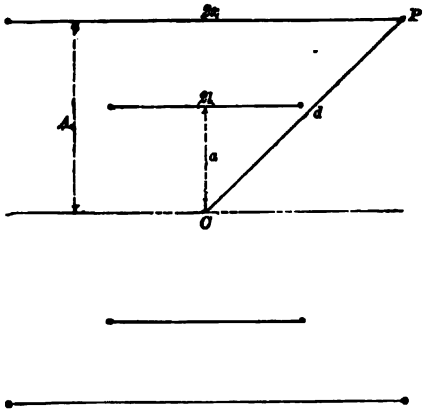


Fig. 28

If the disk or circle be in the center of a solenoid of length $2x$ (Fig. 28), the value of M is of course double that given by using x. If it be not quite in the center, the value of M must be calculated for each end separately.

For illustrations and tests of the above formulas, see examples 48 to 51, pages 103–110.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SOLENOID

EXAMPLE 48. ROSA'S FORMULA (56) COMPARED WITH JONES'S FORMULA (54)

Professor Jones gave the calculations by formula (54) of the constant of the Lorenz apparatus made for McGill University, obtaining the values given below, the second value being that obtained after the plate had been reground and again measured.

A calculation[57] of the same two cases by formula (56) gives very closely agreeing results.

	1st value	2nd value, disk slightly smaller
By formula (54)	$M = 18{,}056.36$	18,042.52
" " (56)	$M = 18{,}056.46$	18,042.74
Difference	$-.10$	$-.22$

These differences, amounting to five parts in a million in the first case and twelve parts in a million in the second case, are wholly negligible in the most refined experimental work.

EXAMPLE 49. FORMULA (56) COMPARED WITH JONES'S FIRST FORMULA

Take as a second example the case given by Jones[58] to illustrate his first formula.

$$A = 10 \text{ inches} \qquad a = 5 \text{ inches} \qquad x = 2 \text{ inches}$$

$$d^2 = 104 \qquad \frac{a^2A^2}{d^4} = \frac{2500}{10816} \qquad \log\frac{a^2A^2}{d^4} = \bar{1}.3638733$$

	1st term	$= 1.0000000$
	2 "	$= .0866771$
$X_2 = 2.8400$	3 "	$= .0118537$
$X_4 = 2.1064$	4 "	$= .0017781$
$X_6 = 1.5208$	5 "	$= .0002670$
$X_8 = 1.0173$	6 "	$= .0000379$
$X_{10} = 0.5815$	7 "	$= .0000046$
	Sum	$= 1.1006184$

$$\frac{2\pi^2a^2}{d} = 48.38972$$

[57] This Bulletin, 3, p. 218; 1907.

[58] Phil. Mag., 27, p. 61; 1889. In this example, P_0 should be 0.654870 instead of 0.54870, as printed in Jones's article.

$\therefore M = 53.25861\ N$, N being the number of turns of wire on the coil.

Jones gives $M = 53.25879\ N$.

The difference between these values is three parts in a million.

EXAMPLE 50. CALCULATION OF CONSTANT OF AYRTON-JONES CURRENT BALANCE BY FORMULAS (54) AND (56)

As a further test of the formulas let us calculate the constant of an electro-dynamometer or current balance of the Ayrton-Jones type,[50] of which AB, Fig. 29, is the upper fixed coil and ED is the moving coil, the circle S at the upper end lying in the plane through the middle of AB and the circle R at the lower end of ED lying in the middle plane of the lower fixed coil BC.

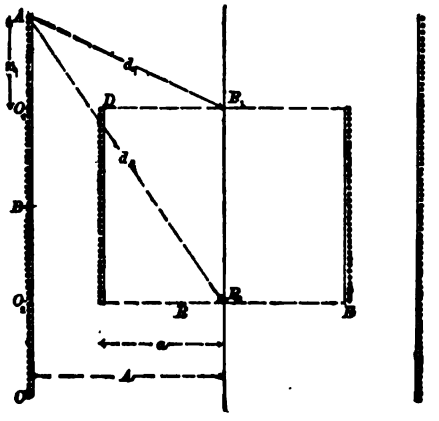


Fig. 29

Assume the dimensions as follows:

$A =$ 16 cm = radius of fixed coil, Fig. 29.
$a =$ 10 cm = radius of moving coil.
$x_1 =$ 8 cm = half length of AB = O_1A
$x_2 =$ 24 cm = 1.5 times AB = O_2A
$n_1 =$ 10 = number of turns per cm
$N_1 =$ 80 = number of turns in distance $O_1A = x_1$, Fig. 29.
$N_2 =$ 240 = number of turns in distance $O_2A = x_2$

[50] This Bulletin, 8, p. 226; 1907.

$$d_1 = \sqrt{A^2 + x^2_1} = 8\sqrt{5} = \text{diagonal } AP_1, \text{ Fig. 29.}$$

$$d_2 = \sqrt{A^2 + x^2_2} = 8\sqrt{13} = \text{diagonal } AP_2$$

We have to determine two mutual inductances, namely, M_S between the coil O_1A of 80 turns on the circle S, and M_R between the coil O_2A of 240 turns on the circle R. In each case the circle is in the plane passing through the lower end of the coil.

Formula (56) will be used, taking N_1, x_1, and d_1 in the first case and N_2, x_2, and d_2 in the second case.

	For M_S	For M_R
A	16 cm	16 cm
a	10	10
x	8	24
A^2	256	256
x^2	64	576
$N = nx$	80	240
d^2	320	832
$\log d^2$	2.5051500	2.9201233
$\log \frac{a^2A}{d^2}$	$\bar{1}$.3979400	$\bar{2}$.5679934
$\log \frac{x^2}{A^2}$	$\bar{1}$.3979400	0.1760913
X_2	+2.000	− 6.00
X_4	+0.250	+ 0.25
X_6	−0.9375	+23.5
X_8	−1.203	−45.7
X_{10}	−0.562	−49.0
1st term	1.0000000	1.0000000
2d "	+ .0937500	+ .0138683
3d "	+ .0097656	− .0006411
4th "	+ .0002670	+ .0000009
5th "	− .0002253	+ .0000027
6th "	− .0000662	− .0000002
7th "	− .0000072	.0000000
Sum = S	1.1034839	1.0132306

$\log S_1$	=	0.0427660	$\log S_2$	=	0.0057083
" $2\pi^2$	=	1.2953298	" $2\pi^2$	=	1.2953298
" $a^2(=100)$	=	2.0000000	" $a^2(=100)$	=	2.0000000
" $N_1(=80)$	=	1.9030900	" $N_2(=240)$	=	2.3802112
		5.2411858			5.6812493
" d_1	=	1.2525750	" d_2	=	1.4600616
$\log M_S$	=	3.9886108	$\log M_R$	=	4.2211877
$\therefore M_S$	=	9741.16	M_R	=	16641.32

THE SAME EXAMPLE BY JONES'S FORMULA

We will now calculate M_S and M_R by Jones's second formula given above, using also the following equation to find $F-\Pi$:

$$\frac{k'^2 \sin\beta\cos\beta(F-\Pi)}{c} = F(k)E(k',\beta)+E(k)F(k',\beta)-F(k)F(k',\beta)-\frac{\pi}{2}$$

	For M_S	For M_R
A	16 cm	16 cm
a	10	10
x	8	24
$\Theta = 2\pi N$	$160\,\pi$	$480\,\pi$
$c=\frac{2\sqrt{Aa}}{A+a}$	0.9730085	0.9730085
$c'=\sqrt{1-c^2}$	0.2307692	0.2307692
$k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+x^2}}$	0.9299812	0.7149701
$k'=\sqrt{1-k^2}$	0.3676073	0.6991550
$\log\sin\beta\left(\sin\beta=\frac{c'}{k'}\right)$	9.7977938	9.5186043
$F(k)$	2.4373371	1.8636661
$E(k)$	1.1323456	1.3449927
$\frac{F-E}{k^2}$	1.5088957	1.0146546
$F(k',\beta)$	0.6852557	0.3394833
$E(k',\beta)$	0.6721988	0.3333201
$\frac{k'^2\sin\beta\cos\beta(F-\Pi)}{c}$	−0.8266738	−1.1256799
$\frac{c'^2}{c^2}(F-\Pi)$	−0.6851799	−0.4045298

$\log\left\{\frac{F-E}{k^2}+\frac{c'^2}{c^2}(\mathrm{F}-\Pi)\right\}$	$\bar{1}.9157773$	$\bar{1}.7854187$
$\log(\Theta(A+a)ck)$	4.0728340	4.4357689
$\log M$	3.9886113	4.2211876
	$M_S = 9741.17$ cm	$M_R = 16641.32$

M_S differs from the value obtained by formula (16) by one part in a million, M_R is identical.

M_S is the mutual inductance of the winding O_1A on S. The inductance M_1 of the whole coil AB on S is twice as much, that is

$$M_1 = 19482.34$$

The inductance of AB on R is M_R above, minus the inductance of O_2B on R which is the same as that of O_1A on S, that is, M_S. Therefore,

$$M_2 = 16641.32 - 9741.17 = 6900.15$$

Hence $M_1 - M_2 = 12582.19$ cm.

The force of attraction of the one winding AB in dynes is

$$\frac{1}{2}f = i_1 i_2 n_2 (M_1 - M_2).$$

The force due to the second winding BC is equal to this. Suppose $i_1 = i_2 = 1$ ampere $= 0.1$ c.g.s. unit of current and $n_2 = 10$ turns per cm. Then

$i_1 i_2 n_2 = 0.10$

$\therefore f = 0.20 \times 12582.19$ dynes
$= 2516.438$ dynes
$2f = 5032.876$ dynes = change of force on reversal of current
$= 5.1356$ gms where $g = 980$.

If there are two sets of coils, one on each side of the balance, as in the ampere balance built for the National Physical Laboratory, the force would be doubled again.

In calculating the mutual inductance of the disk and surrounding solenoid in the Lorenz apparatus the series (56) will be more convergent when the winding is long, and of course more convergent when the disk is not of too great diameter.

EXAMPLE 51. MUTUAL INDUCTANCE OF CAMPBELL'S FORM OF STANDARD BY FORMULAS (55) AND (56)

A cylinder 20 cm in diameter has two coils of 50 turns each wound as shown in Fig. 30, each covering 5 cm (=AB) with an interval of 10 cm between (=AA′). A secondary coil of 1000 turns of finer wire is wound in a channel S, with a mean radius of 14.5 cm. The magnetic field near S, due to the double solenoid, is very weak, and is zero at some point; at this place M will be a maximum, and variations in M due to small changes in A will be very small. To calculate M for the solenoid AB and the coil S, we take two cases, as in the preceding example. First, M for S and a winding O_2B of 100 turns; second, M for S and O_2A of 50 turns. The difference will be M for S and the actual winding AB. Or, supposing

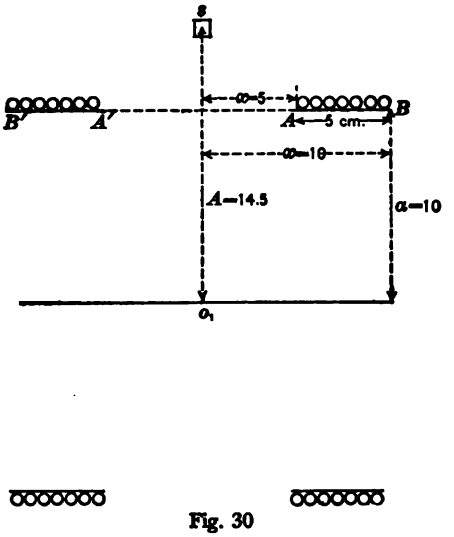


Fig. 30

AB to have 100 turns, M will be the same as for AB of 50 and A′ B′ of 50. Using formula (55) we have the following values:

	For M_1		For M_2
$a=$	10	$=$	10
$A=$	14.5	$=$	14.5
$x=b=$	10	$=$	5.0
$\log k=$	1.9590874	$=$	1.98366715
$\gamma=$	$65°31'7''.32$	$=$	$74°23'38''.88$
$k'=$	$\sqrt{0.1717243}$	$=$	$\sqrt{0.0723711}$
$\beta=$	$26°18'36''.85$	$=$	$43°3'33''.06$
$\gamma'=$	$24°28'52''.68$	$=$	$15°36'21''.12$
$F=$	2.3267801	$=$	2.7312000
$E=$	1.1590043	$=$	1.0812388
$\frac{c}{k}(F-E)=$	1.2613045	$=$	1.6839704
$F(k',\beta)=$	0.4618972	$=$	0.7561693
$E(k',\beta)=$	0.4565314	$=$	0.7469284
$\psi=$	-1.0479404	$=$	-0.7784352
$\frac{A-a}{b}\psi=$	-0.4715732	$=$	-0.7005918
$\frac{c}{k}(F-E)+\frac{A-a}{b}\psi=$	0.7897313	$=$	0.9833786
$n_1n_2=$	200,000	$=$	100,000
$M_1=$	24,313,940 cm	$M_2=$	15,137,940 cm
$=$	24.31394 millihenrys	$=$	15.13794 millihenrys

$M=$ $M_1-M_2=9.1760$ millihenrys.

Campbell gives[60] the value of M as 9.1762 millihenrys, but for want of any particulars of his calculation we do not know wherein the difference lies.

We have worked this problem out also by formula (56) with the following results:

$$M_1=24.31387 \text{ millihenrys}$$
$$M_2=\underline{15.13857} \quad \text{" "}$$
$$M=\ 9.17530 \quad \text{" "}$$

The value of M_1 agrees with that found by (55) to about two parts in a million. M_2 is, however, a little larger, making M smaller. This is due to the fact that formula (56) is not as convergent for

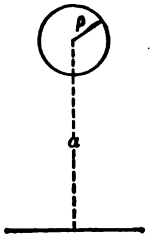
[60] A. Campbell, Proc. Roy. Soc., **79**, p. 428; 1907.

$x = 5$ in this problem as for $x = 10$, and hence the terms neglected after the seventh are appreciable. Hence, for so short a coil as this, formula (54) or (55) will give a more accurate result than (56).

5. THE SELF-INDUCTANCE OF A CIRCULAR RING OF CIRCULAR SECTION

KIRCHHOFF'S FORMULA

The formula for the self-inductance of a *circle* was first given by Kirchhoff[61] in the following form:



$$L = 2l\left\{\log\frac{l}{\rho} - 1.508\right\} \qquad [58]$$

where l is the circumference of the circular conductor and ρ is the radius of its cross section. This is equivalent to the following:

$$L = 4\pi a\left\{\log\frac{8a}{\rho} - 1.75\right\} \qquad [59]$$

a being the radius of the circle, Fig. 31. These formulas are approximate, being more nearly correct as the ratio ρ/a is smaller

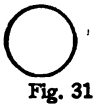


Fig. 31

MAXWELL'S FORMULA

A more accurate expression, obtained by means of Maxwell's principle of the geometrical mean distance, is the following:

$$L = 4\pi a\left\{\left(1 + \frac{3}{16}\frac{R^2}{a^2}\right)\log\frac{8a}{R} - \left(2 + \frac{R^2}{16a^2}\right)\right\} \qquad [60]$$

Substituting in this equation the value of the geometrical mean distance for a circular area, $R = \rho e^{-\frac{1}{4}} = .7788\rho$, we obtain[62]

$$L = 4\pi a\left\{\left(1 + 0.1137\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho} - .0095\frac{\rho^2}{a^2} - 1.75\right\} \qquad [61]$$

This is a very accurate formula for circles in which the radius of section ρ is very small in comparison with the radius a of the circle. The geometrical mean distance R has, however, been computed on the supposition of a linear conductor, and can only

[61] Pogg. Annalen, **121**, p. 551; 1864.
[62] Wied. Annalen, **58**, p. 935; 1894.

be applied to circles of relatively small value of ρ/a, and the square of the geometrical mean distance is used for the arithmetical mean square distance in the second order terms. We must therefore expect an appreciable error in formula (61) when the ratio ρ/a is not very small. Formulas (58), (59), and (61) have been deduced on the supposition of a uniform distribution of the current over the cross section of the ring.

If the ring is a hollow, circular, thin tube, or if the current in the ring is alternating and of extremely high frequency, so that it can be regarded as flowing on the surface of the ring, the geometrical mean distance for the section would be the radius ρ, and we should have instead of (61) the following by substituting $R=\rho$,

$$L=4\pi a\left\{\left(1+\frac{3}{16}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-\frac{\rho^2}{16a^2}-2\right\} \qquad [62]$$

In the case of solid rings carrying alternating currents of moderate frequency the value of L would be somewhere between the values given by (61) and (62).

RAYLEIGH AND NIVEN'S FORMULA

Rayleigh and Niven gave,[63] without proof, the following formula for a *circular coil* of n turns and of circular section,[64] which is more nearly exact than either of the preceding:

$$L=4\pi n^2 a\left\{\left(1+\frac{\rho^2}{8a^2}\right)\log\frac{8a}{\rho}+\frac{\rho^2}{24a^2}-1.75\right\} \qquad [63]$$

When $n=1$, this will be the self-inductance of a single circular ring.[65] This formula neglects higher powers of $\frac{\rho}{a}$ than the second,

[63] Rayleigh's Collected Papers, Vol. II, p. 15.

[64] Neglecting the correction for effect of insulation and shape of section of the separate wires.

[65] Max Wien, Wied. Annalen, 58, p. 928, 1894, derived by direct integration of Maxwell's formula (12) over the cross section of the ring, the formula

$$L=4\pi a\left\{\left(1+\frac{\rho^2}{8a^2}\right)\log\frac{8a}{\rho}-0.0083\frac{\rho^2}{a^2}-1.75\right\}$$

It was shown, however, by Terezawa, Tokyo Math. Phys. Soc., 5, p. 84, 1909, that this formula is in error, the correct result being identical with that of Rayleigh and Niven (63). This result was verified by Mr. Cohen at the Bureau of Standards in 1909, and quite recently independently by Mr. T. J. Bromwich of Cambridge, England. The error of Wien's expression is in practical cases of no importance.

and its error therefore depends on the magnitude of the ratio of the radius of the cross section to the radius of the ring. Assuming, as is probably justified, that the coefficients of the terms in $\left(\frac{\rho}{a}\right)^4$, are of the same magnitude, or smaller, than those of the terms in $\left(\frac{\rho}{a}\right)^2$, the error will not be greater than $\frac{1}{100000}$ even for $\frac{\rho}{a}=0.1$, an exceptionally unfavorable case.

If used for a coil of more than one turn, the expression for L must be corrected for the space occupied by the insulation between the wires and for the shape of the section.[66]

SELF-INDUCTANCE OF A TUBE BENT INTO A CIRCLE

Suppose that the cross section of the ring is not solid, but is an annulus bounded by two concentric circles of radii ρ_1 and ρ_2, ρ_2 being the larger. Then assuming the current to be uniformly distributed over the cross section, we find[67] by means of Wien's method

$$L=4\pi a\left[\left(1+\frac{\rho_1^2+\rho_2^2}{8a^2}\right)\log\frac{8a}{\rho_2}-1.75+\frac{2\rho_2^2+\rho_1^2}{32a^2}\right.$$
$$-\frac{\rho_1^2}{2(\rho_2^2-\rho_1^2)}+\frac{\rho_1^4}{(\rho_2^2-\rho_1^2)^2}\left(1+\frac{\rho_1^2}{8a^2}\right)\log\frac{\rho_2}{\rho_1} \qquad [64]$$
$$\left.-\frac{\rho_2^4+\rho_1^2\rho_2^2+\rho_2^4}{48a^2(\rho_2^2-\rho_1^2)}\right]$$

In this formula terms of higher order than $\frac{\rho_2^2}{a^2}$ and $\frac{\rho_1^2}{a^2}$ have been neglected. Expanding (64) in terms of $\frac{\rho_2^2-\rho_1^2}{a^2}$ and letting ρ_1 approach ρ_2 we find for the case of a tube with infinitely thin walls, or of a tube carrying a current of infinitely high frequency,

$$L=4\pi a\left[\left(1+\frac{\rho^2}{4a^2}\right)\log\frac{8a}{\rho}-2\right] \qquad [65]$$

[66] See Rosa, this Bulletin, **8**, p. 1; 1907.
[67] Grover, Phys. Rev., **30**, p. 787; 1910.

a result which was also found by direct integration,[68] and which was subsequently communicated to us by Mr. T. J. Bromwich.

This corresponds to Maxwell's equation (62), but as might be expected gives a slightly greater value for the inductance.

If we expand (64) in terms of $\frac{\rho_1}{\rho_2}$ and let ρ_1 approach zero, we find for the limiting case of a ring with a solid cross section, the same formula (63) as was derived by directly performing the integration for this case.

An important case is that of a ring of solid cross section, where the current is not distributed uniformly over the cross section, but the current density is proportional to the distance from the axis of the ring. This would apply to the case of a ring revolving about a diameter in a uniform magnetic field. For this Wien (loc. cit.) derived the formula

$$L=4\pi a\left\{\left(1+\frac{3}{8}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.092\frac{\rho^2}{a^2}-1.75\right\} \qquad [66]$$

J. J. THOMSON'S FORMULA FOR RING OF ELLIPTICAL SECTION

If the circular ring has an elliptical section the approximate formula for its self-inductance (corresponding to (59) for a circular section) is[69]

$$L=4\pi a\left\{\log\frac{16a}{\alpha+\beta}-1.75\right\} \qquad [67]$$

where α and β are the semiaxes of the ellipse, and a is the mean radius of the circular ring.

The formulas of Minchin,[70] Hicks,[71] and Bláthy[72] we have elsewhere[73] shown to be incorrect, and hence they are not here given.

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[68] Russell also gives equation (65) but without the term in $\frac{\rho^2}{a^2}$ in Phil. Mag., **18**, p. 430; 1907.

[69] J. J. Thomson, Phil. Mag., **23**, p. 384; 1886.

[70] Phil. Mag., **37**, p. 300; 1894.

[71] Phil. Mag., **38**, p. 456; 1894.

[72] London Electrician, **24**, p. 630; Apr. 25, 1890.

[73] This Bulletin, **4**, p. 149; 1907.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE SELF-INDUCTANCE OF CIRCULAR RINGS OF CIRCULAR SECTION

EXAMPLE 52. COMPARISON OF FOUR FORMULAS FOR THE SELF-INDUCTANCE OF CIRCLES

For a circle of radius $a = 25$ cm and $\rho = 0.05$ cm we obtain from the four formulas the following values of L:

By Kirchhoff's formula (59)	$L = 654.40496\pi$ cm
By Maxwell's formula (61)	$L = 654.40533\pi$ cm
By Rayleigh and Niven's (63)	$L = 654.40548\pi$ cm
By Wien's second formula (66)	$L = 654.40617\pi$ cm.

Thus for so small a value of $\frac{\rho}{a}$ as 1/500 any of these formulas is sufficiently accurate, the greatest difference being less than one in a million, except in the case of formula (66).

EXAMPLE 53. SECOND COMPARISON OF FOUR FORMULAS FOR CIRCLES

For a circle of radius $a = 25$ cm, $\rho = 0.5$ cm, $\frac{\rho}{a}$ being 1/50.

By Kirchhoff's formula (59)	$L = 424.1464\pi$ cm
By Maxwell's formula (61)	$L = 424.1734\pi$ cm
By Rayleigh and Niven's formula (63)	$L = 424.1781\pi$ cm
By Wien's second formula (66)	$L = 424.2326\pi$ cm.

EXAMPLE 54. THIRD COMPARISON OF FOUR FORMULAS FOR CIRCLES

For a circle of radius $a = 10$ cm, $\rho = 1.0$, $\frac{\rho}{a} = 1/10$.

By Kirchhoff's formula (59)	$L = 105.281\pi$ cm
By Maxwell's formula (61)	$L = 105.476\pi$ cm
By Rayleigh and Niven's formula (63)	$L = 105.517\pi$ cm
By Wien's second formula (66)	$L = 105.902\pi$ cm.

It will be seen that for the smallest ring of radius 10 cm and diameter of section 2 cm Maxwell's formula gives a result 1 part in 2500 too small, while the simple approximate formula of Kirchhoff is in error by one in four hundred. For the larger rings the differences are much smaller.

Wien's second formula gives appreciably larger values than the others, as it should do.

EXAMPLE 55. COMPARISON OF FORMULAS (62) AND (65) FOR VERY THIN WALLED TUBES

(*a*) $a=25$ $\rho=0.05$ cm

By Maxwell's formula (62)	$L=629.40556\pi$ cm
By Formula (65)	$L=629.40579\pi$ cm
Solid ring (63)	$L=654.40548\pi$ cm

(*b*) $a=25$ $\rho=0.5$ cm

By Maxwell's formula (62)	$L=399.1889\pi$ cm
By Formula (65)	$L=399.2064\pi$ cm
Solid ring (63)	$L=424.1781\pi$ cm

(*c*) $a=10$ $\rho=1.0$ cm

By Maxwell's formula (62)	$L=95.585\pi$ cm
By Formula (65)	$L=95.719\pi$ cm
Solid ring (63)	$L=105.517\pi$ cm.

Maxwell's expression is nearly correct for the larger ring, but the error increases rapidly as the ratio $\frac{\rho}{a}$ is increased.

EXAMPLE 56. FORMULA (64) FOR A TUBULAR RING

$a=20$ $\rho_2=0.5$ cm = external radius of the cross section.

The calculation has been carried through for different thicknesses of the walls of the tube $(\rho_2-\rho_1)$ ranging from zero (infinitely thin-walled tube) to ρ_2 (solid cross section).

ρ_1	$\frac{\rho_1}{\rho_2}$		L cm
0	0	Solid ring	1010.032
0.125	¼		1003.210
0.25	½		987.528
0.375	¾		968.045
0.5	1	Infinitely thin walls	947.308

In formula (64), next to the first two terms, the fourth and fifth terms are the most important.

6. THE SELF-INDUCTANCE OF A SINGLE LAYER COIL OR SOLENOID

The following approximate formula for the self-inductance of a long solenoid is often given:

$$L = 4\pi^2 a^2 n_1^2 b \qquad [68]$$

where a is the mean radius, n_1 is the number of turns of wire per cm, and b is the length, supposed great in comparison with a. There is a considerable error in this formula, due to the end effect, but the variations in L due to changes in l are almost exactly proportional to the changes in l, and hence this formula may be used for calculating the corresponding variations in L.

RAYLEIGH AND NIVEN'S FORMULAS

The following formula[74] for the self-inductance of a single layer winding on a solenoid is very accurate when the length b is small compared with the radius a, Fig. 32:

$$L_s = 4\pi a n^2 \left\{ \log \frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2}\left(\log \frac{8a}{b} + \frac{1}{4} \right) \right\} \qquad [69]$$

n is the whole number of turns of wire on the coil, and the radius is measured to the center of the wire. The length b is the *mean over-all length including the insulation on the first and last wires* if the coil is wound closely with insulated wire. (See also p. 97.)

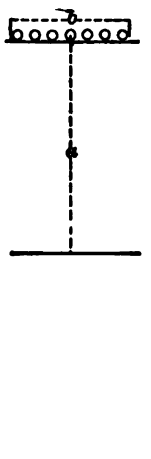

Fig. 32

The self-inductance L_s is, however, not the actual self-inductance of the coil, but the current sheet value; that is, it is the value of the self-inductance if the winding were of infinitely thin tape, so that the current would cover the entire length b. To get the actual self-inductance L for any given case one must correct L_s by formula (80) below. The same remark applies to all the formulas in this section for L_s. The approximate formula (68) is too rough to make it worth while to apply such a correction.

For a coil in which the axial dimension b is zero and the radial depth is c, the following current sheet formula of Rayleigh and Niven gives the self-inductance:

[74] Proc. Roy. Soc., 32, pp. 104–141; 1881. Rayleigh's Collected Papers, 2, p. 15.

$$L_s = 4\pi a n^2\left\{\log\frac{8a}{c} - \frac{1}{2} + \frac{c^2}{96a^2}\left(\log\frac{8a}{c} + \frac{43}{12}\right)\right\} \qquad [70]$$

This is not an important case in practice.

Formulas (69) and (70) may be obtained from (88) by making first $c=0$ and then $b=0$.

COFFIN'S FORMULA

Coffin[75] has extended formula (69) so that it is very accurate for coils of length as great as the radius, and sufficiently accurate for most purposes for coils considerably longer than this.

$$L_s = 4\pi a n^2\left\{\log\frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2}\left(\log\frac{8a}{b} + \frac{1}{4}\right) - \frac{1}{1024}\frac{b^4}{a^4}\left(\log\frac{8a}{b} - \frac{2}{3}\right)\right.$$
$$\left. + \frac{10}{131072}\frac{b^6}{a^6}\left(\log\frac{8a}{b} - \frac{109}{120}\right) - \frac{35}{4194304}\frac{b^8}{a^8}\left(\log\frac{8a}{b} - \frac{431}{420}\right)\right\} \qquad [71]$$

LORENZ'S FORMULA

Lorenz first gave[76] an exact formula for the self-inductance of a

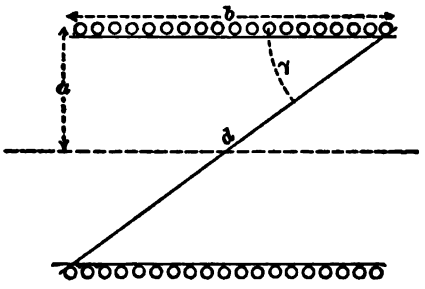

Fig 33

single layer solenoid. It is, like the others, a current sheet formula, and requires correction by (80) for a winding of wire, but applies to a solenoid of any length. Changing the notation slightly Lorenz's formula as originally given is as follows:

[75] This Bulletin, 2, p. 113; 1906.

[76] Wied. Annal., 7, p. 161; 1879. Oeuvres Scientifiques de L. Lorenz, Tome, 2, 1, p. 196.

$$L_s = \frac{32}{3}\frac{\pi n^2 a^2}{b^2}\left\{\frac{2k^2-1}{k^2}E + \frac{1-k^2}{k^2}F - 1\right\} \qquad [72]$$

where $k^2 = \frac{4a^2}{4a^2+b^2}$ and F and E are complete elliptic integrals of the first and second kind of modulus k, and a, b, and n are the radius (Fig. 33), length, and whole number of turns of wire, respectively. By simple substitutions the formula may be put into the following form, where d is the diagonal of the solenoid $= \sqrt{4a^2+b^2}$;

$$L_s = \frac{4\pi n^2}{3b^2}\left\{d(4a^2-b^2)E + db^2F - 8a^2\right\} \qquad [73].$$

Coffin derived[77] an expression for L in elliptic integrals which is equivalent to (73), and also obtained (73) from an expression[78] attributed to Kirchhoff.

Formula (73) may be written

$$L_s = an^2\left[\frac{8\pi}{3}\left\{\sqrt{1+\frac{b^2}{4a^2}}\left(\frac{4a^2}{b^2}-1\right)E + \sqrt{1+\frac{b^2}{4a^2}}F - \frac{4a^2}{b^2}\right\}\right] \qquad [74]$$

or $\quad L_s = an^2 Q$

where a is the radius of the solenoid, n is the whole number of turns on the coil, and Q is the function of $\frac{2a}{b}$ $(=\tan\gamma)$ contained in the square brackets. We have calculated Q for various values of $\tan\gamma$ from 0.2 to 4.0 and given them in Table IV, page 194. This table will be found useful in calculating L_s for solenoids when $\tan\gamma$ has one of the values given in the table, as all calculation of elliptic integrals is avoided. In problems where the length and diameter can be chosen at will, as in the designing of apparatus, this method of calculating L will be most frequently useful. The values of the constant Q given in the table have been computed with great care, so that they give very accurate values of L_s, for long as well as short solenoids.

In calculating the value of L_s by means of formula (69), (71), (73), or (74) and the following, one should use for the length b the *over-all*

[77] This Bulletin, 2, p. 123, equation (31); 1906.

[78] This Bulletin, 2, p. 127, equation (36). The notation is slightly different.

length including the insulation (A B, Fig. 34, and not *a b*) for a close winding of insulated wire, or *n times the pitch* for a uniform winding of bare or covered wire, which is, of course, the same as the length from center to center of $n+1$ turns. The radius a is the mean radius to the center of the wire. The same method of taking the breadth and depth b and c applies in the formulas of section 7. (See also remarks under example 47.)

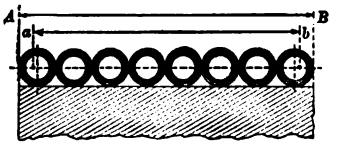

Fig. 34

NAGAOKA'S FORMULAS AND TABLES

In a recent paper[79] Nagaoka has derived formulas and prepared tables by which the self-inductance of a cylindrical current sheet of any dimensions whatever may be accurately and conveniently calculated. Starting from his absolute formula (45) for the mutual inductance of coaxial solenoids, he passes to the special case that the two solenoids coincide, and shows that the resulting expression for the self-inductance is equivalent to Lorenz's absolute formula (73), which he then expands in terms of q or q_1 functions.

He expresses the inductance of a coil of finite length by means of the expression (68) for an infinitely long coil, introducing a correction factor K, which is less than unity, to take account of the effect of the ends of the coil.

Thus

$$L = 4\pi^2 a^2 n_1^2 bK = 4\pi^2 a^2 \frac{n^2}{b} K \qquad [75]$$

where K is a function of half the angular aperture θ of the coil at the center. Nagaoka has prepared tables giving K with θ as argument and also as function of the $\frac{\text{diameter}}{\text{length}} = \frac{2a}{b}$. These tables are reproduced here as Tables XX and XXI, and enable K to be obtained by interpolation with all the accuracy that will usually be required. In case, however, it becomes necessary to obtain a more accurate value of K than can be obtained from these tables, or in such cases as fall outside the range of the tables, or in a portion where the function is changing so rapidly as to make interpolation difficult, the following formulas may be used to calculate K directly.

[79] Jour. Coll. Sci. Tokyo, 27, art. 6, pp. 18–33; 1909.

For short solenoids

$$K=\frac{1}{3\pi\sqrt{q_1}(1+\alpha_1)^2}\left[1-\frac{k'^2}{k^2}+\left\{\frac{k'^2}{k^2}\left(1+\frac{8\beta_1}{1+\alpha_1}\right)+\frac{8\gamma_1}{1-\delta_1}\right\}\frac{1}{2}\log_e\frac{1}{q_1}\right]-\frac{4}{3\pi}\frac{k}{k'}$$

where [76]

$$\alpha_1=q_1^2+q_1^6+q_1^{12}+\cdots\cdots$$

$$\beta_1=q_1^2+3q_1^6+6q_1^{12}+\cdots\cdots \qquad k^2=\frac{4a^2}{4a^2+b^2}$$

$$\gamma_1=q_1-4q_1^4+9q_1^9-\cdots\cdots \qquad k'^2=\frac{b^2}{4a^2+b^2}$$

$$\delta_1=2q_1-2q_1^4+2q_1^9-\cdots\cdots$$

$$q_1=\frac{l_1}{2}+2\left(\frac{l_1}{2}\right)^5+15\left(\frac{l_1}{2}\right)^9+\cdots\cdots$$

$$l_1=\frac{1-\sqrt{k}}{1+\sqrt{k}}=\frac{k'^2}{(1+k)(1+\sqrt{k})^2}$$

For relatively long coils

$$K=\frac{2}{3(1-\delta)^2}\left\{1+\frac{8\beta}{1+\alpha}+\frac{k'^2}{k^2}\cdot\frac{8\gamma}{1-\delta}\right\}-\frac{4}{3\pi}\frac{k}{k'} \qquad [77]$$

where k and k' have the same values as in (76) and

$$q=\frac{l}{2}+2\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+\cdots\cdots$$

$$l=\frac{1-\sqrt{k'}}{1+\sqrt{k'}}=\frac{k^2}{(1+k')(1+\sqrt{k'})^2}$$

and α, β, γ, δ are given by the same equations as α_1, β_1, γ_1, δ_1 in (76) substituting q in place of q_1. Table XV will be found convenient in obtaining q and q_1 from $\frac{l}{2}$ and $\frac{l_1}{2}$. The more complicated expressions for the latter are to be used only when it becomes difficult to obtain $1-\sqrt{k'}$ and $1-\sqrt{k}$ without carrying out the calculation of k and k' to an inconvenient number of decimal places.

For relatively long coils, for which the angle $\theta=\tan^{-1}\frac{2a}{b}$ is not greater than 45°, the simple formula

$$K = 1 - \frac{4}{3\pi}\frac{k}{k'} + 2q + 12q^2 + 44q^3 + 116q^4 + 260q^5$$
$$+ 576q^6 + \frac{3760}{3}q^7 + \cdots \qquad [78]$$

will give values of K correct to a few parts in ten million in the most unfavorable case.

The formulas (76), (77), and (78) between them cover the entire range of values of θ with all the precision desired, since the general terms of the series are known. The formula (76) for short coils is the least convenient to use, and for very short coils (69) is preferable. However, by including terms in q^9 in (77) the range of its applicability may be extended to $\theta = 80°$, so that (76) need not be used except as a check.

THE WEBSTER-HAVELOCK FORMULA

Webster[80] in 1905 by the evaluation of a definite integral, involving Bessel functions, derived a formula for the inductance of relatively long solenoids, which is very simple in form. Havelock[81] gives the same formula as a special application of his formulas for the values of certain integrals of Bessel functions, and stated that the first four terms had already been found by Russell,[82] but seems to have been unacquainted with the work of Webster. This formula is

$$L = 4\pi^2\frac{a^2n^2}{b}\left\{1 - \frac{8}{3\pi}\frac{a}{b} + \frac{1}{2}\frac{a^2}{b^2} - \frac{1}{4}\frac{a^4}{b^4} + \frac{5}{16}\frac{a^6}{b^6}\right.$$
$$\left. - \frac{35}{64}\frac{a^8}{b^8} + \frac{147}{128}\frac{a^{10}}{b^{10}} - \cdots\right\} \qquad [79]$$

Both Webster and Havelock gave the same expression for the general term of this series, viz:

$$\frac{(-1)^s(2s)!(2s+2)!}{s!(s+2)!\{(s+1)!\}^2 2^{2s+1}}\left(\frac{a}{b}\right)^{2s+2}$$

[80] Bull. of Amer. Math. Soc., **14**, No. 1, p. 1; 1907.
[81] Phil. Mag., **15**, p. 332; 1908.
[82] Phil. Mag., **18**, eq. (48), p. 445; 1907.

but in all the terms of Webster's final equation (20), except the first two, a factor 2 has been omitted in the denominator of the coefficients.

This expression (79) is in the form adopted by Nagaoka, the expression in the brackets being equivalent to the correction for the ends K tabulated by Nagaoka.

ROSA'S CORRECTION FORMULA

Rosa has shown[83] that the above formulas (69 to 79) apply accurately only to a winding of infinitely thin strip which completely covers the solenoid (the successive turns being supposed to meet at the edges without making electrical contact) and so realizing the uniform distribution of current over the cylindrical surface which has been assumed in the derivation of all the formulas. A winding of insulated wire or of bare wire in a screw thread may have a greater or less self-inductance than that given by the current sheet formulas above according to the ratio of the diameter of the wire to the pitch of the winding. Putting L for the actual self-inductance of a winding and L_s for the current sheet value given by one of the above formulas,

$$L = L_s - \Delta L$$

The correction ΔL is given by the following expression:

$$\Delta L = 4\pi a n \, [A + B] \qquad [80]$$

where as above a is the radius, n the whole number of turns of wire and A and B are constants given in Tables VII and VIII, pages 197 and 199.

The correction term A depends on the size of the (bare) wire (of diameter d) as compared with the pitch D of the winding; that is, on the value of the ratio d/D. For values of d/D less than 0.58, A is negative, and in such cases when the numerical values of A are greater than the value of B, which is always positive, the correction ΔL will be negative, and hence L will be *greater* than L_s. (See examples 58 and 63.)

[83] This Bulletin, 2, pp. 161–187; 1906.

THE SUMMATION FORMULA FOR L[84]

If we have a single layer winding on a cylinder (Fig. 35), the self-inductance is equal to the sum of the self-inductances of the separate turns plus the sum of the mutual inductances of each wire on all the others. Thus, if there are n turns

$$L = nL_1 + 2(n-1)M_{12} + 2(n-2)M_{13} + 2(n-3)M_{14} + \cdots + 2M_{1n} \quad [81]$$

where L_1 is the self-inductance of a single turn, M_{12} is the mutual inductance of the first and second turns or any two adjacent turns, M_{13} is the mutual inductance of the first and third or of any two turns separated by one, etc., and M_{1n} is the mutual inductance of the first and last turns. For a coil of four turns this becomes

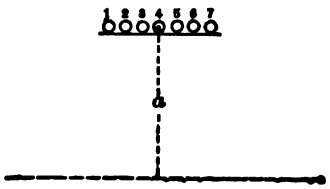

Fig. 35

$$L = 4L_1 + 6M_{12} + 4M_{13} + 2M_{14}$$

L_1 should be calculated by formula (63) or any formula for a circular ring and M_{12}, etc., by (12) or (13). When the number of turns on the coil is small, formula (81) is very convenient, and gives very accurate results.

STRASSER'S FORMULA

Strasser[85] has derived a formula for the self-inductance of a single layer coil of few turns from (81) by substituting for L_1 its value as given by formula (59) and for the various M's their values as given by (12). Strasser's formula with slight correction and some changes in notation is as shown on next page:[86]

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[84] Kirchhoff, Gesammelte Abhandlungen, p. 177.

[85] Wied. Annal., **17**, p. 763; 1905.

[86] Strasser uses the formula for L as: $L = 4\pi a\left(\log\frac{a}{\rho} + 0.333\right)$. This is not quite correct. It should be

$$L_1 = 4\pi a\left(\log\frac{8a}{\rho} - 1.75\right) = 4\pi a \log\left(\frac{a}{\rho} - 1.75 + \log_e 8\right) = 4\pi a\left(\log\frac{a}{\rho} + 0.32944\right).$$

$$L = 4\pi a\left[n\left(\log\frac{8a}{\rho} - 1.75\right) + n(n-1)\left(\log\frac{8a}{d} - 2\right) - A + \frac{d^2}{8a^2}\left\{\left(3\log\frac{8a}{d} - 1\right)\left(\frac{n^2(n^2-1)}{12}\right) - B\right\}\right] \qquad [82]$$

where n is the whole number of turns, d is the pitch, or distance between the centers of two adjacent turns, a is the mean radius of the coil, ρ is the radius of the section of the wire, and A and B are constants given by Table V, page 195, for values of n up to 30. For coils of a larger number of turns (or indeed any number of turns) the value of L can be accurately calculated by (90) and (93) or by (73) and (80).

SELF-INDUCTANCE OF TOROIDAL COIL OF RECTANGULAR SECTION

The first approximation to the self-inductance of a toroidal coil (that is, a circular solenoid) of rectangular section, wound with a single layer of n turns of wire is

$$L_s = 2n^2 h \log\frac{r_2}{r_1} \qquad [83]$$

where h is the axial depth of the coil, and r_1 and r_2 are the inner and outer radii of the ring, Fig. 36. Formula (83) is exact for a toroidal

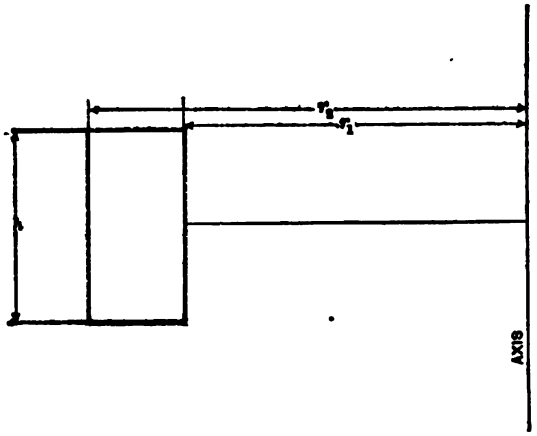

Fig. 36

core enveloped by a current sheet, or for a winding of n turns of infinitely thin tape covering the core completely, the core within

the current sheet being h cm in axial height and $(r_2 - r_1)$ cm in radial breadth.

When the core is wound with round insulated wire, the self-inductance is affected by those lines of force within the cross section of the wire itself, and by those linked with each separate turn of wire in addition to those running through the core. Rosa has shown[87] that the total self-inductance may be more or less than the current sheet value given by (83) according to the size of the wire and the pitch of the winding. In every case, however, the correct value of the self-inductance is derived from the current sheet value L_s by subtracting a correction term ΔL, which is equal to twice the length of the wire multiplied by the sum of two quantities A and B. Thus

$$L = L_s - 2nl(A + B) \qquad [84]$$

where n is the whole number of turns in the winding, l is the length of one turn, A is a quantity, depending on the diameter of the wire and the pitch of the winding, given in Table VII, and B is 0.332. When A is negative and greater than B, L is greater than L_s. This occurs when the pitch of the winding is more than 2.5 times the diameter of the (uncovered) wire.

Fröhlich's formula[88] based on the assumption that a winding of round wires is equivalent to a thick current sheet has been shown to be incorrect.[89]

CHOICE OF FORMULAS

For a coil of only a few turns the summation formula (81), or Strasser's formula (82) give the inductance with great accuracy without the necessity of correction by Tables VII or VIII. Strasser's formula is, however, accurate only for short solenoids, so that the pitch of the winding can not be very great.

For very short solenoids Rayleigh and Niven's formula (69) will give values correct to one in ten thousand for coils whose axial length is as great as one-quarter the diameter of the coil; Coffin's

[87] This Bulletin, 4, p. 141; 1907.
[88] Wied. Annal., 63, p. 142; 1897.
[89] This Bulletin, 4, p. 141; 1907.

extension of this expression (71) gives as great an accuracy for coils as long as one-half the diameter. These two formulas are probably the most convenient for very short solenoids.

For solenoids longer than about one-fifth their diameter the inductance may perhaps most readily be calculated by Nagaoka's formula (75), and the Tables XX and XXI. Havelock's formula (79) is accurate and convenient for coils whose axial length is greater than about one and a quarter times the diameter.

For purposes of great precision, formulas (76), (77), and (78) may be used, (76) being indicated for coils shorter than about one-fifth the diameter, (77) for coils longer than this, and (78) for coils longer than the diameter. Lorenz's absolute formula (73) is of course applicable to coils of all lengths. The interpolation of the elliptic integrals is, however, most easily carried out for coils whose length ranges between one-fifth of the diameter and equality with the latter. The form of this formula is such as to make it necessary in some cases to calculate the separate terms to a greater number of places than are required in the result.

It must be remembered that all these formulas, with the exception of Strasser's and the summation formula (81) give values for a current sheet, and must be corrected to reduce to the actual winding of round wires. This requires the use of formula (80) and Tables VII and VIII.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE INDUCTANCE OF SINGLE LAYER SOLENOIDS

EXAMPLE 57. RAYLEIGH AND NIVEN'S FORMULA (69) AND CORRECTION FORMULA (80) COMPARED WITH THE SUMMATION FORMULA (81)

$a = 25$ cm, $b = 1$ cm, $n = 10$ turns Fig. 37. Suppose the bare wire is 0.8 mm diameter, the covered wire 1.0 mm.

By formula (69)

$$L_s = 4\pi \times 25 \times 100\left\{\log_e 200 - \frac{1}{2} + \frac{1}{20{,}000}\left(\log_e 200 + \frac{1}{4}\right)\right\}$$

$$= 10{,}000\,\pi \times 4.798595$$

$$= 47{,}985.95\,\pi \text{ cm}$$

which is the value of L for a current sheet.

The correction ΔL by formula (80) is $\Delta L = 1000\,\pi\,(A+B)$
Since $D = 1.0$ mm and $d = 0.8$ mm, $d/D = 0.8$

By Table VII, $A = 0.3337$
" " VIII, $B = 0.2664$

$$A + B = 0.6001$$

$$\therefore \quad \Delta L = 600.1\,\pi \text{ cm.}$$

The value of ΔL calculated to one place more of decimals is $\Delta L = 600.16\,\pi$ cm

$$\therefore \quad L = 47985.95\,\pi - 600.16\,\pi$$

or, $$L = 47385.79\,\pi \text{ cm.}$$

The value of L may also be calculated by the summation formula (81), using Rayleigh and Niven's formula (63) for L_1 and Maxwell's formula (12), for the M's. The following are the values of the ten terms of (81) and the resulting value of L:

$$\begin{aligned}
10\,L_1 &= 6767.196\,\pi \text{ cm} \\
18\,M_{12} &= 10081.664\,\pi \\
16\,M_{13} &= 7852.535\,\pi \\
14\,M_{14} &= 6303.439\,\pi \\
12\,M_{15} &= 5057.868\,\pi \\
10\,M_{16} &= 3991.888\,\pi \\
8\,M_{17} &= 3047.787\,\pi \\
6\,M_{18} &= 2193.465\,\pi \\
4\,M_{19} &= 1408.982\,\pi \\
2\,M_{110} &= 680.982\,\pi \\
\hline
\text{Sum} = L &= 47385.806\,\pi \text{ cm.}
\end{aligned}$$

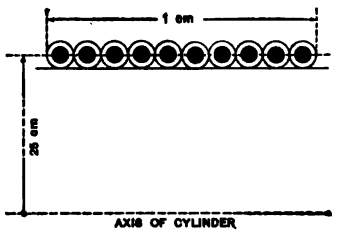

Fig. 37

The difference of less than one in a million between the results obtained by formulas (69) and (80) combined and formula (81) is a good check on the corrections of (80), which amount in this case to more than 1 per cent of the value of the self-inductance. Formula (69) for as short a coil as this is very accurate, the next term, the fourth term of (71), being inappreciable.

If we attempt to use Lorenz's formula in the above example we notice, first, that γ is nearly 89°. The elliptic integrals must consequently be calculated by the series formulas (3), which give their value with all the accuracy desired. We meet, however, with the difficulty that the first and third terms are very nearly equal to one

another and are several hundred times as large as the second term and the sum of the three terms. Consequently, using seven-place logarithms, it is impossible to obtain the self-inductance closer than about five parts in one hundred thousand.

This is also an unfavorable case for (76). Using seven-place logarithms we find

$$K = 21.281755 - 21.220657 = 0.061098$$

and consequently

$$L_e = 47986.27\pi$$

which is about one part in one hundred thousand larger than the correct value.

EXAMPLE 58

As an extreme case to test the use of formulas (69) and (80) we may calculate the self-inductance of a single turn of wire. Let us take the particular case already calculated by Maxwell's and Rayleigh and Niven's formulas (61) and (63), example 52. The radius $a = 25$ cm, the diameter of the bare wire $= 1$ mm. We may now assume that the wire is covered and that the diameter D is 2 mm. Then $\frac{d}{D} = 0.5$. In using Rayleigh's current sheet formula we take the length of the equivalent current sheet as equal to D. We thus have

$$L_e = 4\pi a\left\{\log_e\frac{200}{0.2} - \frac{1}{2} + \frac{0.04}{20000}\left(\log_e\frac{200}{0.2} + \frac{1}{4}\right)\right\}$$

$$= 100\pi\left\{6.907755 - 0.5 + \frac{7.16}{500000}\right\}$$

$$= 640.777\pi \text{ cm.}$$

From Tables VII and VIII $A = -0.1363$ and $B = 0$. Carrying the value of A to one place of decimals more the value is $A = -0.13628$. Thus, since $n = 1$, $\Delta L = 4\pi a\ (-0.13628) = -13.628\pi$, and being negative is added to L_e. Hence

$$L = (640.777 + 13.628)\pi$$
$$= 654.405\pi.$$

This is identical with the value given by the other formulas, example 52.

If we had taken the bare wire of diameter 0.1 cm as equivalent to a current sheet 0.1 cm long in the above formulas for L_n, we should have obtained a different value for L_s, but in that case $\frac{d}{D}$ would be unity and A would be $+.5568$. The resulting value of L would, however, be the same as above.

EXAMPLE 59. COFFIN'S FORMULA (71) COMPARED WITH LORENZ'S (73)

We will use for this case a single layer coil wound on an accurately measured marble cylinder belonging to the Bureau of Standards.

Length of winding, $l = 30.5510$ cm $= b$ in formula (73)
Radius " " $a = 27.0862$ cm
Number of turns $n = 440$

By (71)

$$L_s = 4\pi\overline{440}^2 \times 27.0862 \left\{ 1.4590686 + 0.0878241 - 0.0020427 + .0001651 - 0.0000204 \right\}$$

$$= 4\pi\overline{440}^2 \times 27.0862 \times 1.5449947$$
$$= 10180999 \text{ cm} = 0.10180999 \text{ henry.}$$

By (73)

$$d^2 = 4a^2 + b^2 = 3868.0128$$
$$4a^2 - b^2 = 2001.2858$$
$$\gamma = 60^\circ\ 34'\ 43.''\ 655$$
$$\log F = 0.3369388$$
$$\text{"}\ E = 0.0811833$$

Then

$$L_s = \frac{4\pi.\overline{440}^2}{3\,(30.551)^2} \cdot \left\{ 150050.12 + 126105.36 - 158977.00 \right\}$$

or, $L_s = 101810100$ cm $= 0.10181010$ henry.

The correction to be applied to these values is as follows, the diameter of the bare wire being 0.0634 cm, and consequently $\frac{d}{D}$ $=0.9135$:

$$A=0.4664$$
$$B=0.3353$$
$$(A+B)=0.8017$$
$$4\pi na=108.3448\times 440\pi=47671.7\,\pi$$
$$\therefore\ \Delta L=120067 \text{ cm}$$

and

$L=0.10168992$ henry by Coffin's formula
$L=0.10169003$ " by Lorenz's formula.

The agreement between these two formulas is very satisfactory, although in Coffin's formula b is greater than a. For shorter coils the accuracy of this formula is better; for longer coils the error rapidly increases.

EXAMPLE 60. NAGAOKA'S FORMULAS (75) AND (77)

We will take for this the coil in the preceding example

$$a=27.0862 \qquad b=30.5510 \qquad n=440$$

Here $\frac{2a}{b}=1.77318$ and by interpolation in Table XXI using third differences we find

$$K=0.557885-.003165-.000023-.000001$$
$$=0.554696$$

For this case $\theta=60°34'43.''655=60°.57879$, which gives, by interpolation in Table XX,

$$K=0.560382-.005712+.000027-.000001$$
$$=0.554696$$

Substituting this value of K in (75) we find

$$L_s=0.10181013$$

which differs only three parts in ten million from the value found by Lorenz's formula.

Calculating K by (77) we find

$$\sqrt{k'}=0.70087516$$
$$\frac{l}{2}=0.087932623$$

$$q = 0.087943142$$
$$q^2 = 0.007733997$$
$$q^4 = 5.9815 \times 10^{-5}$$
$$q^6 = 4.626 \times 10^{-7}$$

$$\therefore\ \alpha = 0.00773446 \qquad \gamma = 0.087703884$$
$$\beta = 0.007735385 \qquad 1 - \delta = 0.82423335$$

$$\frac{8\gamma}{(1-\delta)} \cdot \frac{k'^2}{k^2} = 0.27074040$$
$$1 + \frac{8\beta}{1+\alpha} = 1.06140809$$
$$\text{Sum} = 1.33214849$$

multiplied by
$$\frac{2}{3(1-\delta)^2} = 1.3072568$$
$$\frac{4}{3\pi}\frac{k}{k'} = 0.7525609$$
$$\therefore K = 0.5546959$$

If we make the calculation with formula (76)

$$1 + \sqrt{k} = 1.93329106$$
$$1 + k = 1.87103210$$
$$\log_{10} k'^2 = \bar{1}.3825629$$
$$\therefore \frac{l_1}{2} = \frac{k'^2}{(1+k)(1+\sqrt{k})^2} = 0.017252700$$

$$q_1 = 0.017252703$$
$$q_1^2 = 0.0002976555$$
$$q_1^4 = 8.86 \times 10^{-8}$$

$$\log_{10} \frac{1}{q_1} = 1.7631429 \qquad \therefore \frac{1}{2} \log_e \frac{1}{q_1} = 2.0298933$$
$$1 + \alpha_1 = 1.00029766 \qquad 1 + \frac{8\beta_1}{1+\alpha_1} = 1.00238054$$
$$\frac{k'^2}{k^2}\left(1 + \frac{8\beta_1}{1+\alpha_1}\right) = 0.31880658$$
$$\frac{8\gamma_1}{1-\delta_1} = 0.14295127$$
$$\text{Sum} = 0.46175785$$
$$\times \frac{1}{2} \log_e \frac{1}{q_1} = 0.93731902$$

$$1-\frac{k'^2}{k^2}=\underline{0.68195058}$$

$$\text{Sum}=1.61926960$$

multiplied by $\dfrac{1}{3\pi\sqrt{q_1}(1+\alpha_1)^2}=1.3072565$

$$\frac{4}{3\pi}\frac{k}{k'}=\underline{0.7525609}$$

$$\therefore\ K=0.5546956$$

The two formulas give the same value of K within about one part in two million.

The corresponding values of L_s are:

$$L_s=0.10181010 \text{ by } (77)$$
$$L_s=0.10181005 \text{ " } (76)$$

the former value being identical with that found by Lorenz's formula. This example illustrates well the advantage of obtaining K from Tables XX and XXI rather than by calculation. The accuracy of these tables is ordinarily more than sufficient.

The correction to be applied to these current sheet values L_s to obtain the self-inductance L, is the same as that calculated in the preceding example.

EXAMPLE 61. WEBSTER-HAVELOCK FORMULA (79) COMPARED WITH NAGAOKA'S FORMULA (78). LONG COIL

$$a=10 \qquad b=40 \qquad N=400$$

and suppose the diameter of the bare wire to be 0.05 cm

$$1+\frac{1}{2}\frac{a^2}{b^2}= \quad 1.03125000$$

$$-\frac{1}{4}\frac{a^4}{b^4}= -0.00097656$$

$$\frac{5}{16}\frac{a^6}{b^6}= \quad 0.00007629$$

$$-\frac{35}{64}\frac{a^8}{b^8}= -0.00000834$$

$$\frac{147}{128}\frac{a^{10}}{b^{10}}= \quad 0.00000110$$

$$-\frac{693}{512}\frac{a^{12}}{b^{12}}= -\underline{0.00000008}$$

$$\text{Sum}= \quad 1.03034241$$

$$-\frac{8}{3\pi}\frac{a}{b} = -0.21220657$$
$$K = 0.81813584$$

which gives

$$L_e = 0.012919483 \text{ henry}$$

By (78)

$$k^2 = \frac{1}{5} \qquad k'^2 = \frac{4}{5} \qquad \sqrt{k'} = 0.94574152$$
$$1 + k' = 1.89442714$$
$$\therefore \frac{l}{2} = \frac{1}{2(1+\sqrt{k'})^2} \frac{k^2}{(1+k')} = 0.013942859$$
$$q = 0.013942860$$
$$1 + 2q = 1.02788572$$
$$12q^2 = 0.00233284$$
$$44q^3 = 0.00011926$$
$$116q^4 = 0.00000438$$
$$260q^5 = 0.00000014$$
$$576q^6 = 0.00000000$$
$$\text{Sum} = 1.03034234$$
$$\frac{4}{3\pi}\frac{k}{k'} = 0.21220657$$
$$K = 0.81813577$$
$$\therefore L_e = 0.012919482 \text{ henry}$$

which differs by only one part in ten million from the value by the Webster-Havelock formula. The value of K found by interpolation in Nagaoka's tables is $K = 0.818136$.

If we solve this problem by means of Lorenz's formula we are met by the difficulty that $\gamma = 26°$, and therefore the integrals F and E must be taken from Table XII where their values can not be found more accurately than one part in a million.

We find

$$d(4a^2 - b^2)E = -\ 87909.94$$
$$db^2F = 118752.95$$
$$-8a^2 = -\ 8000.00$$
$$\text{Sum} = 30843.01$$
$$\therefore L_e = 0.01291949 \text{ henry.}$$

To find the correction to the current sheet value we have $\frac{d}{D}=0.5$, $n=400$

$$A=-0.1363$$
$$B=+0.3351$$
$$A+B=\quad 0.1988$$
$$4\pi na(A+B)=\quad 9999 \text{ cm}$$
$$=0.00001000 \text{ henry},$$

which must be subtracted from the values of L_e to obtain the self-inductance.

EXAMPLE 62. STRASSER'S FORMULA (82) COMPARED WITH (69) AND (80) AND WITH (81)

Take the coil of 10 turns used in example 57

$$a=25, \quad d=0.10 \quad \rho=0.04, \quad n=10.$$

From Table V, $\quad A=97.9226 \quad B=4241.59$

Substituting in (82),

$$L=100\pi\left[10(\log_e\frac{200}{.04}-1.75)+90(\log_e\frac{200}{0.1}-2)-97.9226\right.$$
$$\left.+\frac{0.01}{5000}\left\{(3\log_e\frac{200}{0.1}-1)\frac{9900}{12}-4241.59\right\}\right]$$

or, $$L=100\pi\left[473.8306+0.0275\right]=47385.81\pi \text{ cm}.$$

This very close agreement with the results by the other two methods (see example 57) is a confirmation of the accuracy of the constants A and B of Table V. Of course, a close agreement with (81) is to be expected, for (82) is derived directly from (81).

EXAMPLE 63. FORMULAS (83) AND (84) FOR TOROIDAL COILS

Professor Fröhlich's standard of self-inductance had the following dimensions:

$r_2=35.05377$ cm = outer mean radius.
$r_1=24.97478$ cm = inner mean radius.
$h=20.08455$ cm = height, center to center of wire.
$\rho=0.011147$ cm = radius of wire.
$n=2738$ = whole number of turns.

These values substituted in (83) give

$$L_s = 0.1020893 \text{ henry.}$$

The correction $\Delta L = -2nl\,(A+B)$ to be substituted in (84) to give the true value of L is found as follows:

The mean spacing of the winding is $D = \pi\frac{r_1+r_2}{n} = 0.0689$

The diameter of the bare wire $\quad d = 2\rho \quad = 0.0223$

$\therefore \quad d/D \quad = 0.324$

From Table VII,

$$A = -0.572$$
$$B = +0.332^{90}$$
$$\therefore A + B = -0.240$$

$2nl = 2 \times 2738 \times 60.327 = 330300$ cm = whole length of wire in winding.

$$-2nl(A+B) = +79{,}300 \text{ cm}$$
$$= 0.0000793 \text{ henry}$$
$$L_s = 0.1020893 \quad ''$$
$$L = 0.1021686 \quad ''$$

Thus, the correction increases the value of the self-inductance. If the insulation were thinner and the wire thicker (with the same pitch) the correction might be of opposite sign. Thus, if ρ were 0.02 and hence d/D were 0.58, A would be $+0.012$ and ΔL would then be 0.0001130 and $L = 0.1019763$ henry, considerably less than the preceding value.

7. THE SELF-INDUCTANCE OF A CIRCULAR COIL OF RECTANGULAR SECTION

MAXWELL'S APPROXIMATE FORMULA

Maxwell first gave[91] an approximate formula for the important case of a circular coil or conductor of rectangular section, Fig. 38, as follows:

$$L = 4\pi a n^2\left(\log\frac{8a}{R} - 2\right) \qquad [85]$$

[90] This Bulletin, 4, p. 141; 1907. This value applies to any toroidal coils, of 24 turns or more.

[91] Elect. and Mag., Vol. II, § 706.

where R is the geometrical mean distance of the cross section of the coil or conductor. The current is supposed uniformly distributed over this section.

The value of R for any given shape of rectangular section is given by (124). Its value for several particular cases is given in the table on page 168. It is very nearly proportional to the perimeter of the rectangle and approximately equal to 0.2235 $(\alpha+\beta)$ where α and β are the length and breadth of the rectangle.

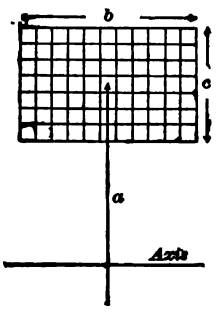

Fig. 38

Formula (85) is derived from (11) by putting R, the geometrical mean distance of the area of the section of the coil from itself, in place of r, the distance between two circles. If we use (12) instead of (11) for this purpose, we shall have a closer approximation to the value of L. Thus,

$$L=4\pi an^2\left\{\log\frac{8a}{R}\cdot\left(1+\frac{3R^2}{16a^2}\right)-\left(2+\frac{R^2}{16a^2}\right)\right\} \qquad [86]$$

We have placed R^2 in place of d^2 in the second order terms, which is of course not strictly correct, as we should use an arithmetical mean square distance instead of a geometrical mean square distance. (See p. 171.) Nevertheless, (86) is a much closer approximation than (85).

PERRY'S APPROXIMATE FORMULA

Professor Perry has given[92] the following empirical expression for the self-inductance of a short circular coil of rectangular section:

$$L=\frac{4\pi n^2a^2}{0.2317a+0.44b+0.39c} \qquad [87]$$

in which n is the whole number of turns of wire, a the mean radius, b the axial breadth, c the radial depth. As in all the formulas of this paper, the dimensions are in centimeters and the value of L is in centimeters. This formula gives a good approximation to L as long as b and c are small compared with a.

[92] John Perry, Phil. Mag., **30**, p. 223; 1890.

WEINSTEIN'S FORMULA

Maxwell's more accurate expression for the self-inductance of a circular coil of rectangular section[93] was not quite correct. The investigation was repeated by Weinstein,[94] who gave the following formula:

$$L_u = 4\pi a n^2 (\lambda + \mu)$$

where

$$\lambda = \log \frac{8a}{c} + \frac{1}{12} - \frac{\pi x}{3} - \frac{1}{2}\log(1+x^2) + \frac{1}{12x^2}\log(1+x^2)$$
$$+ \frac{1}{12}x^2 \log\left(1+\frac{1}{x^2}\right) + \frac{2}{3}\left(x - \frac{1}{x}\right)\tan^{-1}x$$
$$\mu = \frac{c^2}{96a^2}\left[\left(\log\frac{8a}{c} - \frac{1}{2}\log(1+x^2)\right)(1+3x^2) + 3.45x^2 + \frac{221}{60}\right. \qquad [88]$$
$$\left. - 1.6\pi x^3 + 3.2x^3\tan^{-1}x - \frac{1}{10}\frac{1}{x^2}\log(1+x^2) + \frac{1}{2}x^4\log\left(1+\frac{1}{x^2}\right)\right]$$

b and c are the breadth and depth of the coil and $x = \frac{b}{c}$.

Weinstein's formula for the case of a square section, where $b = c$ reduces to the following simpler expression:

$$L_u = 4\pi a n^2\left\{\left(1 + \frac{c^2}{24a^2}\right)\log\frac{8a}{c} + .03657\frac{c^2}{a^2} - 1.194914\right\} \qquad [89]$$

This is a very accurate formula as long as c/a is a small quantity. The current is supposed distributed uniformly over the section of the coil, and hence for a winding of round insulated wire, correction must be made by formula (93).

STEFAN'S FORMULA

Stefan[95] simplified Weinstein's expression (88) by collecting together terms depending on the ratio of b to c and computing two short tables of constants y_1 and y_2. His formula is as follows:

$$L = 4\pi a n^2\left\{\left(1 + \frac{3b^2 + c^2}{96a^2}\right)\log\frac{8a}{\sqrt{b^2+c^2}} - y_1 + \frac{b^2}{16a^2}y_2\right\} \qquad [90]$$

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[93] Phil. Trans., 1865, and Collected Works.
[94] Wied. Annal., 21, p. 329; 1884.
[95] Wied. Annal., 22, p. 113; 1884.

The values of y_1 and y_2 are given in Table VI, page 196, as functions of $x=b/c$ or c/b; that is, x is the ratio of the breadth to the depth of the section, or vice versa, being always less than unity. This formula must be corrected by the quantity $\Delta_2 L$ as shown below.

For the method of taking the dimensions b and c of the cross section, see page 116, section 6; also example 47, page 97.

LONG COIL OF RECTANGULAR SECTION; I. E., SOLENOID OF MORE THAN ONE LAYER

ROSA'S METHOD

When the coil is so long that the formula of Stefan is no longer accurate, the self-inductance may be accurately calculated by a method given by Rosa.[96]

In Figs. 39, 40, and 41 are shown three coils, having the same length and mean radius. The first is a single winding of thin tape and the self-inductance, calculated by a current sheet formula, is L_s. The second is a single layer of wire of square section (length b, depth c, and b/c turns) and its self-inductance is L_u, the current being supposed uniformly distributed over the area of the square conductors. The third is a winding of round insulated wire of length b, depth c, and any number of layers, and its self-inductance is L. These different self-inductances are related as follows:

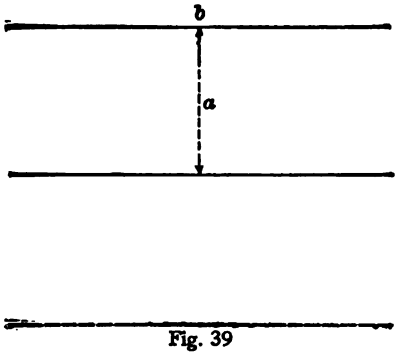

Fig. 39

$$L_s - \Delta_1 L = L_u$$
$$L_u + \Delta_2 L = L$$
$$\therefore L = L_s - \Delta_1 L + \Delta_2 L$$

L_s is calculated by any current sheet formula as (69), (71), (72), or (73). The correction $\Delta_1 L$ for the depth of the coil is given by the following formula:

$$\Delta_1 L = 4\pi a n' \; [A_s + B_s] \qquad [91]$$

[96] This Bulletin, 4, p. 369; 1907.

This formula has the same form as (80), but some of the quantities have a different meaning; a is the mean radius as before, n' is b/c, the number of square conductors in the length b, Fig. 40, and A_s and B_s are given in Tables IX and X.

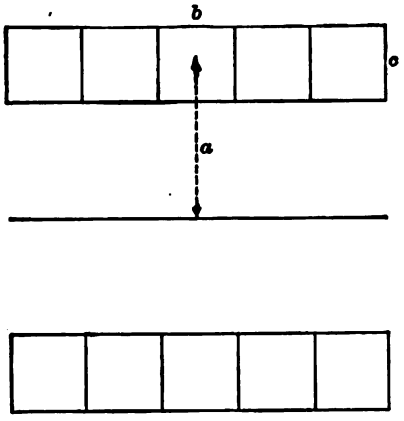

Fig. 40

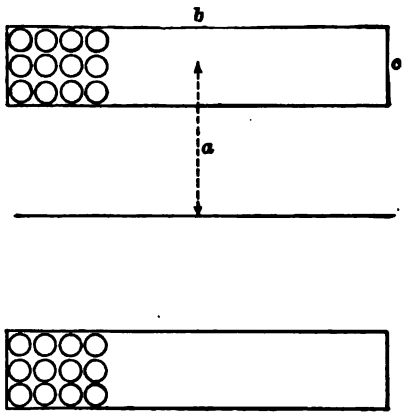

Fig. 41

The correction $\Delta_2 L$ is calculated in precisely the same way as for a short coil, as described below, formula (93). The above formula for $\Delta_1 L$ gives a very accurate value of the correction to be applied to L_s to obtain L_u, and permits a test to be made for the error of Stefan's formula when applied to longer coils than the latter is intended for. Such a calculation shows that for a coil as long as its diameter Stefan's formula (and Weinstein's also, of course) is 1 per cent in error, giving too large a value.

COHEN'S APPROXIMATE FORMULA

Cohen has given the following approximate formula[97] for the self-inductance of a long coil or solenoid of several layers:

[97] This Bulletin, 4, p. 389; 1907.

$$L = 4\pi^2 n^2 m\left\{\frac{2a_0^4 + a_0^2 l^2}{\sqrt{4a_0^2 + l^2}} - \frac{8a_0^3}{3\pi}\right\}$$
$$+ 8\pi^2 n^2\left[\left\{(m-1)a_1^2 + (m-2)a_2^2 + \cdots\right\}\left(\sqrt{a_1^2 + l^2} - \frac{7}{8}a_1\right)\right.$$
$$+ \frac{1}{2}\left\{m(m-1)a_1^2 + (m-1)(m-2)a_2^2 + \cdots\right\}\left(\frac{a_1 \delta a}{\sqrt{a_1^2 + l^2}} - \delta a\right)$$
$$\left. - \frac{1}{2}\left\{m(m-1)a_1^2 + (m-2)(m-3)a_2^2 \cdots\right\}\frac{\delta a}{8}\right] \qquad [92]$$

where a_0 is the mean radius of the solenoid, $a_1, a_2, \cdots a_m$ are the mean radii of the various layers in the order of their magnitudes, m is the number of layers and δa is the distance between centers for any two consecutive layers, and n is the number of turns per unit length.

For long solenoids, where the length is, say, four times the diameter, we can neglect the last term in equation (92).

This formula is sufficiently accurate for most purposes; it will give results accurate to within one-half of 1 per cent even for short solenoids, where the length is only twice the diameter.

MAXWELL'S CORRECTION FORMULA[98]

GIVING THE VALUE OF $\Delta_2 L$

Maxwell has shown that when a coil of rectangular section (Fig. 41) is wound with round insulated wire and the self-inductance is calculated by a formula in which the current is assumed to be distributed uniformly over the section, as in Weinstein's and Stefan's, the calculated value L_u is subject to three corrections, each of which tends to increase the calculated value of the self-inductance. Thus:

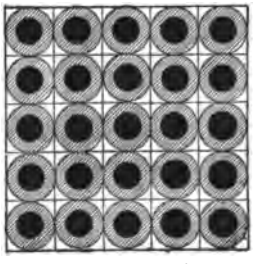

Fig. 42

$$L = L_u + \Delta_2 L$$

and $$\Delta_2 L = 4\pi a n\left\{\log_e \frac{D}{d} + 0.13806 + E\right\} \qquad [93]$$

Maxwell showed that the first term takes account of the effect of the insulation, d and D being the diameters of the bare and covered wire, respectively, Fig. 42. The second correction term (0.13806)

[98] Elect. and Mag., Vol. II, § 693.

reduces from a square section to a circular section for the conductor. The third correction term E takes account of the differences in the mutual inductances of the separate turns of wire on one another when the wire has a round section from what the mutual inductances would be if the wire were of square section and no space was occupied by insulation. This term was stated by Maxwell to be equal to -0.01971; it was subsequently stated by Stefan to be equal to $+0.01688$. Rosa has shown[99] that its value is variable, depending on the number of turns of wire in the coil and the shape of the cross section of the latter, and has given the values of E for a number of particular cases.

From the following table one can interpolate for E for any particular case not included in the table.

Summary of the values of E found for the various cases considered:

2	turns		$E=$ . . .	0.006528
3	"	(one layer)	$E=$	.009045
4	"	(two layers)	$E=$	.01691
4	"	(one layer)	$E=$	.01035
8	"	(two layers)	$E=$	.01335
10	"	(one layer)	$E=$	.01276
20	"	(one layer)	$E=$	.01357
16	"	(four layers)	$E=$	.01512
100	"	(ten layers)	$E=$	.01713
400	"	(20×20)	$E=$	.01764
1,000	"	(50×20)	$E=$	.01778
Infinite number of turns			$E=$	.01806

The correction $\Delta_2 L$ is much smaller than $\Delta_1 L$, and can be neglected except when the highest accuracy is sought. The value

[99] This Bulletin, 8, p. 37; 1907.

of L_s and $\Delta_1 L$ can be calculated with accuracy if the dimensions are accurately known, and this is possible if one uses enameled wire of uniform section and takes proper care in winding and measuring the coil. However, such a coil can not be recommended for a standard of the highest precision, and the full theory is given for the sake of completeness and to show the magnitude of the smaller corrections, rather than because all the corrections are likely to be generally needed in practice.

CHOICE OF FORMULAS

If the dimensions of the cross section be very small relatively to the mean radius, formula (86) may be used. Formula (85) is a still rougher approximation, as is also (87).

For somewhat larger cross section Weinstein's formula (88) will give good results. Stefan's form (90) of Weinstein's expression is more convenient to use. Formula (89) is convenient and accurate for coils of square cross section. All these formulas assume that the current is uniformly distributed over the cross section of the coil, and must consequently be corrected by formula (93) to reduce to a winding of round wires.

The formulas (88) and (90) begin to be in error for long coils. Cohen's formula (92), however, is most accurate for long solenoids, whose length is more than about four times the diameter.

The most accurate formulas are those of Rosa's method (91) and (93). Since the current sheet value may be very accurately obtained by any of the suitable formulas in section 6, this method may be applied to any solenoidal coil whatever.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE SELF-INDUCTANCE OF CIRCULAR COILS OF RECTANGULAR SECTION

EXAMPLE 64. MAXWELL'S APPROXIMATE FORMULAS (85), (86) AND PERRY'S APPROXIMATE FORMULA (87) COMPARED WITH WEINSTEIN'S FORMULA (89)

Suppose a coil of mean radius 4 cm, with 100 turns of insulated wire, wound in a square channel 1×1 cm. (Fig. 43.)

Substituting in (85) $a=4$, $n=10$, $R=0.44705$ (the g. m. d. of a square 1 cm on a side) we have

$$L = 4\pi.4.\overline{100}^2\left[\log_e \frac{32}{.44705} - 2\right]$$
$$= 1.141 \text{ millihenrys.}$$

This is a first approximation to the self-inductance of the coil. Formula (86) gives a second approximation as follows:

$$L = 4\pi.4.\overline{100}^2\left[\log_e \frac{32}{0.44705}\left(1 + \frac{3\times\overline{0.447}^2}{256}\right) - \left(2 + \frac{\overline{0.447}^2}{256}\right)\right]$$
$$= 1.146 \text{ millihenrys.}$$

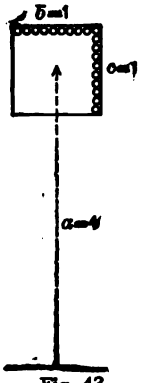

Fig. 43

Perry's approximate formula, which applies only to relatively short coils, happens to give a very close approximation for this case. Substituting in (87), the above values, and also $b = c = 1$,

$$L = \frac{4\pi\ \overline{100}^2 \times 16}{0.9268 + 0.44 + 0.39}$$
$$= 1.144 \text{ millihenrys.}$$

Substituting in the more accurate formula (89) of Weinstein we shall obtain a value with which to compare the above approximations.

$$L = 160000\pi\left[\left(1 + \frac{1}{384}\right)\log_e \frac{32}{1} + 0.03657 \times \frac{1}{16} - 1.194914\right]$$
$$= 1.147 \text{ millihenrys.}$$

For $a = 4$, $b = 2$, $c = 1$ $n = 200$

Formula	(85)	gives	3.750	millihenrys
"	(86)	"	3.787	"
"	(87)	"	3.661	"
"	(89)	"	3.805	"

For $a = 10$, $b = 1$, $c = 1$, $n = 100$

Formula	(85)	gives	4.005	millihenrys
"	(86)	"	4.007	"
"	(87)	"	3.994	"
"	(89)	"	4.008	"

It will be seen that formula (87) does not give as close approximations as the others, except in the case of the first example, where it

happens to give a value very close to that given by (89). All the values, those of (89) included, are subject to correction by (93) when the coil is wound with round insulated wire.

EXAMPLE 65. FORMULAS (89) AND (90) COMPARED WITH CURRENT-SHEET FORMULAS

As a test of these formulas we may calculate the self-inductance of a single turn of wire, using the case already calculated in example 52; that is, a circle of radius $a = 25$ cm, and the diameter of the bare wire is 1 mm. Substituting these values in (89) we have

$$L = 100\pi\left[\left(1 + \frac{.01}{15000}\right)\log_e 2000 + \frac{.03657}{(250)^2} - 1.194914\right]$$

$$= 640.5995\,\pi \text{ cm.}$$

Substituting in (90),

$$L = 100\pi\left[\left(1 + \frac{.01}{15000}\right)\log_e \frac{200}{\sqrt{.02}} - 0.848340 + \frac{.01 \times .8162}{10000}\right]$$

$$= 640.5995\,\pi \text{ cm,}$$

agreeing with the value by (89).

Fig. 44

These values are for a conductor of square cross section (Fig. 44). To reduce to a circular section of same diameter (0.1 cm) we must apply the second correction term of (93); that is, add to the above value

$$\Delta L = 4\pi a \times 0.138060$$

$$\text{Thus, } L = (640.5995 + 13.8060)\,\pi$$

$$= 654.4055\pi \text{ cm,}$$

which agrees with the value found for the self-inductance of a round wire 0.1 cm diameter, bent into a circle of 25 cm radius, by formula (63) example 52 and formulas (69) and (80), example 58.

EXAMPLE 66. STEFAN'S FORMULA (90) COMPARED WITH (69) BY MEANS OF ROSA'S CORRECTION FORMULA (91)

Suppose a coil of mean radius 10 cm, wound with 100 turns in a square channel 1×1 cm. Assuming the current uniformly distributed we obtain from (90), in which $y_1 = 0.848340$, $y_2 = 0.8162$,

$$\log_e \frac{8a}{\sqrt{b^2+c^2}} = \log_e \frac{80}{\sqrt{2}} = 4.03545$$

$$L_u = 4\pi \times 100{,}000\left[\left(1+\frac{4}{9600}\right)4.03545 - 0.84834 + 0.00051\right]$$
$$= 4\pi \times 318{,}930 \text{ cm}$$
$$= 4.00779 \text{ millihenrys.}$$

By formula (69) we have for the self-inductance of a current sheet for which $a = 10$, $b = 1$, $n = 1$,

$$L_s = 4\pi \times 38.83475 \text{ cm.}$$

This is larger than the value for the coil of section 1×1 by $\Delta_1 L$, the value of the latter being given by formula (91).

By Table IX, $A_s = 0.6942$. More closely, it is 0.69415.[100]

By Table X, $B_s = 0$. In this case $n' = 1$. Hence,

$$\Delta_1 L = 4\pi \times 10 \times 0.69415 = 4\pi \times 6.9415 \text{ cm}$$
$$\therefore L_1 = 4\pi\ (38.83475 - 6.9415) = 400.782 \text{ cm.}$$

This is the value of the self-inductance for one turn only, the current being uniformly distributed. For 100 turns L is 10^4 times as great.

$$\therefore L_u = 4.00782 \text{ millihenrys.}$$

This value agrees with the above value by Stefan's formula within less than one part in one hundred thousand.

For a coil of insulated round wires, this result must be corrected by formula (93).

For a coil of the same radius, but of length $b = 10$ cm, $c = 1$ cm, wound with 10 layers of 100 turns each, we have the following values:

By Stefan's formula, $y_1 = 0.59243$, $y_2 = 0.1325$

$$L_u = 4\pi \times 10 \times \overline{1000}^2 \times 1.55536$$
$$= 195.452 \text{ millihenrys.}$$

[100] This Bulletin, 4, p. 369; 1907.

By (69) the current sheet value of L for 10 turns is

$$L_{10} = 4\pi \times 10 \times 100 \times 1.65095$$
$$= 4\pi \times 1650.95.$$

The correction for depth of section by (91) is, since by Tables IX and X, $A_s = 0.6942$, $B_s = 0.2792$, and therefore $A_s + B_s = 0.9734$

$$\Delta_1 L = 4\pi 10 \times 10 \times 0.9734$$
$$= 4\pi \times 97.34$$
$$\therefore L_u = L_{10} - \Delta_1 L = 4\pi(1650.95 - 97.34)$$
$$= 4\pi \times 1553.61 \text{ cm for 10 turns.}$$

For $n = 1000$ turns the self-inductance will be $\overline{100}^2$ times as great.

$$L_u = 4\pi \times 15.5361 \times 10^6 \text{ cm}$$
$$= 195.232 \text{ millihenrys.}$$

This value is about 1 part in 900 smaller than the above value, showing that Stefan's formula gives too large results by that amount for a coil of this length. If the coil were twice as long, the error would be about ten times as great.

It is interesting to obtain by this method an estimate of the error by Stefan's formula for coils longer than those for which it is intended. For short coils it is seen to be very accurate, subject always to the corrections of formula (93), and for longer coils it gives a good approximation. The method of (91), however, applies to coils of any length.

EXAMPLE 67. STEFAN'S FORMULA (90) COMPARED WITH (81) AND WITH STRASSER'S (82) FOR COILS OF FEW TURNS, USING THE CORRECTION FORMULA (93)

Coil of 2 turns of wire, 0.4 mm diameter, wound in a circle of 1.46 cm radius with a pitch of 2 mm. Stefan's formula assumes a uniform distribution over a rectangular section. Suppose a section as shown in Fig. 45, 4×2 mm, with one turn of wire in the center of each square. For the rectangular section, with the current uniformly distributed, the self-inductance by Stefan's formula is with $a = 1.46$, $c/b = 0.5$, $y_1 = 0.7960$, $y_2 = 0.3066$, $L_u = 4\pi a n^2 \times 2.4763 = 4\pi a n \times 4.9526$, n being 2. To reduce this to the case of a

winding of 2 turns of wire as shown we must apply the corrections given by (93) thus:

$$\begin{aligned} \log D/d = \log_e 5 &= 1.60944 \\ \text{second term} &= 0.13806 \\ \text{third term } E &= 0.00653 \\ \hline & \ 1.7540 \end{aligned}$$

$$\therefore \Delta_2 L = 4\pi an \times 1.7540$$
$$L = L_u + \Delta_2 L = 4\pi an \times 6.7066$$
$$= 246.1 \text{ cm.}$$

Fig. 45

By the summation formula (81) we have in this case

$$L = 2L_1 + 2M_{12}$$
$$= 4\pi a[9.2400 + 4.1606]$$
$$= 245.86 \text{ cm.}$$

The value by Strasser's formula is the same as by the summation formula to which it is equivalent. We have also used formulas (69) and (80) for this case and have obtained 246.0.

This is one of several problems calculated by Drude[101] by Stefan's formula. Drude concluded that Stefan's formula was inapplicable to such coils, as it gave results from 10 to 25 per cent too large. His trouble was, however, due to taking the length of the coil as the distance between the center of the first wire and the center of the last (instead of n times the pitch) and neglecting the correction terms of formula (93). As we have seen above, Stefan's formula when properly used can be depended upon to give accurate results for short coils, and results within less than 1 per cent for coils of length equal to the radius of the coil.

We have calculated several other cases given by Drude and give below the results, together with his experimental values. The radius is the same in each case, and the numbers in the first column are the number of turns in the several coils.

[101] Wied. Annal., 9, p. 601; 1902.

n	By Stefan's Formula (90) and (93)	By Rayleigh's Formula (69) and (80)	By Strasser's Formula (82) or (81)	Drude's Observed Values (Values of L in Centimeters)
2	246.1	246.0	245.9	238.5
4	711.9	711.1	710.8	697.9
6	1298.7	1297.7	1297.8	1271.4
9	2318.0	2313.0	2315.7	2300.1

It will be seen that the values by the different formulas agree very closely, and that the experimental values agree as closely as could be expected for such small inductances.

EXAMPLE 68. FORMULAS (69) AND (80) COMPARED WITH (90) AND (93) FOR COIL OF 20 TURNS WOUND WITH A SINGLE LAYER

$$a=25 \qquad b=2 \text{ cm} \qquad c=0.1 \text{ cm} \qquad n=20.$$

Diameter of bare wire 0.6 mm, of covered wire 1.0 mm.

In the last case we obtained the self-inductance of the coil by two distinct methods, the first being the method of summation, the second by assuming the current uniformly distributed over the section, and then applying the three corrections C, F, E. In this problem we may first calculate L by use of the current sheet formula (69), and then apply the corrections for section, A and B formula (80); and, second, by Stefan's formula for uniform distribution, and apply the three corrections C, F, E, which give the value for a winding of round insulated wires.

Rayleigh's formula for this example gives:

$$L=4\pi an^2\left\{\log_e 100-0.5+\frac{4}{20{,}000}\left(\log_e 100+\frac{1}{4}\right)\right\}$$

$$\log_e 100=4.605170$$

$$\frac{4}{20{,}000}\left(\log_e 100+\frac{1}{4}\right)=\begin{array}{r}0.000971\\ \hline 4.606141\\ -0.500000\\ \hline 4.106141\end{array}$$

$$4\pi an^2=40{,}000\pi \qquad \therefore L_s=164\,245.64\pi \text{ cm.}$$

This is the self-inductance of a winding of 20 turns of infinitely thin tape, each turn being 1 mm wide, with edges touching without

making electrical contact, which arrangement fulfills the conditions of a current sheet. To reduce this to the case of round wires we must apply the corrections A and B for self and mutual induction.[102]

$$\begin{aligned} &\text{By Table VII, for } d/D = 0.6,\ A = 0.0460 \\ &\text{By Table VIII, for } n = 20,\quad B = \underline{0.2974} \\ &\qquad\qquad\qquad\qquad A + B = 0.3434 \\ &\qquad\qquad 4\pi an = \quad 2{,}000\pi \\ &\Delta L = 4\pi an(A+B) = \quad 686.8\pi \text{ cm} \\ &\qquad L = L_s - \Delta L = 163\ 558.84\pi \text{ cm.} \end{aligned}$$

By Stefan's formula we find, sustituting the above values of a, n, b, c, and taking $y_1 = 0.548990$ and $y_2 = 0.1269$

$$L_u = 162\ 234.60\pi \text{ cm.}$$

The correction E for a single layer coil of 20 turns is given on page 141. The three corrections are then as follows:

$$\begin{aligned} C &= 0.13806 \\ F &= 0.51082 = \log_e \frac{10}{6} \\ E &= \underline{0.01357} \\ \text{Sum} &= 0.66245 \\ \therefore \Delta L &= 4\pi an(C+F+E) = 1324.90\pi \text{ cm.} \\ \therefore L &= L_u + \Delta L = 163\ 559.50\pi \text{ cm.} \end{aligned}$$

This value of L is greater than the value found by the other method by only four parts in a million. Thus we see that the method of calculating L_u by Stefan's or Weinstein's formula and applying the corrections C, F, E gives practically identical results with the method of summation and also with the current sheet method *for short coils.* When, however, the coils are longer, the agreement is not so good, for the reason that the formula of Weinstein (and Stefan's, derived from it) is not as accurate when the section of the coil is greater. Thus if the coil in the above problem had been 5 cm long and 2.5 mm deep and wound with 20 turns of heavier wire, the difference would have been one part in twenty-five thousand (still very good agreement), and if it were 10 cm long and

[102] Rosa, this Bulletin, 2, p. 161; 1906.

0.5 cm deep (the radius being 25 cm) it would have been one part in two thousand two hundred. For most experimental work, therefore, Stefan's formula is amply accurate.

EXAMPLE 69. COHEN'S FORMULA (92) COMPARED WITH (91)

A solenoid of length $l = 50$ cm, mean radius 5 cm, depth of winding 0.4 cm, is wound with 4 layers of wire of 500 turns each. Substituting these values in (92) we have ($n = 10$)

$$L_s = 16\pi^2 n^2 (1144.3 + 3336.0 - 10.84 - 1.04)$$
$$= 70.562 \text{ millihenrys.}$$

By the second method we first find L_s by (69), then $\Delta_1 L$ by (91), and $\Delta_2 L$ by (93)

$$\begin{aligned} L_s &= 72.648 \text{ millihenrys} \\ -\Delta_1 L &= -2.167 \quad \text{"} \\ \Delta_2 L &= \underline{\ \ 0.048} \quad \text{"} \\ L &= 70.529 \quad \text{"} \end{aligned}$$

This shows a very close agreement between (92) and (91).

In calculating L_s we may use Table IV. Since $d/l = 0.2$

$$Q = 3.6324, \quad an^2 = 5 \times \overline{2000}^2 = 20{,}000{,}000$$
$$L_s = 3.6324 \times 20{,}000{,}000 \text{ cm}$$

or,

$$L_s = 72.648 \text{ millihenrys.}$$

8. SELF AND MUTUAL INDUCTANCE OF LINEAR CONDUCTORS [103]

SELF-INDUCTANCE OF A STRAIGHT CYLINDRICAL WIRE

The self-inductance of a length l of straight cylindrical wire of radius ρ is

$$L = 2\left[l \log \frac{l + \sqrt{l^2 + \rho^2}}{\rho} - \sqrt{l^2 + \rho^2} + \frac{l}{4} + \rho \right] \qquad [94]$$

$$= 2l\left[\log \frac{2l}{\rho} - \frac{3}{4} \right] \text{ approximately.} \qquad [95]$$

Where the permeability of the wire is μ, and that of the medium outside is unity, (95) appears in the form

$$L = 2l\left[\log \frac{2l}{\rho} - 1 + \frac{\mu}{4} \right] \qquad [96]$$

[103] See paper by E. B. Rosa, this Bulletin, 4, p. 301; 1907.

This formula was originally given by Neumann.

For a straight cylindrical tube of infinitesimal thickness, or for alternating currents of great frequency, when there is no magnetic field within the wire, the self-inductance is

$$L = 2l\left[\log\frac{2l}{\rho} - 1\right] \qquad [97]$$

This is obtained by subtracting from (95) $l/2$ or from (96) $\mu\, l/2$, the magnetic flux within the conductor due to unit current.

THE MUTUAL INDUCTANCE OF TWO PARALLEL WIRES

The mutual inductance of two parallel wires of length l, radius ρ, and distance apart d is the number of lines of force, due to unit current in one, which cut the other when the current disappears.

This is

$$M = 2\left[l\log\frac{l+\sqrt{l^2+d^2}}{d} - \sqrt{l^2+d^2} + d\right] \qquad [98]$$

$$\therefore M = 2l\left[\log\frac{2l}{d} - 1 + \frac{d}{l}\right] \text{ approximately} \qquad [99]$$

when the length l is great in comparison with d.

Equation (98), which is an exact expression when the wires have no appreciable cross section, is not an exact expression for the mutual inductance of two parallel cylindrical wires, but is not appreciably in error even when the section is large and d is small if l is great compared with d.

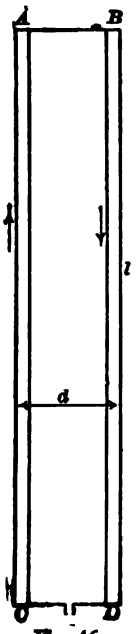

Fig. 46

THE SELF-INDUCTANCE OF A RETURN CIRCUIT

If we have a return circuit of two parallel wires each of length l (the current then flowing in opposite direction in the two wires) the self-inductance of the circuit, neglecting the effect of the end connections shown by dotted lines, Fig. 46, will be very approximately

$$L = 4l\left[\log\frac{d}{\rho} + \frac{\mu}{4} - \frac{d}{l}\right] \qquad [100]$$

In the usual case of $\mu = 1$ this will be, when d/l is small

$$L = 4l\left[\log \frac{d}{\rho} + \frac{1}{4}\right] \qquad [101]$$

If the end effect is large, as when the wires are relatively far apart, use the expression for the self-inductance of a rectangle below (107); or, better, add to the value of (100) the self-inductance of AB + CD, using equation (94) in which $l = 2$AB.

Experimental work at the Bureau of Standards, not yet published, has shown that formula (100), and therefore (94) and (98) are consistent with the formula (63) for the inductance of a circular ring.

[This is equivalent to the following formula in which the logarithms are common:

$$L = 0.7411 \log_{10} \frac{d}{\rho} + .0805 \text{ in millihenrys per mile of conductor,}$$

$$= 0.4605 \log_{10} \frac{d}{\rho} + .050 \text{ in millihenrys per kilometer of conductor,}$$

d and ρ being expressed in centimeters, inches, or any other unit.]

MUTUAL INDUCTANCE OF TWO LINEAR CONDUCTORS IN THE SAME STRAIGHT LINE

The mutual inductance of two adjacent linear conductors of lengths l and m in the same straight line is

$$M_{lm} = l \log \frac{l+m}{l} + m \log \frac{l+m}{m}, \text{ approximately.} \qquad [102]$$

This approximation is very close indeed if the radius of the conductor (which has been assumed zero) is very small.

THE SELF-INDUCTANCE OF A STRAIGHT RECTANGULAR BAR

The self-inductance of a straight bar of rectangular section is, to within the accuracy of the approximate formula (99), the same as the mutual inductance of two parallel straight filaments of the same length separated by a distance equal to the geometrical mean distance of the cross section of the bar. Thus,

$$L = 2l\left[\log \frac{2l}{R} - 1 + \frac{R}{l}\right] \qquad [103]$$

where R is the geometrical mean distance of the cross section of the rod or bar. If the section is a square, $R = 0.447\ \alpha$, α being the side of the square. If the section is a rectangle, the value of R is given by Maxwell's formula (124).

This is equivalent to the following:

$$L = 2l\left[\log\frac{2l}{\alpha+\beta} + \frac{1}{2} + \frac{0.2235(\alpha+\beta)}{l}\right] \qquad [104]$$

In the above formula L is the self-inductance of a straight bar or wire of length l and having a rectangular section of length α and breadth β.

TWO PARALLEL BARS. SELF AND MUTUAL INDUCTANCE

The mutual inductance of two parallel straight, square, or rectangular bars is equal to the mutual inductance of two parallel wires or filaments of the same length and at a distance apart equal to the geometrical mean distance of the two areas from one another. This is very nearly equal in the case of square sections to the distance between their centers for all distances, the g. m. d. being a very little

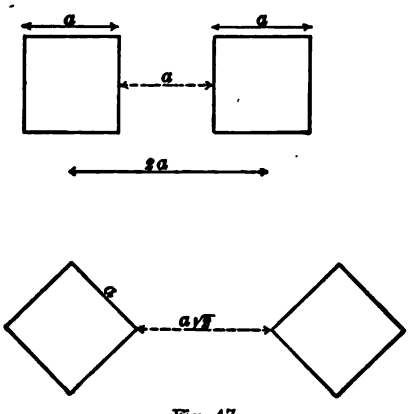

Fig. 47

greater for parallel squares, and a very little less for diagonal squares[104] (Fig. 47). We should, therefore, use equation (99) with d equal to g. m. d. of the sections from one another; that is, substantially, to the distances between the centers.

[104] Rosa, this Bulletin, 8, p. 1; 1907.

The self-inductance of a return circuit of two such parallel bars is equal to twice the self-inductance of one minus twice their mutual inductance. That is,

$$L=2[L_1-M]$$

in which L_1 is calculated by (104) and M by (99).

SELF-INDUCTANCE OF A SQUARE

The self-inductance of a square may be derived from the expressions for the self and mutual inductance of finite straight wires from the consideration that the self-inductance of the square is the sum of the self-inductances of the four sides minus the mutual inductances. That is,

$$L=4L_1-4M$$

the mutual inductance of two mutually perpendicular sides being zero. Substituting a for l and d in formulas (94) and (98) we have,

neglecting ρ^2/a^2, $L=8a(\log\frac{a}{\rho}+\frac{\rho}{a}-.524)$ [105]

where a is the length of one side of the square and ρ is the radius of the wire. If we put $l=4a=$ whole length of wire in the square,

$$L=2l\left(\log\frac{l}{\rho}+\frac{4\rho}{l}-1.910\right)$$

or, $L=2l\left(\log\frac{l}{\rho}-1.910\right)$ approximately. [106]

Formulas (105) and (106) were first given by Kirchhoff[105] in 1864.

SELF-INDUCTANCE OF A RECTANGLE

(a) *The conductor having a circular section*

The self-inductance of the rectangle of length a and breadth b is

$$L=2(L_a+L_b-M_a-M_b)$$

where L_a and L_b are the self-inductances of the two sides of length a and b taken alone, M_a and M_b are the mutual inductances of the two opposite pairs of length a and b, respectively.

From (94) and (98) we therefore have, neglecting ρ^2/a^2, and putting d for the diagonal of the rectangle $=\sqrt{a^2+b^2}$

[105] Gesammelte Abhandlungen, p. 176. Pogg. Annal., **121**, 1864.

$$L=4\left[(a+b)\log\frac{2ab}{\rho}-a\log(a+d)-b\log(b+d)\right.$$
$$\left.-\frac{7}{4}(a+b)+2(d+\rho)\right] \qquad [107]$$

(*b*) *The conductor having a rectangular section*

For a rectangle made up of a conductor of rectangular section $\alpha\times\beta$,

$$L=4\left[(a+b)\log\frac{2ab}{\alpha+\beta}-a\log(a+d)-b\log(b+d)\right.$$
$$\left.-\frac{a+b}{2}+2d+0.447(\alpha+\beta)\right] \qquad [108]$$

where as before d is the diagonal of the square. This is equivalent to Sumec's exact formula[106] (6*a*).

For $a=b$, a square,

$$L=8a\left[\log\frac{a}{\alpha+\beta}+0.2235\frac{\alpha+\beta}{a}+0.726\right] \qquad [109]$$

If $\alpha=\beta$, that is, the section of the conductor is a square,

$$L=8a\left[\log\frac{a}{\alpha}+0.447\frac{\alpha}{a}+0.033\right] \qquad [110]$$

MUTUAL INDUCTANCE OF TWO EQUAL PARALLEL RECTANGLES

For two equal parallel rectangles of sides a and b and distance apart d the mutual inductance, which is the sum of the several mutual inductances of parallel sides, is,

$$M=4\left[a\log\left(\frac{a+\sqrt{a^2+d^2}}{a+\sqrt{a^2+b^2+d^2}}\cdot\frac{\sqrt{b^2+d^2}}{d}\right)\right.$$
$$\left.+b\log\left(\frac{b+\sqrt{b^2+d^2}}{b+\sqrt{a^2+b^2+d^2}}\cdot\frac{\sqrt{a^2+d^2}}{d}\right)\right]$$
$$+8\left[\sqrt{a^2+b^2+d^2}-\sqrt{a^2+d^2}-\sqrt{b^2+d^2}+d\right] \qquad [111]$$

[106] Elektrotech. Zs., 27, p. 1175; 1906.

For a square, where $a=b$, we have

$$M=8\left[a\log\left(\frac{a+\sqrt{a^2+d^2}}{a+\sqrt{2a^2+d^2}}\cdot\frac{\sqrt{a^2+d^2}}{d}\right)\right]$$
$$+8\left[\sqrt{2a^2+d^2}-2\sqrt{a^2+d^2}+d\right] \qquad [112]$$

Formula (111) was first given by F. E. Neumann[107] in 1845.

The case of two rectangles symmetrically placed about a common vertical axis, the horizontal sides of the smaller rectangle being equidistant from those of the larger rectangle, has been discussed by Martens[108] and a formula derived which enables the mutual inductance to be found for any angle ξ between the planes of the rectangles. This formula is, however, very elaborate and calculations therewith laborious.

SELF AND MUTUAL INDUCTANCE OF THIN TAPES

The self-inductance of a straight, thin tape of length l and breadth b (and of negligible thickness), Fig. 48 (1), is equal to the mutual inductance of two parallel lines of distance apart R_1 equal to the geometrical mean distance of the section, which is $0.22313b$, or

$\log R_1=\log b-\frac{3}{2}$.

Thus we have approximately

$$L=2l\left[\log\frac{2l}{R_1}-1\right]$$
$$=2l\left[\log\frac{2l}{b}+\frac{1}{2}\right] \qquad [113]$$

If the thickness of the tape is not negligible, this formula becomes, when a is the thickness of the tape,

$$L=2l\left[\log\frac{2l}{b}-\frac{a}{b}+\frac{1}{2}\right] \qquad [114]$$

A closer approximation to L is given by (104), in which α is the thickness and β is the breadth of the tape. For two such tapes in the same plane, coming together at their edges with-

[107] Allgemeine Gesetze der Inducirten Ströme, Abh. Berlin Akad.
[108] Ann. der Phys. **29**, p. 963; 1909.

out making electrical contact, Fig. 48 (2), the mutual inductance is

$$M = 2l\left[\log\frac{2l}{R_2} - 1\right]$$
$$= 2l\left[\log\frac{2l}{b} - 0.8863\right] \qquad [115]$$

where R_2 is the geometrical mean distance of one tape from the other, which in this case is $0.89252b$. For a return circuit made up of these two tapes the self-inductance is

$$L = 2L_1 - 2M$$
$$= 4l\left(\log\frac{R_2}{R_1}\right) = 4l\log_e 4 \qquad [116]$$
$$= 5.545 \times \textit{length of one tape.}$$

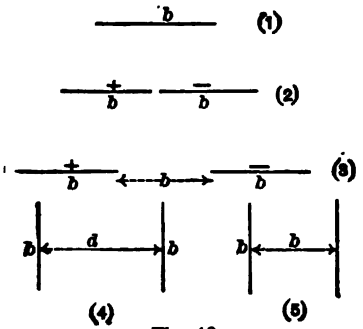

Fig. 48

Thus the self-inductance of such a circuit is independent of the width of the tapes. If the tapes are separated by the distance b, Fig. 48 (3), equal to the width of the tapes, $R_2 = 1.95653b$ and $L = 8.685l$.

If the two tapes are not in the same plane, but parallel, Fig. 48 (4),

$$L = 2L_1 - 2M = 4l\log\frac{R_2}{R_1} \qquad [117]$$

and when the distance apart is equal to the breadth of the tapes, Fig. 48 (5), we have

$$\log\frac{R_2}{R_1} = \frac{\pi}{2}$$

and

$$L = 4l\frac{\pi}{2} = 2\pi\, l\,. \qquad [118]$$

In this case, also, the self-inductance [2π cm per unit of length] of the pair of thin strips is independent of their width so long as the distance apart is equal to their width. Formula (117) with (132)

may be employed to calculate the self-inductance of a noninductive shunt made up of a sheet of thin metal doubled on itself.

CONCENTRIC CONDUCTORS

The self-inductance of a thin, straight tube of length l and radius a_2, when a_2/l is very small, is given by (97),

$$L_2 = 2l\left[\log \frac{2l}{a_2} - 1\right]$$

The mutual inductance of such a tube on a conductor within it is equal to its self-inductance, since all the lines of force due to the outer tube cut through the inner when they collapse on the cessation of current. The self-inductance of the inner conductor, supposed a solid cylinder, is

$$L_1 = 2l\left[\log \frac{2l}{a_1} - \frac{3}{4}\right]$$

If the current goes through the latter and returns through the outer tube, the self-inductance of the circuit is

$$L = L_1 + L_2 - 2M = L_1 - L_2$$

since M equals L_2

$$\therefore\ L = 2l\left[\log \frac{a_2}{a_1} + \frac{1}{4}\right] \qquad [119]$$

This result can also be obtained by integrating the expression for the force outside a_1 between the limits a_1 and a_2, and adding the term for the field within a_1, there being no magnetic field outside a_2.

If the outer tube has a thickness $a_3 - a_2$ and the current is distributed uniformly over its cross section the self-inductance will be a little greater, the geometrical mean distance from a_1 to the tube, which is more than a_2 and less than a_3, being given by the expression

$$\log a_g = \frac{a^2_3 \log a_3 - a^2_2 \log a_2}{a^2_3 - a^2_2} - \frac{1}{2}$$

Putting this value of log a in (119) in place of log a_2, we should have the self-inductance of the return circuit.

If the current is alternating and of very high frequency, the current would flow on the outer surface of a_1 and on the inner surface

of the tube, and L for the circuit would be

$$L = 2l \log \frac{a_2}{a_1} \qquad [120]$$

MULTIPLE CONDUCTORS

If a current be divided equally between two wires of length l, radius ρ and distance d apart, the self-inductance of the divided conductor is the sum of their separate self-inductances plus twice their mutual inductance.

Thus, when d/l is small,

$$L = 2l\left[\log \frac{2l}{(\rho d)^{\frac{1}{2}}} - \frac{7}{8}\right] = 2l\left[\log \frac{2l}{(r_g d)^{\frac{1}{2}}} - 1\right] \qquad [121]$$

where r_g, the g. m. d. of the section of the wire is 0.7788ρ for a round section.

If there are three straight conductors in parallel and distance d apart, the self-inductance is similarly

$$L = 2l\left[\log \frac{2l}{(r_g d^2)^{\frac{1}{3}}} - 1\right] \qquad [122]$$

The expression $(r_g d^2)^{\frac{1}{3}}$ is the g. m. d. of the multiple conductor.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE SELF AND MUTUAL INDUCTANCE OF LINEAR CONDUCTORS

EXAMPLE 70. FORMULAS (94), (95), (96), AND (97)

A straight copper wire 100 cm long and 0.2 cm diameter will have a self-inductance by formula (95) of

$$L = 200\left(\log_e \frac{200}{0.1} - \frac{3}{4}\right) = 1370.18 \text{ cm.}$$

If it were twice as long

$$L = 400\left(\log_e \frac{400}{0.1} - \frac{3}{4}\right) = 3017.62 \text{ cm.}$$

The more exact formula (94) gives practically the same result where ρ is so small compared with l.

If the wire were of iron with a permeability of 1000, we should have in the first case for $l = 100$

$$L = 200\,(\log_e 2000 - 1 + 250) = 51320 \text{ cm.}$$

For sufficiently rapid oscillations so that the current may be considered to be confined to the surface of the wire

$$L = 200\,(\log_e 2000 - 1) = 1320.18 \text{ cm.}$$

If the length of the conductor were 10 meters and the diameter 0.2 cm as before, the self-inductance by (95) would be

$$L = 2000\left(\log_e 20000 - \frac{3}{4}\right) = 18307.0 \text{ cm}$$

$$= 18.307 \text{ microhenrys.}$$

EXAMPLE 71. FORMULAS (98) AND (99)

Two parallel copper wires of length 100 cm and distance apart 200 cm will have a mutual inductance of

$$M = 2\left[100 \log_e \frac{100 + 100\sqrt{5}}{200} - 100\sqrt{5} + 200\right]$$

$$= 200\left[\log_e \frac{1 + \sqrt{5}}{2} - \sqrt{5} + 2\right]$$

$$= 200\,(\log_e 1.61803 - 0.2361)$$
$$= 49.02 \text{ cm.}$$

If the length of each conductor were 200 cm and the distance apart 100 cm, then

$$M = 400\left[\log_e \frac{2 + \sqrt{5}}{1} - \frac{\sqrt{5}}{2} + \frac{1}{2}\right] = 330.24 \text{ cm.}$$

The approximate formula (99) is only applicable when the length of the conductors is great compared with their distance apart. Suppose two conductors 10 meters long are 10 cm apart, then by (99)

$$M = 2000\left[\log_e \frac{2000}{10} - 1 + \frac{10}{1000}\right]$$
$$= 2000\,[5.2983 - 0.9900]$$
$$= 8616.6 \text{ cm} = 8.6166 \text{ microhenrys.}$$

The formula (98) gives a value less than two parts in one hundred thousand greater.

EXAMPLE 72. FORMULAS (100) AND (101)

Suppose a return circuit of two parallel wires, each 10 meters long and 0.2 cm diameter, distant apart 10 cm, center to center, Fig. 49. The self-inductance of the circuit, neglecting the ends, is by (100)

$$L = 4000\left[\log_e \frac{10}{0.1} + \frac{1}{4} - \frac{10}{1000}\right]$$
$$= 4000 \times 4.8452$$
$$= 19380.8 \text{ cm} = 19.3808 \text{ microhenrys.}$$

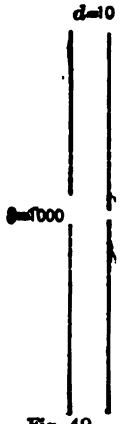

Fig. 49

We have already calculated (example 70) the self-inductance of one of these two wires by itself. Doubling the value we have 36.6140 microhenrys as the self-inductance of two wires in series. In example 71 we calculated the mutual inductance of these two wires. Doubling the value for M we have 17.2332 microhenrys. The resultant self-inductance of the circuit (neglecting the ends) is

$$L = 2L_1 - 2M = 36.6140 - 17.2332$$
$$= 19.3808 \text{ microhenrys.}$$

as found above by formula (100).

Taking account of the ends neglected above, we should find that $2L_1$ for the two ends by (95) is 181.9 cm and $2M$ by (98) is practically zero. Hence the self-inductance of the circuit is, including the ends,

$$L = 19.5627 \text{ microhenrys.}$$

EXAMPLE 73. FORMULA (102) FOR THE MUTUAL INDUCTANCE OF ADJACENT CONDUCTORS IN THE SAME STRAIGHT LINE

When the two conductors are of equal length, $l = m$, and (102) becomes

$$M = 2\,l \log_e 2 = 2\,l \times 0.69315 \text{ cm.}$$

If $l = 1000$ cm, $M = 1386.3$ cm.

If $m = 1000\ l$, (81) gives

$$M = l \log_e 1001 + 1000\ l \log 1.001$$
$$= l \log_e 1001 + l \text{ approximately.}$$

If $l = 1$ cm, we have

$$M = \log_e 1001 + 1000 \log_e 1.001$$
$$= 6.909 + 0.999 = 7.908 \text{ cm.}$$

The self-inductance of the short wire AB, supposed 1 cm long and of 1 mm radius, is

$$L = 2\left(\log_e \frac{2}{0.1} - .75\right) = 2\ (2.9957 - .75) = 4.4915 \text{ cm,}$$

which is a little more than one-half of the mutual inductance of AB and BC, BC being one thousand times the length of AB.

In closed circuits, all the magnetic lines due to a circuit are effective in producing self-inductance, and hence the self-inductance is always greater than the mutual inductance of that circuit with any other, assuming one turn in each. But with open circuits, as in this case, we may have a mutual inductance between two single conductors greater than the self-inductance of one of them.

EXAMPLE 74. FORMULA (104) FOR THE SELF-INDUCTANCE OF A RECTANGULAR BAR

In formula (104), substituting $l = 1000$, and $\alpha + \beta = 2$ for a square bar 1000 cm long and 1 square cm section, we have, neglecting the small last term,

$$L = 2000\left[\log_e \frac{2000}{2} + \frac{1}{2}\right]$$
$$= 2000\ (6.908 + 0.5) = 14816 \text{ cm}$$
$$= 14.816 \text{ microhenrys.}$$

This would also be the self-inductance for any section having $\alpha + \beta = 2$ cm.

EXAMPLE 75. FORMULAS (105) AND (106) FOR THE SELF-INDUCTANCE OF A SQUARE MADE UP OF A ROUND WIRE

If the side of the square is 1 meter, $a = 100$ cm, $\rho = 0.1$ cm, we have from (105)

$$L = 800\ (\log_e 1000 - 0.524)$$
$$= 5107 \text{ cm} = 5.107 \text{ microhenrys.}$$

If $\rho = .05$ cm,

$$L = 5662 \text{ cm} = 5.662 \text{ microhenrys.}$$

That is, the self-inductance of such a rectangle of round wire is about 11 per cent greater for a wire 1 mm in diameter than for one 2 mm in diameter.

If l/ρ is constant, L is proportional to l, that is, if the thickness of the wire is proportional to the length of the wire in the square, the self-inductance of the square is proportional to its linear dimensions.

EXAMPLE 76. FORMULA (107) FOR THE SELF-INDUCTANCE OF A RECTANGLE OF ROUND WIRE

Suppose a rectangle 2 meters long and 1 meter broad.

Substituting $a = 200$ cm, $b = 100$, $\rho = 0.1$, in (107) we have

$$L = 8017.1 \text{ cm} = 8.017 \text{ microhenrys.}$$

We can obtain the same result from the values of self and mutual inductances calculated in examples 70 and 71. That is, the resultant self-inductance of the rectangle is the sum of the self-inductances of the four sides, minus twice the mutual inductances of the two pairs of opposite sides. Thus

$$L = (L_1 + L_3) + (L_2 + L_4) - 2M_{13} - 2M_{24}$$

By example 70, $L_1 + L_3 = 6035.24$

$L_2 + L_4 = 2740.36 \quad 8775.60$

By example 71, $2M_{13} = 660.48$

$2M_{24} = 98.04 \quad 758.52$

$\therefore L = 8017.08$ cm

$= 8.0171$ microhenrys.

The agreement of this result with that obtained from formula (107) serves as a check on the latter formula, and also illustrates how the values of the self and mutual inductances of open circuits may be combined to give the self-inductance of a closed circuit.

EXAMPLE 77. FORMULAS (108), (109), AND (110) FOR THE SELF-INDUCTANCE OF A RECTANGLE OR SQUARE MADE UP OF A BAR OF RECTANGULAR SECTION

$$\text{Let } a=200 \qquad b=100 \qquad \alpha=\beta=1.0 \text{ cm.}$$

Substituting these values in (108) we obtain

$$\begin{aligned} L &= 4(2971.05-1209.76-577.95-150+447.21+0.99) \\ &= 5926.16 \text{ cm.} \end{aligned}$$

For a square 10 meters on a side, made of square bar 1 sq cm cross section we have $a=1000$, $\alpha=1$; substituting in (110)

$$\begin{aligned} L &= 8000(6.908+.033) \\ &= 8000\times 6.941 \text{ cm} = 55.53 \text{ microhenrys.} \end{aligned}$$

For a circular section, diameter 1 cm, $\rho=0.5$; substituting in (105)

$$\begin{aligned} L &= 8000\left(\log_e 2000+\frac{1}{2000}-0.524\right) \\ &= 8000\times 7.076 \text{ cm} = 56.61 \text{ microhenrys,} \end{aligned}$$

a little more than for a square section, as would be expected.

EXAMPLE 78. FORMULA (112) FOR THE MUTUAL INDUCTANCE OF PARALLEL SQUARES

Suppose two parallel squares each 1 meter on a side, 10 cm distant from one another.

$a=100$, $d=10$. Substituting in (112),

$$\begin{aligned} M &= 8\left[100\log_e\left(\frac{1+\sqrt{1.01}}{1+\sqrt{2.01}}\cdot\frac{\sqrt{101}}{1}\right)+\sqrt{20100}-2\sqrt{10100}+10\right] \\ &= 800\left[\log_e\left(\frac{10.1+\sqrt{101}}{1+\sqrt{2.01}}\right)+\sqrt{2.01}-2\sqrt{1.01}+0.1\right] \\ &= 1142.5 \text{ cm} = 1.1425 \text{ microhenrys.} \end{aligned}$$

EXAMPLE 79. FORMULAS (113), (114), AND (115) FOR THE SELF AND MUTUAL INDUCTANCE OF THIN STRAIGHT STRIPS OR TAPES

Let the tape of thin copper be 10 meters long and 1 cm wide.

Substituting $l = 1000$ and $b = 1$ in (113) we have

$$L = 2000\left(\log_e 2000 + \frac{1}{2}\right)$$
$$= 2000 \times 8.1009 = 16202 \text{ cm}$$
$$= 16.202 \text{ microhenrys,}$$

as the self-inductance when the conducting strip is very thin. If the tape is 2 mm thick we may allow for the effect of the thickness by using (114) and we find

$$L = 2000 \times 7.9009 \text{ cm} = 15.802 \text{ microhenrys,}$$

which differs slightly from the preceding value.

Two such tapes edge to edge in one plane will have a mutual inductance by (115) of

$$M = 2000\ (\log_e 2000 - 0.8863)$$
$$= 2000 \times 6.7146 \text{ cm}$$
$$= 13.429 \text{ microhenrys.}$$

EXAMPLE 80. FORMULA (117) FOR THE SELF-INDUCTANCE OF A RETURN CIRCUIT OF TWO PARALLEL SHEETS; NONINDUCTIVE SHUNTS.

Suppose the dimensions of a thin manganin sheet which has been doubled on itself be as follows:

$$l = 30 \text{ cm} \qquad b = 10 \text{ cm} \qquad d = 1 \text{ cm.}$$

By (132) $\log R_2 = 1.0787$

$$\log R_1 = \log_e 10 - \frac{3}{2} = 0.8026$$
$$L = 4l\ (\log R_2 - \log R_1)$$
$$= 120 \times 0.2761$$
$$= 33.13 \text{ cm}$$
$$= .0331 \text{ microhenrys.}$$

EXAMPLE 81. FORMULA (122), 3 CONDUCTORS IN MULTIPLE

Suppose three cylindrical conductors, each 10 meters long and 4 mm diameter, the distance apart of their centers being 1 cm. Substitute in (122) as follows:

$$l = 1000 \text{ cm} \qquad \rho = 2 \text{ mm} \qquad d = 1 \text{ cm.}$$

Then $(r_g a^2)^{\frac{1}{3}} = 0.538$ cm

and $$L = 2000\left(\log_e \frac{2000}{0.538} - 1\right)$$

$$= 2000 \times 7.221 \text{ cm} = 14.442 \text{ microhenrys.}$$

If the whole current flowed through a single one of the three conductors the self-inductance would be

$$L = 2000\left(\log_e \frac{2000}{0.2} - \frac{3}{4}\right) = 17.92 \text{ microhenrys,}$$

or about 25 per cent more than when divided among the three.

9. FORMULAS FOR GEOMETRICAL AND ARITHMETICAL MEAN DISTANCES

GEOMETRICAL MEAN DISTANCES

Maxwell showed how to calculate mutual and self-inductances in several important cases by means of what he called the geometrical mean distance, either of one conductor from another or of a conductor from itself. On account of the importance of this method we give below some of the most useful of these formulas. The geometrical mean distance of a point from a line is the *n^{th} root of the product of the n distances from the point P to the various points in the line, n being increased to infinity in determining the value of R.* Or, the logarithm of R is the *mean value of log d for all the infinite values of the distance d.* Similarly, the geometrical mean distance of a line from itself is the *n^{th} root of the product of the n distances between all the various pairs of points in the line, n being infinity.*[109]

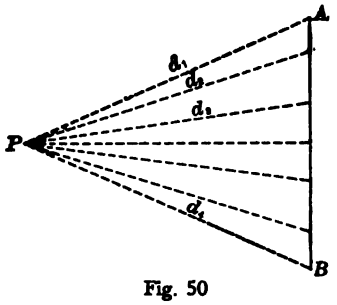

Fig. 50

Similar definitions apply to the g. m. d. of one area from another, or of an area from itself.

[109] Rosa, this Bulletin, 4, p. 325; 1907.

The geometrical mean distance R of a *line* of length a from itself is given by

$$\log R = \log a - \frac{3}{2}$$

$$R = ae^{-\frac{3}{2}} \qquad [123]$$

$$\text{or } R = 0.22313a$$

The g. m. d. of a *rectangular area* of sides a and b from itself is given by

$$\log R = \log\sqrt{a^2+b^2} - \frac{1}{6}\frac{a^2}{b^2}\log\sqrt{1+\frac{b^2}{a^2}} - \frac{1}{6}\frac{b^2}{a^2}\log\sqrt{1+\frac{a^2}{b^2}}$$

$$+\frac{2}{3}\frac{a}{b}\tan^{-1}\frac{b}{a} + \frac{2}{3}\frac{b}{a}\tan^{-1}\frac{a}{b} - \frac{25}{12} \qquad [124]$$

When the area is a *square*, and hence $a = b$,

$$\log R = \log a + \frac{1}{3}\log 2 + \frac{\pi}{3} - \frac{25}{12} \qquad [125]$$

$$\therefore\ R = 0.44705\ a$$

For a *circular area* of radius a,

$$\log R = \log a - \frac{1}{4}$$

$$R = ae^{-\frac{1}{4}} \qquad [126]$$

$$R = 0.7788\ a$$

For an *ellipse* of semi-axes a and b,

$$\log R = \log\frac{a+b}{2} - \frac{1}{4} \qquad [127]$$

An approximate expression for the g. m. d. of a *rectangular area* of length a and breadth b is

$$R = 0.2235(a+b) \qquad [128]$$

which is nearly true for all values of a and b; that is, the geometrical mean distance of the rectangular area from itself is approximately proportional to the perimeter of the rectangle. The following table gives the ratio $R/(a+b)$ for a series of rectangles of different proportions, from a square to a ratio of 20 to 1 between length and breadth, and finally when the breadth is infinitesimal in comparison with the length. By interpolating for any other case between the

values given in the table one can obtain a quite accurate value without the trouble of calculating it by formula (124).

Geometrical Mean Distances of Rectangles of Different Proportions

[a and b are the Length and Breadth of the Rectangles. R is the Geometrical Mean Distance of its Area]

Ratio	R	$\frac{R}{a+b}$
1 :1	0.44705a	0.22353
1.25:1	0.40235a	0.22353
1.5 :1	0.37258a	0.22355
2 :1	0.33540a	0.22360
4 :1	0.27961a	0.22369
10 :1	0.24596a	0.22360
20 :1	0.23463a	0.22346
1 :0	0.22315a	0.22315

The g. m. d. of an *annular area* of radii a_1 and a_2 from itself is given by

$$\log R = \log a_1 - \frac{a_2^4}{(a_1^2-a_2^2)^2}\log\frac{a_1}{a_2} + \frac{1}{4}\frac{3a_2^2-a_1^2}{a_1^2-a_2^2} \qquad [129]$$

The g. m. d. of a *line* of length a from a second line of the same length, distant in the same straight line na, center to center, Fig. 51, is given by the following formula:

Fig. 51

$$\log R_n = \frac{(n+1)^2}{2}\log(n+1)a - n^2\log na + \frac{(n-1)^2}{2}\log(n-1)a - \frac{3}{2} \qquad [130]$$

This formula is equivalent to the following, which is more convenient for calculation for all values of n greater than one.[110]

$$\log R_n = \log n - \left[\frac{1}{12n^2} + \frac{1}{60n^4} + \frac{1}{168n^6} + \frac{1}{360n^8} + \frac{1}{660n^{10}} + \cdots\cdots\right] \qquad [131]$$

This formula is very convergent, and only two or three terms are generally required.

[110] Rosa, this Bulletin, 2, p. 168: 1906.

The following values of the geometrical mean distances (calling a unity) were calculated from the above formulas, all after the second being obtained by (131):

$R_0 = 0.22313$	$R_5 = 4.98323$
$R_1 = 0.89252$	$R_6 = 5.98610$
$R_2 = 1.95653$	$R_7 = 6.98806$
$R_3 = 2.97171$	$R_8 = 7.98957$
$R_4 = 3.97890$	$R_9 = 8.99076$

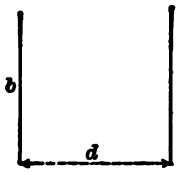

Fig. 52

If the lines are parallel and at distance d, Fig. 52, the g. m. d. is given by

$$\log R = \frac{d^2}{b^2}\log d + \frac{1}{2}\left(1 - \frac{d^2}{b^2}\right)\log (b^2 + d^2) + 2\frac{d}{b}\tan^{-1}\frac{b}{d} - \frac{3}{2} \qquad [132]$$

If $d = b$,

$$\log R = \log b + \frac{\pi}{2} - \frac{3}{2} \qquad [133]$$

The g. m. d. from a point O_2, Fig. 53, outside a circle to the circumference of the circle, or to the entire area of the circle *is the distance d from O_2 to the center of the circle.*

(1) The g. m. d. from the center O_1 to the circumference is of course the radius a. (2) The g. m. d. of any point (as O_3) within the circle from the circumference is also a. (3) The g. m. d. of any point on the circumference (as O_4) from all other points of the circumference is also a.

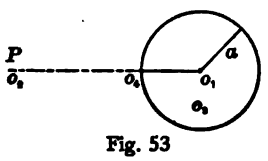

Fig. 53

(4) Therefore the g. m. d. of a circular line of radius a from itself is a; that is,

$$R = a \qquad [134]$$

for each of the four cases named above.

The g. m. d. of a point outside a circular ring, Fig. 54, from the ring *is the distance d to the center of the ring.* The g. m. d. of any point O_1, O_2, etc., within the ring is given by

$$\log R = \frac{a_1^2 \log a_1 - a_2^2 \log a_2}{a_1^2 - a_2^2} - \frac{1}{2} \qquad [135]$$

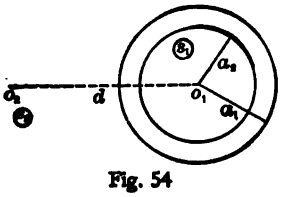

Fig. 54

The same expression gives the g. m. d. of any figure, as S_1, within the ring from the ring. The g. m. d. of an external figure,

as S_2, from the annular ring is equal to the g. m. d. of the center O_1 from the figure S_2.

The g. m. d. from one circular area to another is the distance between their centers; that is,

$$R = d \qquad [136]$$

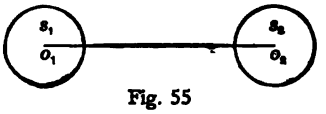

Fig. 55

for the area S_1 with respect to S_2 as it is for the point O_1 with respect to S_2.

The g. m. d. of a line of length a from a second parallel line of length a' located symmetrically (Fig. 56) is given by Gray[111], equation (114). The g. m. d. of a line from a parallel and symmetrically situated rectangle is given by Gray's equation (112). The g. m. d. of two unequal rectangles from one another is given by Gray's equation (113).[112]

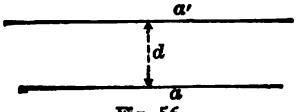

Fig. 56

The g. m. d. of two adjacent rectangles and of two obliquely situated rectangles are given by Rosa,[113] equations (8*a*) and (17). As these expressions are somewhat lengthy and not often required they are not repeated here. The values of the g. m. d. for two equal squares in various relative positions to one another have been accurately calculated[114] by these formulas, and the results used in the determination[115] of the correction term E of formula (93).

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[111] Absolute Measurements, Vol. II, Part I.

There are a number of misprints in equations 104, 109, 111, and 113 of Gray. The sign of the first term of equation 104 should be +. The signs before p^2 in the coefficients of the log in the first four terms of equation 109 should be all minus; thus $\frac{1}{4}(\beta^2-p^2)$, $-\frac{1}{4}(\alpha^2-p^2)$, $-\frac{1}{4}[(\alpha-\beta)^2-p^2]$, $+\frac{1}{4}[(a-\alpha)^2-p^2]$. Similarly in equation 111 the coefficients of the first two terms should be $\frac{1}{2}(\beta^2-p^2)$ and $-\frac{1}{2}(\alpha^2-p^2)$. In equation 113 the coefficient of β^4 in each of the first four terms should be $\frac{1}{6}$ instead of $\frac{1}{2}$ and the first term should have log $[(p+b+b')^2+\beta^2]$ instead of log $[(p+b+b')^2-\beta^2]$.

[112] Also by Rosa, equation (8) this Bulletin, 8, p. 6; 1907.

[113] This Bulletin, 8, pp. 7 and 12; 1907.

[114] This Bulletin, 8, pp. 9–19; 1907.

[115] This Bulletin, 8, p. 37; 1907.

ARITHMETICAL MEAN DISTANCES

In the determination of self and mutual inductances by the method of geometrical mean distances it has been shown[116] that more accurate formulas can be obtained by the use of certain arithmetical mean distances and arithmetical mean square distances taken in connection with geometrical mean distances.

The arithmetical mean distance of a point from a line is the arithmetical mean of the n distances of the point from the various points of the line, n being infinite. Similarly, the arithmetical mean distance of a line from itself is the *arithmetical mean of the distances of the n pairs of points in the line from one another, n being infinite.*

The a. m. d. of a line of length b from itself is[117]

$$S_2 = \frac{b}{3} \qquad [137]$$

that is, while the g. m. d. of a line from itself is 0.22313 times its length, the a. m. d. is one-third the length.

The arithmetical mean square distance of a line from itself is of course larger than the square of the a. m. d. Putting S_2^2 for the arithmetical mean square distance (a. m. s. d.).

$$S_2^2 = \frac{b^2}{6} \text{ or } \sqrt{S_2^2} = \frac{b}{\sqrt{6}} \qquad [138]$$

The arithmetical mean distance of a point in the circumference of a circle from the circle is the same as the a. m. d. of the circle from itself; that is, for a circle of radius a,

$$S_1 = S_2 = \frac{4}{\pi}a \qquad [139]$$

The arithmetical mean square distance is

$$S_2^2 = 2a^2 \text{ and } \sqrt{S_2^2} = a\sqrt{2} \qquad [140]$$

(The g. m. d. for this case is $R = a$, equation (134).)

[116] Rosa, this Bulletin, 4, pp. 326–32; 1907.
[117] Rosa, this Bulletin, 4, p. 326; 1907.

The arithmetical mean distance of an external point P from the circumference of a circle, Fig. 57, is

$$S_1=\sqrt{d^2+a^2} \qquad [141]$$

which is the distance PA.

The arithmetical mean distance from P to the entire area of the circle is

$$S_1=\sqrt{d^2+\frac{a^2}{2}} \qquad [142]$$

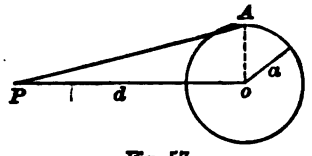

Fig. 57

(The g. m. d. for each of these cases is $R=d$, equation (136).)

For the proof of these and other expressions for the arithmetical mean distances and applications of their use see the article referred to above.

10. HIGH-FREQUENCY FORMULAS

Excepting in a very few specified cases, the formulas of the preceding sections apply only to conductors carrying direct current or alternating currents of frequencies so low that the error, due to the assumption that the current is uniformly distributed over the cross section of the wire, is negligible.

In the case of standards of mutual inductance the inductance may be regarded as sensibly independent of the frequency, unless the two coils are very close together, and even then the capacity between the coils will be a more potent source of error than the departure of the current from a uniform distribution over the cross section of the wire.

The self-inductance of a coil or conductor, on the other hand, depends appreciably on the field in the cross section of the conductor, and any deviation of the distribution of the current in the wire from uniformity gives rise to a *decrease* in the inductance. The amount of this change depends on the frequency of the current and the radius of the cross section of the conductor, as well as on the conductivity and permeability of the material of which it is composed.

This *decrease* of the inductance is accompanied by an *increase* in the resistance of the conductor. Whereas, however, the inductance

with increasing frequency approaches a limiting value, the resistance increases indefinitely as the frequency approaches an infinite value. The change of resistance is always relatively much larger than the change in inductance.

The eddy current effects just described are, for the most part, negligible at low frequencies, except in the case of heavy conductors and in coils wound with stout wire in several layers. In the latter case, however, the diminution of the inductance, due to the irregular distribution of the current, is marked, to a greater or less degree, by the effect of the capacity between the windings of the coil, which gives rise to an *increase* of the inductance with the frequency. For the same reason the resistance is increased more than it would be by the eddy currents alone.

Unfortunately, the rigorous or approximate solution of the problem at high frequencies for the various cases for which the inductance with steady currents may be calculated is in many instances very difficult, if not impossible. Some of the simpler cases, however, because of their great importance, have received much attention, with the result that the changes of inductance and resistance may be calculated with a good degree of precision.

STRAIGHT CYLINDRICAL WIRES

This is the most important case of all, since the solution is rigorous, and the results may be applied to the construction of practical, absolute standards for high-frequency work. The problem has been treated successively by Maxwell,[118] Heaviside,[119] Rayleigh,[120] and Kelvin.[121]

Putting l = length of conductor
ρ = radius of conductor
σ = specific resistance of its material
μ = permeability
f = frequency, $p = 2\pi f$
R' = resistance with current of frequency f
L' = inductance " " " " f

[118] Elect. and Mag., II, § 690.
[119] Elect. Papers, II, p. 64.
[120] Phil. Mag., 21, p. 381; 1886.
[121] Math. and Phys. Papers, III, p. 491; 1889.

R = resistance with direct current
L = inductance " " "
$x = 2\rho\sqrt{\frac{\pi p \mu}{\sigma}}$

Thus

$$\frac{R'}{R} = \frac{x}{2}\frac{W}{Y} \qquad [143]$$

$$L' = 2l\left[\log\frac{2l}{\rho} - 1 + \frac{\mu}{4}\left(\frac{4}{x}\frac{Z}{Y}\right)\right] \qquad [144]$$

where

$$W = \text{ber}\, x\ \text{bei}'\, x - \text{bei}\, x\ \text{ber}'\, x$$
$$Y = (\text{ber}'\, x)^2 + (\text{bei}'\, x)^2$$
$$Z = \text{ber}\, x\ \text{ber}'\, x + \text{bei}\, x\ \text{bei}'\, x \qquad [144a]$$

Since from (96)

$$L = 2l\left[\log\frac{2l}{\rho} - 1 + \frac{\mu}{4}\right]$$

we find

$$\Delta L = L' - L = -2l\frac{\mu}{4}\left[1 - \frac{4}{x}\frac{Z}{Y}\right] \qquad [145]$$

$$\frac{\Delta L}{L} = -\frac{\frac{\mu}{4}\left(1 - \frac{4}{x}\frac{Z}{Y}\right)}{\log\frac{2l}{\rho} - 1 + \frac{\mu}{4}} \qquad [146]$$

For nonmagnetic material the equation (146) takes the form

$$\frac{\Delta L}{L} = -\frac{\left(1 - \frac{4}{x}\frac{Z}{Y}\right)}{4\log\frac{2l}{\rho} - 3} \qquad [147]$$

In these expressions, ber x and bei x are functions introduced by Lord Kelvin, being respectively the real and imaginary parts of the ordinary Bessel function of order zero, J_0, having for its argument $xi\sqrt{i}$, where x is a real quantity, and $i = \sqrt{-1}$. These functions are given by the series

$$\left.\begin{aligned} \operatorname{ber} x &= 1 - \frac{x^4}{2^2 4^2} + \frac{x^8}{2^2 4^2 6^2 8^2} - \cdots\cdots \\ \operatorname{bei} x &= \frac{x^2}{2^2} - \frac{x^6}{2^2 4^2 6^2} + \frac{x^{10}}{2^2 4^2 6^2 8^2 10^2} - \cdots\cdots \end{aligned}\right\} \qquad [148]$$

and ber$'x$ and bei$'x$ are their differential coefficients with respect to x.

These series are very convergent, but the calculation, naturally, becomes laborious for large values of x. To lighten the labor of calculation Russell[122] and Savidge[123] have developed asymptotic expressions for ber x, bei x, ber $'x$, bei $'x$ and the auxiliary quantities W, Y, and Z, which give their numerical values with an accuracy of about one part in ten thousand for values of x greater than about 6, but whose accuracy increases rapidly as x becomes larger.

Savidge[123] has, in addition, calculated extensive tables of the above functions and the allied ker and kei functions to four places of decimals, and for values of the argument ranging between 1 and 30 in steps of one unit. These tables will be found very useful in the solution of a variety of problems. For calculation with the formulas (143) to (147), however, it seemed desirable to construct tables in which the argument advances by smaller steps than in the tables of Savidge. For this purpose ber x, bei x, ber $'x$ and bei $'x$ were calculated directly from their series, for arguments from 0.1 to 5.0, in steps of 0.1, and from 5 to 7 in steps of 0.2. From these were obtained directly by (144*a*) the values of W, Y, Z. For the larger values of x, the quantities W, Y, Z were calculated by asymptotic formulas, and checked at a few points by the direct series. Thus the interval from 5 to 10 was covered in steps of 0.2, the interval 10 to 15 in steps of 0.5, and from 15 to 50 in steps of one unit, the aim being to keep the differences of the same order of magnitude. It will, probably, seldom be necessary to make calculations for values of x greater than about 50. If such calculations are occasionally required, they may be made with little trouble by the asymptotic formulas given below.

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[122] Phil. Mag., **17**, p. 524; 1909.
[123] Phil. Mag., **19**, p. 49; 1910.

Since making the above calculations, we have determined the expressions for the general terms in Russell's equations (8), (9), and (10), (loc. cit., p. 529), thus materially increasing their range of applicability.

These equations, thus extended, are

$$\left.\begin{aligned}
W&=\frac{x}{2}\left\{1+\frac{6}{(\underline{|3})^2}\left(\frac{x}{2}\right)^4+\frac{30}{(\underline{|5})^2}\left(\frac{x}{2}\right)^8+\frac{140}{(\underline{|7})^2}\left(\frac{x}{2}\right)^{12}+\cdots\right.\\
&\qquad\left.+\frac{1\cdot2\cdot3\cdots\quad(2n+1)}{1^2\,2^2\,3^2\cdots\;n^2(\underline{|2n+1})^2}\left(\frac{x}{2}\right)^{4n}+\cdots\right\}\\
Y&=\frac{x^2}{4}\left\{1+\frac{3}{(\underline{|3})^2}\left(\frac{x}{2}\right)^4+\frac{10}{(\underline{|5})^2}\left(\frac{x}{2}\right)^8+\frac{35}{(\underline{|7})^2}\left(\frac{x}{2}\right)^{12}+\cdots\right.\\
&\qquad\left.+\frac{1}{(n+1)}\,\frac{1\cdot2\cdot3\cdots\cdots(2n+1)}{1^2\,2^2\,3^2\cdots n^2(\underline{|2n+1})^2}\left(\frac{x}{2}\right)^{4n}+\cdots\right\}\\
Z&=\frac{x^2}{16}\left\{1+\frac{3}{2(\underline{|3})^2}\left(\frac{x}{2}\right)^4+\frac{10}{3(\underline{|5})^2}\left(\frac{x}{2}\right)^8+\frac{35}{4(\underline{|7})^2}\left(\frac{x}{2}\right)^{12}+\cdots\right.\\
&\qquad\left.+\frac{1}{(n+1)^2}\,\frac{1\cdot2\cdot3\cdots\cdots(2n+1)}{1^2\,2^2\,3^2\cdots n^2(\underline{|2n+1})^2}\left(\frac{x}{2}\right)^{4n}+\cdots\right\}
\end{aligned}\right\}\quad[149]$$

It would have been simpler to have calculated the values of W, Y, and Z by these formulas than by the more indirect process actually used. The formulas (149) have, however, shown themselves of great service in checking the results. For completeness the asymptotic formulas of Savidge used have also been appended. They give results to one in one hundred thousand for $x \geqq 10$ and may be used with an error of not greater than one in ten thousand down to $x=6$.

$$\left.\begin{aligned}
W&=\frac{e^{x\sqrt{2}}}{2\pi x}\left[\frac{1}{\sqrt{2}}+\frac{1}{8x}+\frac{9}{(8x)^2\sqrt{2}}+\frac{39}{(8x)^3}+\frac{75}{2(8x)^4\sqrt{2}}-\cdots\right]\\
Y&=\frac{e^{x\sqrt{2}}}{2\pi x}\left[1-\frac{6}{8x\sqrt{2}}+\frac{9}{(8x)^2}+\frac{150}{(8x)^3\sqrt{2}}+\frac{2475}{2(8x)^4}+\cdots\right]\\
Z&=\frac{e^{x\sqrt{2}}}{2\pi x}\left[\frac{1}{\sqrt{2}}-\frac{3}{8x}-\frac{15}{(8x)^2\sqrt{2}}-\frac{45}{(8x)^3}+\frac{315}{2(8x)^4\sqrt{2}}+\cdots\right]
\end{aligned}\right\}[150]$$

The results are given in Table XXII which gives the values, to one in one hundred thousand, of not only the quantities $\frac{x}{2}\frac{W}{Y}$ and

$\frac{4}{x}\frac{Z}{Y}$ required in the preceding formulas, but of $\frac{W}{Y}$ and $\frac{Z}{Y}$ also. These will be found useful in allied problems, and it may seem preferable in some cases to interpolate the values of these latter quantities to obtain the former. For example, with $x > 2.5$ the first differences, and in some places the second differences also, are smaller with $\frac{W}{Y}$ than with $\frac{x}{2}\frac{W}{Y}$. The accuracy of the Table XXII may be regarded as greater than will usually be required, and should suffice for the most precise work.

In addition to the general formulas of Kelvin (143) and (144), Rayleigh[124] has given expansions holding for small values of the argument x. These equations, which were extended to another term by Heaviside, are, expressed in the present nomenclature,

$$\left.\begin{aligned} \frac{x}{2}\frac{W}{Y} &= 1 + \frac{1}{12}\left(\frac{x}{2}\right)^4 - \frac{1}{180}\left(\frac{x}{2}\right)^8 + \frac{11}{12\cdot 28\cdot 30}\left(\frac{x}{2}\right)^{12} - \cdots \\ \frac{4}{x}\frac{Z}{Y} &= 1 - \frac{1}{24}\left(\frac{x}{2}\right)^4 + \frac{13}{4320}\left(\frac{x}{2}\right)^8 - \frac{647}{12^2\cdot 360\cdot 56}\left(\frac{x}{2}\right)^{12} + \cdots \end{aligned}\right\} \quad [151]$$

Their applicability is limited to the range of values of x less than about 2, and it will be more convenient to use Table XXII.

For very high frequencies Rayleigh gave also the limiting formulas

$$R' = \sqrt{\frac{1}{2}pl\mu R}$$

$$L' = 2l\left[\log\frac{2l}{\rho} - 1 + \frac{1}{2}\sqrt{\frac{\mu R}{2pl}}\right]$$

In some instances these formulas have been used, as though they were exact, over a considerable range of frequencies, without any statement being made as to the magnitude of the error involved. Expressing these formulas in the present nomenclature, we obtain the following formulas for infinite frequencies:

[124] Phil. Mag., **21**, p. 387; 1886.

$$\left(\frac{R'}{R}\right)_{x=\infty} = \frac{x}{2\sqrt{2}} \qquad [152]$$

$$(L')_{x=\infty} = 2l\left[\log\frac{2l}{\rho} - 1 + \frac{\mu}{4}\left(\frac{4}{x\sqrt{2}}\right)\right]_{x=\infty}$$

$$= 2l\left[\log\frac{2l}{\rho} - 1\right] \qquad [153]$$

These are seen to be in agreement with equations (143) and (144), if we remember that the limiting values of $\frac{W}{Y}$ and $\frac{Z}{Y}$ as the argument x is indefinitely increased are both $\frac{1}{\sqrt{2}}$. (See formulas (150).) From (150) we find that only for values of x greater than about 900 is the error from using (152) as small as one-tenth per cent. For $x = 70$ the error is about 1 per cent, and in many practical cases it is still larger.

The limiting value of the change of inductance is found from (147) to be

$$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -\frac{1}{4\log\frac{2l}{\rho} - 3} \qquad [154]$$

$$(\Delta L)_{x=\infty} = -\frac{l}{2} \qquad [155]$$

The error from using (153) is only about one part in ten thousand for $x = 60$. The error, however, arising from the neglect of the term $\frac{4}{x}\frac{Z}{Y}$ in (147) is more than 5 per cent.

From (154) we obtain the curious result that the limiting value of the fractional change of inductance, as the frequency is indefinitely increased, depends only on the ratio of the length of the wire to the cross section. Table XXIII gives an idea of the way the limiting value falls off as this ratio is increased.

The preceding formulas show that the change of resistance and inductance are functions of the quantity

$$x = 2\rho\sqrt{\frac{\pi\mu p}{\sigma}} = 2\pi\rho\sqrt{2\mu K f} \qquad [156]$$

where K is the conductivity.

Taking the specific resistance of annealed copper at 20° as 1.721 microhms or 1721 in absolute electromagnetic units,

$$K_0 = 5.811 \times 10^{-4}$$

and (156) takes the simple form

$$x = 0.02142\, \rho \sqrt{f}$$

To aid in making approximate calculations, and for purposes of orientation, the auxiliary Table XXIV has been calculated, giving the value $x = x_0$ for copper wire of the above conductivity and of a cross section of 1 mm radius at various frequencies. For the higher frequencies, the corresponding wave length λ in meters has been included as likely to be of service in calculations for wireless-telegraph circuits. The range of this table may be considerably extended by remembering that x varies with $\sqrt{f}$ or $\sqrt{\frac{1}{\lambda}}$. Thus the value of x_0 for 7 500 cycles is found directly from the tabulated value for 750 000 cycles by shifting the decimal point. Similarly, the value for $\lambda = 150$ meters is obtained from the tabulated value for 15 000 meters. It is for this reason that the larger values of λ have been tabulated.

To calculate x for a copper wire of radius r mm, we have $x = x_0 r$, and if the conductivity have any value K, the further factor $\sqrt{\frac{K}{K_0}}$ must be applied. Finally, if the wire is, in addition, of magnetic material of permeability μ, an additional factor $\sqrt{\mu}$ is necessary to obtain the required value of x.

CONCENTRIC MAIN

The simple case of a cylindrical, straight wire may be regarded as a special case of the more general problem of a concentric main; that is, of a solid or tubular inner conductor surrounded by a coaxial tubular outer conductor. This case has been very completely treated by Russell,[125] but as the formulas are not simple they are not given here.

[125] Phil. Mag., 17, p. 524; 1909.

TWO PARALLEL WIRES

Unless the two wires are so near together, relatively to their radius of cross section, that their mutual inductance is appreciably affected by changes in the distribution of the current within the wires, each wire may be treated by the formulas given for a straight, cylindrical wire.

Supposing, therefore, that the wires are alike in every respect

$$L' = 4l\left[\log\frac{d}{\rho} + \frac{\mu}{4}\left(\frac{4}{x}\frac{Z}{Y}\right)\right] \qquad [157]$$

and from (101) we find for wires of nonmagnetic material

$$\frac{\Delta L}{L} = -\frac{\left(1 - \frac{4}{x}\frac{Z}{Y}\right)}{4\log\frac{d}{\rho} + 1} \qquad [158]$$

$$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -\frac{1}{4\log\frac{d}{\rho} + 1} \qquad [159]$$

$$(\Delta L)_{x=\infty} = l \qquad [159a]$$

and

$$\frac{R'}{R} = \frac{x}{2}\frac{W}{Y} \qquad [160]$$

the values of $\frac{Z}{Y}$ and $\frac{W}{Y}$ being taken from Table XXII.

Nicholson[126] has recently given a solution of the problem, when the two wires are so close together that their mutual inductance suffers a sensible change with the frequency. To obtain an idea of the magnitude of this effect, in a practical case, the results by Nicholson's formulas were compared with those of (158) and (160). With $d = 1$ cm and $\rho = 0.1$ cm, and with a frequency of 10^6, (158) gives $\frac{\Delta L}{L} = -8.5$ per cent, the effect of Nicholson's correction being to give a value of $\frac{\Delta L}{L}$ numerically only nine parts in ten thousand smaller.

[126] Phil. Mag., 18, p. 417; 1909.

Similarly for the resistance, (160) gives $\frac{R'}{R} = 7.56$, while Nicholson's formula reduces this to 7.55. Since this example relates to a rather unfavorable case, for a standard whose inductance is to be calculated from the dimensions, these corrections for mutual effect may, in general, be regarded as negligible, and the formulas (158), (159), and (160) may be regarded as sufficiently accurate with the precision usually attainable in the measurement of the dimensions.

It is to be noticed that the maximum possible relative change of inductance, with the frequency, is greater with two parallel wires than with either alone, because this change with the parallel wires depends on the sum of their self-inductances which is greater than the resulting self-inductance of the combination (see p. 151). Table XXIII gives an idea of the values attained by $\left(\frac{\Delta L}{L}\right)_{x=\infty}$ in the case of the two parallel wires. This maximum change of inductance depends only on the ratio of their distance apart to the radius of cross section of the wire.

Evidently, other cases of linear conductors of circular cross section, may likewise be made to depend on the solution for straight wires.

CIRCULAR RING OF CIRCULAR SECTION

The inductance of a circular ring, in which the current is confined wholly to the circumference of the cross section was given in formula (65). Combining this with (63) we find that on the assumption that (65) represents the actual distribution of the current at infinite frequency,

$$(\Delta L)_{x=\infty} = -\pi a\left[1 - \frac{1}{2}\frac{\rho^2}{a^2}\left(\log\frac{8a}{\rho} - \frac{1}{3}\right)\right] \qquad [161]$$

The absolute value of the change of inductance at infinite frequency is, in the case of a straight wire, (see 145)

$$(\Delta L)_{x=\infty} = -\frac{l}{2}$$

which shows that, if the wire of the ring were stretched out straight, the value given in (161) would become

$$(\Delta L)_{x=\infty} = -\frac{1}{2}2\pi a = -\pi a \qquad [162]$$

Equation (161) gives, therefore, the effect of the curvatures of the ring, which for ordinary cases will be seen to be small. The resistance and inductance of the ring must, therefore, very approximately follow the same law of variation with the frequency as the straight wire.

The assumption of formula (65) that at high frequencies the magnetic field is symmetrical around the axis of the cross section of the ring is not strictly true. Actually, it will be a little stronger toward the axis of the ring, so that the amplitude of the current is slightly larger in that part of the cross section which is nearest the axis of the ring. This effect, however, will be extremely small and may be neglected.

We have, therefore, with great approximation

$$\Delta L = -\pi a\left(1 - \frac{4}{x}\frac{Z}{Y}\right)\left[1 - \frac{1}{2}\frac{\rho^2}{a^2}\left(\log\frac{8a}{\rho} - \frac{1}{3}\right)\right] \qquad [163]$$

or, if terms in $\frac{\rho^2}{a^2}$ may be neglected,

$$\frac{\Delta L}{L} = -\frac{\left(1 - \frac{4}{x}\frac{Z}{Y}\right)}{4\log\frac{8a}{\rho} - 7} \qquad [164]$$

$$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -\frac{1}{4\log\frac{8a}{\rho} - 7} \qquad [165]$$

The values of $\left(\frac{\Delta L}{L}\right)_{x=\infty}$ for various values of the determinative ratio $\frac{8a}{\rho}$ are tabulated in Table XXIII.

Neglecting the curvature, the change in resistance will be given by the same expression as for the straight wire, that is

$$\frac{R'}{R} = \frac{x}{2}\frac{W}{Y} \qquad [166]$$

The quantities $\frac{x}{2}\frac{W}{Y}$ and $\frac{4}{x}\frac{Z}{Y}$ are to be taken from Table XXII as before, the argument x being given by (156).

EXAMPLES ILLUSTRATING THE FORMULAS FOR HIGH FREQUENCY

EXAMPLE 82. STRAIGHT WIRE, VERY HIGH FREQUENCY

Let $f = 500000$ cycles per second, and $\therefore \lambda = 600$ meters
$l = 200$ cm
$\rho = 0.125$ cm.

If the wire is of copper, Table XXIV gives $x_0 = 15.146$ for a wire of $\rho = 0.1$ cm. We find, therefore, $x = 15.146 \times 1.25 = 18.932$. Entering Table XXII with this value of x, we find by interpolation, using second differences,

$$\frac{x}{2}\frac{W}{Y} = \frac{R'}{R} = 6.95035 \qquad \frac{4}{x}\frac{Z}{Y} = 0.14923$$

a slightly more accurate value of the latter may be found by making the interpolation for $\frac{Z}{Y}$.

$$\frac{2l}{\rho} = 3200 \text{ and } \therefore 4 \log \frac{2l}{\rho} - 3 = 29.284$$

By (154) $$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -0.034148$$

The value found from Table XXIII by interpolation is 0.03416+.

Formula (147) gives therefore $\frac{\Delta L}{L} = -0.034148(1 - 0.14923)$

$$= -0.029052$$

By (145) $$\Delta L = -85.08 \text{ cm}$$

Recapitulating, the resistance at 500000 cycles per second is 6.95 times as great as with direct current, while the inductance is 85.08 cm or 2.9052 per cent less than the direct current value. This change of the inductance is 85.08 per cent of the possible change of 100 cm (0.034148 of the total inductance).

If the wire had been of manganin, for which the conductivity was one thirtieth of that of copper, the value of x becomes

$$x = 18.932 \times \sqrt{\frac{1}{30}} = 3.4566$$

and we find

$$\frac{R'}{R}=1.47620 \qquad \frac{4}{x}\frac{Z}{Y}=.77255$$

$$1-\frac{4}{x}\frac{Z}{Y}=.22745$$

The resistance is 1.47620 times the value at zero frequency, while the decrease of inductance is only 22.745 per cent of the total possible (0.034148), or 0.007767.

On the other hand, if the wire had been of iron (conductivity one-seventh of that of copper) and the permeability is assumed as low as 100

$$x=\sqrt{100}\sqrt{\frac{1}{7}}\,18.932=71.556$$

$$\frac{R'}{R}=25.551 \qquad \frac{4}{x}\frac{Z}{Y}=0.039526 \qquad 1-\frac{4}{x}\frac{Z}{Y}=.96047$$

By (146) $$\left(\frac{\Delta L}{L}\right)_{x=\infty}=-\frac{100}{128.284}=-0.77950$$

By (145) $$(\Delta L)_{x=\infty}=-10000 \text{ cm}$$

and the actual changes are

$$\Delta L=-9605 \text{ cm}$$

$$\left(\frac{\Delta L}{L}\right)=-0.77950\times.96047=-0.7487$$

The influence of this relatively low permeability is enormous. The resistance is more than twenty-five times its direct current value, while the inductance is less than the direct current value by nearly 75 per cent of the latter, the maximum possible change with this permeability being about 78 per cent.

EXAMPLE 83. STRAIGHT WIRE—LOW FREQUENCY

If we consider the same wires as in the previous example, except that the frequency is assumed as only 1,000 per second.

Then for copper, $x=0.6774\times1.25=0.84675$

$$\frac{R'}{R}=1.00266 \qquad \frac{4}{x}\frac{Z}{Y}=0.99867 \qquad 1-\frac{4}{x}\frac{Z}{Y}=0.00133$$

$$\Delta L = -0.133 \text{ cm}$$

$$\frac{\Delta L}{L} = -0.034148\,(.00133) = -0.000045$$

The resistance increase is only 0.266 per cent and the decrease of inductance is only about forty-five millionths of the total.

For manganin, $x = 0.84675\sqrt{\frac{1}{30}} = 0.1582$

By (151) $\quad \frac{R'}{R} = 1.0000030 \qquad \frac{4}{x}\frac{Z}{Y} = 1 - 0.0000015$

$$1 - \frac{4}{x}\frac{Z}{Y} = \quad 0.0000015$$

The increase in resistance is about three millionths and the decrease in inductance about five hundred-millionths of the direct current values.

For iron, with $\mu = 100$ as before

$$x = 0.84675\sqrt{100}\sqrt{\frac{1}{7}} = 3.2004$$

$$\frac{R'}{R} = 1.38516 \qquad \frac{4}{x}\frac{Z}{Y} = 0.81391 \qquad 1 - \frac{4}{x}\frac{Z}{Y} = 0.18609$$

$$\frac{\Delta L}{L} = -\frac{100}{128.284}(0.18609) = -0.14506$$

That is, the resistance increase is 38.5 per cent, the inductance decrease 14.5 per cent of the direct current values.

EXAMPLE 84. PARALLEL WIRES

Let us take wires of the same diameter and length as in examples 82 and 83 and consider the same frequencies. The values of $\frac{x}{2}\frac{W}{Y}$ and $\frac{4}{x}\frac{Z}{Y}$ will be the same as those in the cases corresponding in the previous examples. Further, assume that the distance between the centers of the wires is $d = 1.5$ cm.

Then $\qquad \frac{d}{\rho} = 12 \qquad 4 \log \frac{d}{\rho} + 1 = 10.9396$

∴ for nonmagnetic material

$$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -\frac{1}{10.9396} = -0.091412$$

as may also be found directly with sufficient precision from Table XXIII.

For iron wires, $\mu = 100$

$$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -\frac{\mu}{4 \log \frac{d}{\rho} + \mu} = -\frac{100}{109.94} = -0.9308.$$

The results for the cases treated in the previous examples are, therefore, for the parallel wires, as follows:

Material	Frequency	$\frac{R'}{R}$	$\frac{\Delta L}{L}$
Copper	500000	6.9504	−0.077772
"	1000	1.00266	−0.000122
Manganin	500000	1.4762	−0.020792
"	1000	1.0000030	−0.00000014
Iron ($\mu = 100$)	500000	25.551	−0.8940
"	1000	1.3852	−0.1732

The following table shows the effect of reducing the radius of cross section to $\rho = 0.01$ cm

Material	Frequency	X	$\left(\frac{\Delta L}{L}\right)_{x=\infty}$	$\frac{R'}{R}$	$\frac{\Delta L}{L}$
Copper	500000	1.5146	0.047522	1.02682	−0.000636
"	1000	0.06774	"	1.0000011	-2.6×10^{-9}
Manganin	500000	0.27652	0.047522	1.000030	-7.2×10^{-7}
"	1000	0.01237	"	$1 + 1.2 \times 10^{-10}$	-3×10^{-12}
Iron (μ=100)	500000	5.7245	0.83303	2.2974	0.4273
"	1000	0.25603	"	1.000022	-9.3×10^{-8}

EXAMPLE 85. CIRCULAR RING

Suppose the ring is of copper and that

$$\rho = 0.1 \text{ cm}, \quad a = 20 \text{ cm}, \quad \lambda = 700 \text{ m}$$

Then from Table XXIV, $x = 14.023$ and from Table XXII,

$$\frac{R'}{R} = 5.2173, \; 1 - \frac{4}{x}\frac{Z}{Y} = 0.79873$$

$$\log \frac{8a}{\rho} = \log 1600 = 7.37776$$

$$\frac{\rho}{a} = 0.005$$

By (162), $(\Delta L)_{x=\infty} = -20\pi = -62.83$ cm.

The correction term in (161) $= 1 - 0.0000880$

By (165) or Table XXIII,

$$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -0.04442$$

and by (164),

$$\frac{\Delta L}{L} = -0.04442 \times 0.79873$$
$$= -0.03548$$

WASHINGTON, January 1, 1911.

NOTE.

After the present paper had gone to press, a third formula for the mutual inductance of coaxial circles was published by Nagaoka (Tokyo Math. Phys., Soc. 6, p. 10; 1911). This formula was given by Nagaoka in the following form:

$$M = 4\pi\sqrt{Aa}\left\{4\pi q^{\frac{1}{2}}\left(\frac{1 - 4q^3 + 9q^8 - \cdots\cdots}{1 - 3q^2 + 5q^6 - \cdots\cdots}\right)\right\}$$

The general term of the numerator being $(-1)^{n-1}n^2q^{n^2-1}$ and that of the denominator $(-1)^m(2m+1)q^{\frac{(2m+1)^2}{4}-\frac{1}{4}}$.

The quantity q is calculated from the modulus k_1', which is complementary to the modulus k_1 of formula (2). Using the same nomenclature as in section 1 of this collection we have

$$q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \cdots$$

$$l = \frac{1 - \sqrt{k_1'}}{1 + \sqrt{k_1'}} = \frac{k_1^2}{(1 + k_1')(1 + \sqrt{k_1'})^2}$$

$$k_1 = \frac{r_1 - r_2}{r_1 + r_2} = \frac{4Aa}{(r_1 + r_2)^2} \qquad k_1' = \frac{2\sqrt{r_1 r_2}}{r_1 + r_2}$$

$$r_1 = \sqrt{(A + a)^2 + d^2} \qquad r_2 = \sqrt{(A - a)^2 + d^2}$$

The general term of the above formula has been given for the sake of completeness. In general, however, the convergence is so rapid that all but the first terms are negligible.

As an example of the use of this formula, the calculation for the circles of examples 4 and 11 above is appended:

$$A = a = 25 \qquad d = 4$$

$$R_1 = \sqrt{2516} = 50.159744 \qquad R_2 = 4$$

$$k_1' = \frac{4\sqrt{2516}}{4+\sqrt{2516}} \qquad \sqrt{k_1'} = 0.72323683$$

$$\frac{l}{2} = 0.080303278 \qquad q = 0.080309959$$

$$1 - 4q^2 + 9q^6 = 0.99792812 \qquad 1 - 3q^2 + 5q^6 = 0.98065227$$

$$\log q^4 = \bar{1}.1785770 \qquad \therefore M = 606.0676\ cm$$

which agrees very closely with the value 606.0674 found in examples 4 and 11.

On expansion the above formula becomes

$$M = 4\pi\sqrt{Aa}\{4\pi q^2(1 + 3q^2 - 4q^3 + 9q^4 - 12q^5 + \cdots)\}$$

which suggests that the quantity q in this expression is equal to the square of the corresponding quantity in formula (8) above. The truth of this proposition may be established by expressing Landen's transformation in terms of q functions.

As regards numerical calculation, therefore, this last formula of Nagaoka is entirely equivalent to his earlier formula (8).

APPENDIX

TABLES OF CONSTANTS AND FUNCTIONS USEFUL IN THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE

TABLE I

Maxwell's Table of Values of $\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}=\left[\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right]$

(For use with Formula (1))

	$\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}$	Δ_1		$\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
60° 0′	$\bar{1}$.499 4780	2 7868	65° 0′	$\bar{1}$.637 6633	2 7508
6′	$\bar{1}$.502 2648	2 7854	6′	$\bar{1}$.640 4141	2 7508
12′	$\bar{1}$.505 0502	2 7840	12′	$\bar{1}$.643 1649	2 7507
18′	$\bar{1}$.507 8342	2 7828	18′	$\bar{1}$.645 9156	2 7507
24′	$\bar{1}$.510 6170	2 7816	24′	$\bar{1}$.648 6663	2 7507
30′	$\bar{1}$.513 3986	2 7803	30′	$\bar{1}$.651 4170	2 7509
36′	$\bar{1}$.516 1789	2 7790	36′	$\bar{1}$.654 1679	2 7510
42′	$\bar{1}$.518 9579	2 7778	42′	$\bar{1}$.656 9189	2 7512
48′	$\bar{1}$.521 7357	2 7765	48′	$\bar{1}$.659 6701	2 7514
54′	$\bar{1}$.524 5122	2 7753	54′	$\bar{1}$.662 4215	2 7516
61° 0′	$\bar{1}$.527 2875	2 7743	66° 0′	$\bar{1}$.665 1731	2 7519
6′	$\bar{1}$.530 0618	2 7734	6′	$\bar{1}$.667 9250	2 7522
12′	$\bar{1}$.532 8352	2 7725	12′	$\bar{1}$.670 6772	2 7524
18′	$\bar{1}$.535 6077	2 7715	18′	$\bar{1}$.673 4296	2 7528
24′	$\bar{1}$.538 3792	2 7705	24′	$\bar{1}$.676 1824	2 7532
30′	$\bar{1}$.541 1497	2 7694	30′	$\bar{1}$.678 9356	2 7535
36′	$\bar{1}$.543 9191	2 7683	36′	$\bar{1}$.681 6891	2 7539
42′	$\bar{1}$.546 6874	2 7672	42′	$\bar{1}$.684 4430	2 7543
48′	$\bar{1}$.549 4546	2 7663	48′	$\bar{1}$.687 1973	2 7548
54′	$\bar{1}$.552 2209	2 7654	54′	$\bar{1}$.689 9521	2 7553
62° 0′	$\bar{1}$.554 9863	2 7645	67° 0′	$\bar{1}$.692 7074	2 7561
6′	$\bar{1}$.557 7508	2 7637	6′	$\bar{1}$.695 4635	2 7567
12′	$\bar{1}$.560 5145	2 7629	12′	$\bar{1}$.698 2202	2 7573
18′	$\bar{1}$.563 2774	2 7622	18′	$\bar{1}$.700 9775	2 7580
24′	$\bar{1}$.566 0396	2 7615	24′	$\bar{1}$.703 7355	2 7587
30′	$\bar{1}$.568 8011	2 7607	30′	$\bar{1}$.706 4942	2 7595
36′	$\bar{1}$.571 5618	2 7598	36′	$\bar{1}$.709 2537	2 7603
42′	$\bar{1}$.574 3216	2 7589	42′	$\bar{1}$.712 0140	2 7610
48′	$\bar{1}$.577 0805	2 7582	48′	$\bar{1}$.714 7750	2 7619
54′	$\bar{1}$.579 8387	2 7575	54′	$\bar{1}$.717 5369	2 7628
63° 0′	$\bar{1}$.582 5962	2 7570	68° 0′	$\bar{1}$.720 2997	2 7637
6′	$\bar{1}$.585 3532	2 7567	6′	$\bar{1}$.723 0634	2 7647
12′	$\bar{1}$.588 1099	2 7563	12′	$\bar{1}$.725 8281	2 7656
18′	$\bar{1}$.590 8662	2 7559	18′	$\bar{1}$.728 5937	2 7667
24′	$\bar{1}$.593 6221	2 7555	24′	$\bar{1}$.731 3604	2 7679
30′	$\bar{1}$.596 3776	2 7549	30′	$\bar{1}$.734 1283	2 7689
36′	$\bar{1}$.599 1325	2 7543	36′	$\bar{1}$.736 8972	2 7701
42′	$\bar{1}$.601 8868	2 7537	42′	$\bar{1}$.739 6673	2 7713
48′	$\bar{1}$.604 6405	2 7533	48′	$\bar{1}$.742 4386	2 7725
54′	$\bar{1}$.607 3938	2 7530	54′	$\bar{1}$.745 2111	2 7737
64° 0′	$\bar{1}$.610 1468	2 7527	69° 0′	$\bar{1}$.747 9848	2 7749
6′	$\bar{1}$.612 8995	2 7524	6′	$\bar{1}$.750 7597	2 7763
12′	$\bar{1}$.615 6519	2 7521	12′	$\bar{1}$.753 5360	2 7778
18′	$\bar{1}$.618 4040	2 7519	18′	$\bar{1}$.756 3138	2 7791
24′	$\bar{1}$.621 1559	2 7516	24′	$\bar{1}$.759 0929	2 7806
30′	$\bar{1}$.623 9075	2 7514	30′	$\bar{1}$.761 8735	2 7821
36′	$\bar{1}$.626 6589	2 7513	36′	$\bar{1}$.764 6556	2 7836
42′	$\bar{1}$.629 4102	2 7512	42′	$\bar{1}$.767 4392	2 7853
48′	$\bar{1}$.632 1614	2 7510	48′	$\bar{1}$.770 2245	2 7871
54′	$\bar{1}$.634 9124	2 7509	54′	$\bar{1}$.773 0116	2 7888
65° 0′	$\bar{1}$.637 6633	2 7508	70° 0′	$\bar{1}$.775 8004	2 7904

TABLE I—Continued

	$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1		$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
70° 0′	$\bar{1}$.775 8004	2 7904	75° 0′	$\bar{1}$.918 5141	2 9472
6′	$\bar{1}$.778 5908	2 7920	6′	$\bar{1}$.921 4613	2 9522
12′	$\bar{1}$.781 3828	2 7938	12′	$\bar{1}$.924 4135	2 9572
18′	$\bar{1}$.784 1766	2 7956	18′	$\bar{1}$.927 3707	2 9623
24′	$\bar{1}$.786 9722	2 7975	24′	$\bar{1}$.930 3330	2 9676
30′	$\bar{1}$.789 7697	2 7995	30′	$\bar{1}$.933 3006	2 9729
36′	$\bar{1}$.792 5692	2 8017	36′	$\bar{1}$.936 2735	2 9783
42′	$\bar{1}$.795 3709	2 8037	42′	$\bar{1}$.939 2518	2 9838
48′	$\bar{1}$.798 1746	2 8056	48′	$\bar{1}$.942 2356	2 9895
54′	$\bar{1}$.800 9802	2 8078	54′	$\bar{1}$.945 2251	2 9951
71° 0′	$\bar{1}$.803 7880	2 8100	76° 0′	$\bar{1}$.948 2202	3 0007
6′	$\bar{1}$.806 5980	2 8124	6′	$\bar{1}$.951 2209	3 0066
12′	$\bar{1}$.809 4104	2 8148	12′	$\bar{1}$.954 2275	3 0127
18′	$\bar{1}$.812 2252	2 8172	18′	$\bar{1}$.957 2402	3 0188
24′	$\bar{1}$.815 0424	2 8195	24′	$\bar{1}$.960 2590	3 0251
30′	$\bar{1}$.817 8619	2 8220	30′	$\bar{1}$.963 2841	3 0316
36′	$\bar{1}$.820 6839	2 8245	36′	$\bar{1}$.966 3157	3 0380
42′	$\bar{1}$.823 5084	2 8270	42′	$\bar{1}$.969 3537	3 0446
48′	$\bar{1}$.826 3354	2 8297	48′	$\bar{1}$.972 3983	3 0514
54′	$\bar{1}$.829 1651	2 8323	54′	$\bar{1}$.975 4497	3 0583
72° 0′	$\bar{1}$.831 9974	2 8349	77° 0′	$\bar{1}$.978 5080	3 0652
6′	$\bar{1}$.834 8323	2 8377	6′	1.981 5731	3 0723
12′	$\bar{1}$.837 6700	2 8406	12′	$\bar{1}$.984 6454	3 0795
18′	$\bar{1}$.840 5106	2 8435	18′	$\bar{1}$.987 7249	3 0869
24′	$\bar{1}$.843 3541	2 8464	24′	$\bar{1}$.990 8118	3 0944
30′	$\bar{1}$.846 2005	2 8494	30′	$\bar{1}$.993 9062	3 1020
36′	$\bar{1}$.849 0499	2 8525	36′	$\bar{1}$.997 0082	3 1099
42′	$\bar{1}$.851 9024	2 8556	42′	0.000 1181	3 1178
48′	$\bar{1}$.854 7580	2 8588	48′	0.003 2359	3 1259
54′	$\bar{1}$.857 6168	2 8620	54′	0.006 3618	3 1341
73° 0′	$\bar{1}$.860 4788	2 8653	78° 0′	0.009 4959	3 1426
6′	$\bar{1}$.863 3441	2 8688	6′	0.012 6385	3 1511
12′	$\bar{1}$.866 2129	2 8723	12′	0.015 7896	3 1598
18′	$\bar{1}$.869 0852	2 8759	18′	0.018 9494	3 1687
24′	$\bar{1}$.871 9611	2 8795	24′	0.022 1181	3 1778
30′	$\bar{1}$.874 8406	2 8831	30′	0.025 2959	3 1871
36′	$\bar{1}$.877 7237	2 8869	36′	0.028 4830	3 1964
42′	$\bar{1}$.880 6106	2 8907	42′	0.031 6794	3 2061
48′	$\bar{1}$.883 5013	2 8946	48′	0.034 8855	3 2159
54′	$\bar{1}$.886 3959	2 8986	54′	0.038 1014	3 2258
74° 0′	$\bar{1}$.889 2945	2 9025	79° 0′	0.041 3272	3 2360
6′	$\bar{1}$.892 1970	2 9066	6′	0.044 5633	3 2465
12′	$\bar{1}$.895 1036	2 9108	12′	0.047 8098	3 2570
18′	$\bar{1}$.898 0144	2 9151	18′	0.051 0668	3 2679
24′	$\bar{1}$.900 9295	2 9194	24′	0.054 3347	3 2789
30′	$\bar{1}$.903 8489	2 9239	30′	0.057 6136	3 2901
36′	$\bar{1}$.906 7728	2 9284	36′	0.060 9037	3 3016
42′	$\bar{1}$.909 7012	2 9329	42′	0.064 2053	3 3132
48′	$\bar{1}$.912 6341	2 9376	48′	0.067 5185	3 3252
54′	$\bar{1}$.915 5717	2 9424	54′	0.070 8437	3 3375
75° 0′	$\bar{1}$.918 5141	2 9472	80° 0′	0.074 1812	3 3500

TABLE I—Continued

	$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1		$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
80° 0′	0.074 1812	3 3500	85° 0′	0.265 4154	4 6004
6′	0.077 5312	3 3628	6′	0.270 0156	4 6499
12′	0.080 8940	3 3760	12′	0.274 6655	4 7015
18′	0.084 2700	3 3892	18′	0.279 3670	4 7553
24′	0.087 6592	3 4027	24′	0.284 1223	4 8109
30′	0.091 0619	3 4165	30′	0.288 9332	4 8689
36′	0.094 4784	3 4307	36′	0.293 8021	4 9293
42′	0.097 9091	3 4452	42′	0.298 7314	4 9924
48′	0.101 3543	3 4601	48′	0.303 7238	5 0585
54′	0.104 8144	3 4752	54′	0.308 7823	5 1274
81° 0′	0.108 2896	3 4906	86° 0′	0.313 9097	5 1995
6′	0.111 7802	3 5064	6′	0.319 1092	5 2751
12′	0.115 2866	3 5226	12′	0.324 3843	5 3544
18′	0.118 8092	3 5392	18′	0.329 7387	5 4375
24′	0.122 3484	3 5561	24′	0.335 1762	5 5250
30′	0.125 9045	3 5735	30′	0.340 7012	5 6172
36′	0.129 4780	3 5912	36′	0.346 3184	5 7143
42′	0.133 0692	3 6094	42′	0.352 0327	5 8168
48′	0.136 6786	3 6280	48′	0.357 8495	5 9254
54′	0.140 3066	3 6470	54′	0.363 7749	6 0404
82° 0′	0.143 9536	3 6667	87° 0′	0.369 8154	6 1624
6′	0.147 6203	3 6869	6′	0.375 9777	6 2923
12′	0.151 3072	3 7076	12′	0.382 2700	6 4306
18′	0.155 0148	3 7287	18′	0.388 7006	6 5786
24′	0.158 7435	3 7503	24′	0.395 2792	6 7370
30′	0.162 4938	3 7722	30′	0.402 0162	6 9072
36′	0.166 2660	3 7949	36′	0.408 9234	7 0904
42′	0.170 0609	3 8183	42′	0.416 0138	7 2884
48′	0.173 8792	3 8425	48′	0.423 3022	7 5031
54′	0.177 7217	3 8673	54′	0.430 8053	7 7373
83° 0′	0.181 5890	3 8926	88° 0′	0.438 5417	7 9921
6′	0.185 4816	3 9185	6′	0.446 5341	8 2723
12′	0.189 4001	3 9452	12′	0.454 8064	8 5816
18′	0.193 3453	3 9728	18′	0.463 3880	8 9247
24′	0.197 3181	4 0013	24′	0.472 3127	9 3079
30′	0.201 3194	4 0308	30′	0.481 6206	9 7389
36′	0.205 3502	4 0606	36′	0.491 3595	10 2275
42′	0.209 4108	4 0915	42′	0.501 5870	10 7868
48′	0.213 5023	4 1236	48′	0.512 3738	11 4341
54′	0.217 6259	4 1565	54′	0.523 8079	12 1932
84° 0′	0.221 7824	4 1904	89° 0′	0.536 0011	13 0958
6′	0.225 9728	4 2255	6′	0.549 0969	14 1917
12′	0.230 1983	4 2617	12′	0.563 2886	15 5520
18′	0.234 4600	4 2991	18′	0.578 8406	17 2914
24′	0.238 7591	4 3379	24′	0.596 1320	19 6050
30′	0.243 0970	4 3778	30′	0.615 7370	22 8537
36′	0.247 4748	4 4192	36′	0.638 5907	27 7976
42′	0.251 8940	4 4621	42′	0.666 3883	36 3882
48′	0.256 3561	4 5065	48′	0.702 7765	55 9176
54′	0.260 8626	4 5526	54′	0.758 6941	
85° 0′	0.265 4154	4 6004			

The above table has been recalculated and some of the values corrected in the last place. The values given are sufficiently accurate to give M within one part in a million.

TABLE II

Giving the Values of Log F and Log E as Functions of tan γ. (*See p. 20*)

tan γ	Log F	F	Log E	E
0.1	0.1971 996	1.5747 065	0.1950 415	1.5669 007
0.2	0.2003 678	1.5862 361	0.1918 928	1.5555 817
0.3	0.2054 261	1.6048 192	0.1869 144	1.5378 514
0.4	0.2120 849	1.6296 146	0.1804 536	1.5151 429
0.5	0.2200 096	1.6596 236	0.1729 048	1.4890 346
0.6	0.2288 634	1.6938 051	0.1646 557	1.4610 185
0.7	0.2383 385	1.7311 652	0.1560 492	1.4323 502
0.8	0.2481 728	1.7708 135	0.1473 640	1.4039 900
0.9	0.2581 561	1.8119 912	0.1388 116	1.3766 121
1.0	0.2681 272	1.8540 745	0.1305 409	1.3506 441
1.5	0.3147 473	2.0641 787	0.0955 992	1.2462 329
2.0	0.3535 711	2.2572 057	0.0713 258	1.1784 897
2.5	0.3852 192	2.4278 352	0.0547 850	1.1344 491
3.0	0.4112 984	2.5780 917	0.0432 738	1.1047 748
4.0	0.4518 237	2.8302 429	0.0289 324	1.0688 885
5.0	0.4821 752	3.0351 154	0.0207 426	1.0489 205
7.5	0.5341 061	3.4206 300	0.0109 567	1.0255 497
10.0	0.5682 672	3.7005 581	0.0068 338	1.0158 598
12.5	0.5932 708	3.9198 622	0.0047 004	1.0108 819

TABLE III

Values of the Constant K as Functions of x/A and a/A

(For use in Formula (57))

x/A = .50	.75	1	1.25	1.50	1.75	2
a/A						
0.50	9.39283	12.30385	14.27982	15.62795	16.56549	17.23299	17.71973
0.55	9.52044	12.40135	14.34594	15.67140	16.59411	17.25215	17.73283
0.60	9.66358	12.50816	14.41766	15.71837	16.62503	17.27286	17.74701
0.65	9.82296	12.62412	14.49474	15.76867	16.65813	17.29504	17.76221
0.70	9.99921	12.74897	14.57688	15.82212	16.69330	17.31865	17.77841
0.75	10.19272	12.88232	14.66377	15.87850	16.73039	17.34357	17.79554

TABLE IV

Values of the Constant Q in Formula (74), $L_e = n^2 a Q$

For the self-inductance of a single-layer winding on a solenoid; n is the whole number of turns of wire in the winding and a is the mean radius. The corrections by Tables VII and VIII must be made to get L from L_e as usual. (See p. 122.)

In the following table $2a$ is the diameter, b is the length, and therefore $2a/b = \tan \gamma$. (See Fig. 33.)

$2\frac{a}{b} = \tan \gamma$	Q	$2\frac{a}{b} = \tan \gamma$	Q
0.20	3.63240	1.80	19.57938
0.30	5.23368	2.00	20.74631
0.40	6.71017	2.20	21.82049
0.50	8.07470	2.40	22.81496
0.60	9.33892	2.60	23.74013
0.70	10.51349	2.80	24.60482
0.80	11.60790	3.00	25.41613
0.90	12.63059	3.20	26.18009
1.00	13.58892	3.40	26.90177
1.20	15.33799	3.60	27.58548
1.40	16.89840	3.80	28.23494
1.60	18.30354	4.00	28.85335

For an explanation of the above formula, see page 118.

TABLE V

Constants A and B for Strasser's Formula (82)

$$A=2\log_e[(n-1)!(n-2)!\cdots 1]$$
$$B=3[(n-2)2^2\log_e 2+(n-3)3^2\log_e 3+\cdots(n-1)^2\log_e(n-1)]$$

n	A	B	n	A	B
1	0	0	16	354.396	35693
2	0	0	17	415.739	46775
3	1.38629	8.318	18	482.75	60314
4	4.96981	46.298	19	555.54	76662
5	11.3259	150.82	20	634.22	96198
6	20.9009	376.05	21	718.89	119330
7	34.0594	794.79	22	809.65	146490
8	51.1097	1499.58	23	906.59	178140
9	72.3189	2603.62	24	1009.81	214760
10	97.9226	4241.59	25	1119.38	256880
11	128.131	6570.33	26	1235.38	305030
12	163.136	9769.51	27	1357.91	359790
13	203.110	14042.2	28	1487.02	421750
14	248.215	19615.3	29	1622.80	491560
15	298.597	26740.1	30	1765.32	569860

We have recently recomputed Strasser's constants, finding several errors which are corrected here.

TABLE VI

Table of Constants for Stefan's Formula (90)

b/c or c/b	y_1	y_2	b/c or c/b	y_1	y_2
0.00	0.50000	0.1250	0.55	0.80815	0.3437
0.05	.54899	.1269	0.60	.81823	.3839
0.10	.59243	.1325	0.65	.82648	.4274
0.15	.63102	.1418	0.70	.83311	.4739
0.20	.66520	.1548	0.75	.83831	.5234
0.25	.69532	.1714	0.80	.84225	.5760
0.30	.72172	.1916	0.85	.84509	.6317
0.35	.74469	.2152	0.90	.84697	.6902
0.40	.76454	.2423	0.95	.84801	.7518
0.45	.78154	.2728	1.00	.84834	.8162
0.50	.79600	.3066			

There is in general no difficulty in obtaining y_1 and y_2 with sufficient accuracy by interpolation from this table, using second and in some cases third differences. The only case which may give trouble is when b/c or c/b is less than 0.1. In such cases, however, Stefan's formula does not give precise results, and the errors in the interpolation will not be important.

TABLE VII

Values of Correction Term A, Depending on the Ratio $\frac{d}{D}$ of the Diameters of Bare and Covered Wire on the Single Layer Coil

(For use in Formula (80))

$$A = \log_e\left(1.7452\frac{d}{D}\right)$$

$\frac{d}{D}$	A	Δ_1	$\frac{d}{D}$	A	Δ_1	$\frac{d}{D}$	A	Δ_1	Δ_2
1.00	0.5568	−100	0.75	0.2691	−134	0.50	−0.1363	−202	− 4
.99	.5468	−101	.74	.2557	−136	.49	− .1565	−206	− 5
.98	.5367	−103	.73	.2421	−138	.48	− .1771	−211	− 4
.97	.5264	−104	.72	.2283	−140	.47	− .1982	−215	− 4
.96	.5160	−105	.71	.2143	−142	.46	− .2197	−219	− 6
0.95	0.5055	−106	0.70	0.2001	−144	0.45	−0.2416	−225	− 5
.94	.4949	−107	.69	.1857	−146	.44	− .2641	−230	− 5
.93	.4842	−108	.68	.1711	−148	.43	− .2871	−235	− 6
.92	.4734	−109	.67	.1563	−150	.42	− .3106	−241	− 6
.91	.4625	−110	.66	.1413	−152	.41	− .3347	−247	− 6
0.90	0.4515	−112	0.65	0.1261	−155	0.40	−0.3594	−253	− 7
.89	.4403	−113	.64	.1106	−157	.39	− .3847	−260	− 7
.88	.4290	−114	.63	.0949	−160	.38	− .4107	−267	− 7
.87	.4176	−116	.62	.0789	−163	.37	− .4374	−274	− 7
.86	.4060	−117	.61	.0626	−166	.36	− .4648	−281	− 9
0.85	0.3943	−118	0.60	0.0460	−168	0.35	−0.4929	−290	− 9
.84	.3825	−120	.59	.0292	−171	.34	− .5219	−299	− 9
.83	.3705	−121	.58	+ .0121	−174	.33	− .5518	−308	− 9
.82	.3584	−123	.57	− .0053	−177	.32	− .5826	−317	− 9
.81	.3461	−124	.56	− .0230	−180	.31	− .6143	−328	−11
0.80	0.3337	−126	0.55	−0.0410	−184	0.30	−0.6471	−339	−12
.79	.3211	−127	.54	− .0594	−187	.29	− .6810	−351	−13
.78	.3084	−129	.53	− .0781	−190	.28	− .7161	−364	−13
.77	.2955	−131	.52	− .0971	−194	.27	− .7525	−377	−15
.76	.2824	−133	.51	− .1165	−198	.26	− .7902	−392	−16
0.75	0.2691		0.50	−0.1363		0.25	−0.8294		

TABLE VII—Continued

$\frac{d}{D}$	A	Δ_1	Δ_2	$\frac{d}{D}$	A	Δ_1	Δ_2
0.25	−0.8294	−408	− 18	0.10	−1.7457	−1054	− 124
.24	− .8702	−426	− 19	.09	−1.8511	−1178	− 157
.23	− .9128	−445	− 20	.08	−1.9689	−1335	− 206
.22	− .9573	−465	− 23	.07	−2.1024	−1541	− 283
.21	−1.0038	−488	− 25	.06	−2.2565	−1824	− 407
0.20	−1.0526	−513	− 28	0.05	−2.4389	−2231	− 646
.19	−1.1039	−541	− 30	.04	−2.6620	−2877	−1177
.18	−1.1580	−571	− 35	.03	−2.9497	−4054	−2878
.17	−1.2151	−606	− 39	.02	−3.3551	−6932	
.16	−1.2757	−645	− 45	.01	−4.0483		
0.15	−1.3402	−690	− 51				
.14	−1.4092	−741	− 60				
.13	−1.4833	−801	− 70				
.12	−1.5634	−870	− 83				
.11	−1.6504	−953	−101				
0.10	−1.7457						

TABLE VIII

Values of the Correction Term B, Depending on the Number of Turns of Wire on the Single Layer Coil

(For use in Formula (80))

$$B=\frac{2}{n}\sum_{1}^{n-1} m \log_e \frac{m}{R_m}$$

where R_m is geometric mean distance of the sections of the current sheet whose centers coincide with those of the wires. (See this Bulletin, 2, p. 168, equat. (11); 1906.)

Number of Turns—n	B	Number of Turns—n	B
1	0.0000	50	0.3186
2	.1137	60	.3216
3	.1663	70	.3239
4	.1973	80	.3257
5	.2180	90	.3270
6	.2329	100	.3280
7	.2443	125	.3298
8	.2532	150	.3311
9	.2604	175	.3321
10	.2664	200	.3328
15	.2857	300	.3343
20	.2974	400	.3351
25	.3042	500	.3356
30	.3083	600	.3359
35	.3119	700	.3361
40	.3148	800	.3363
45	.3169	900	.3364
50	.3186	1000	.3365

TABLE IX

Value of the Constant A_s as a Function of t/a, t being the Depth of the Winding and a the Mean Radius

$$A_s = 0.6949 - \frac{t^2}{96a^2}\left(\log_e \frac{8a}{t} + 2.76\right)$$

(For use in Formula (91))

t/a	A_s
0	0.6949
0.10	0.6942
0.15	0.6933
0.20	0.6922
0.25	0.6909

TABLE X

Values of the Correction Term B_s depending on the Number of Turns of Square Conductor on Single Layer Coil

(For use in Formula (91))

$$B_s = \frac{2}{n}\sum_{1}^{n-1}\left(m \log \frac{R'_m}{R_m}\right)$$

where R'_m=geom. mean distance for the two squares
R_m = " " " " " elements of the current sheet. (See this Bulletin, 4, p. 373; 1907.)

Number of Turns	B_s	Number of Turns	B_s	Number of Turns	B_s
1	0.0000	11	0.2844	21	0.3116
2	.1202	12	.2888	22	.3131
3	.1753	13	.2927	23	.3145
4	.2076	14	.2961	24	.3157
5	.2292	15	.2991	25	.3169
6	.2446	16	.3017	26	.3180
7	.2563	17	.3041	27	.3190
8	.2656	18	.3062	28	.3200
9	.2730	19	.3082	29	.3209
10	.2792	20	.3099	30	.3218

TABLE XI

Table of Napierian Logarithms to Nine Decimal Places for Numbers from 1 to 100

1	0.000 000 000	51	3.931 825 633
2	0.693 147 181	52	3.951 243 719
3	1.098 612 289	53	3.970 291 914
4	1.386 294 361	54	3.988 984 047
5	1.609 437 912	55	4.007 333 185
6	1.791 759 469	56	4.025 351 691
7	1.945 910 149	57	4.043 051 268
8	2.079 441 542	58	4.060 443 011
9	2.197 224 577	59	4.077 537 444
10	2.302 585 093	60	4.094 344 562
11	2.397 895 273	61	4.110 873 864
12	2.484 906 650	62	4.127 134 385
13	2.564 949 357	63	4.143 134 726
14	2.639 057 330	64	4.158 883 083
15	2.708 050 201	65	4.174 387 270
16	2.772 588 722	66	4.189 654 742
17	2.833 213 344	67	4.204 692 619
18	2.890 371 758	68	4.219 507 705
19	2.944 438 979	69	4.234 106 505
20	2.995 732 274	70	4.248 495 242
21	3.044 522 438	71	4.262 679 877
22	3.091 042 453	72	4.276 666 119
23	3.135 494 216	73	4.290 459 441
24	3.178 053 830	74	4.304 065 093
25	3.218 875 825	75	4.317 488 114
26	3.258 096 538	76	4.330 733 340
27	3.295 836 866	77	4.343 805 422
28	3.332 204 510	78	4.356 708 827
29	3.367 295 830	79	4.369 447 852
30	3.401 197 382	80	4.382 026 635
31	3.433 987 204	81	4.394 339 155
32	3.465 735 903	82	4.406 719 247
33	3.496 507 561	83	4.418 840 608
34	3.526 360 525	84	4.430 816 799
35	3.555 348 061	85	4.442 651 256
36	3.583 518 938	86	4.454 347 296
37	3.610 917 913	87	4.465 908 119
38	3.637 586 160	88	4.477 336 814
39	3.663 561 646	89	4.488 636 370
40	3.688 879 454	90	4.499 809 670
41	3.713 572 067	91	4.510 859 507
42	3.737 669 618	92	4.521 788 577
43	3.761 200 116	93	4.532 599 493
44	3.784 189 634	94	4.543 294 782
45	3.806 662 490	95	4.553 876 892
46	3.828 641 396	96	4.564 348 191
47	3.850 147 602	97	4.574 710 979
48	3.871 201 011	98	4.584 967 479
49	3.891 820 298	99	4.595 119 850
50	3.912 023 005	100	4.605 170 186

log 1525=log 25+log 61; log 9.8=log 98−log 10, etc.

TABLE XII

Values of F and E

The following table of elliptic integrals of the first and second kind is taken from Legendre's *Traité des Fonctions Elliptiques*, Volume 2, Table VIII:

	F	Δ_1	Δ_2		E	Δ_1	Δ_2
0°	1.570 796	120	239	0°	1.570 796	− 120	−239
1	1.570 916	359	240	1	1.570 677	− 359	−239
2	1.571 275	599	240	2	1.570 318	− 598	−239
3	1.571 874	839	241	3	1.569 720	− 836	−238
4	1.572 712	1 080	241	4	1.568 884	−1 075	−238
5	1.573 792	1 321	243	5	1.567 809	−1 312	−237
6	1.575 114	1 564	244	6	1.566 497	−1 549	−236
7	1.576 678	1 808	246	7	1.564 948	−1 785	−235
8	1.578 486	2 054	247	8	1.563 162	−2 020	−234
9	1.580 541	2 302	249	9	1.561 142	−2 255	−233
10	1.582 843	2 551	252	10	1.558 887	−2 487	−232
11	1.585 394	2 803	254	11	1.556 400	−2 719	−230
12	1.588 197	3 057	257	12	1.553 681	−2 949	−228
13	1.591 254	3 314	260	13	1.550 732	−3 177	−227
14	1.594 568	3 574	263	14	1.547 554	−3 404	−225
15	1.598 142	3 836	266	15	1.544 150	−3 629	−223
16	1.601 978	4 103	270	16	1.540 521	−3 852	−221
17	1.606 081	4 373	274	17	1.536 670	−4 073	−218
18	1.610 454	4 647	278	18	1.532 597	−4 291	−216
19	1.615 101	4 925	283	19	1.528 306	−4 507	−214
20	1.620 026	5 208	288	20	1.523 799	−4 721	−211
21	1.625 234	5 495	293	21	1.519 079	−4 932	−208
22	1.630 729	5 788	298	22	1.514 147	−5 140	−205
23	1.636 517	6 087	304	23	1.509 007	−5 345	−202
24	1.642 604	6 391	311	24	1.503 662	−5 547	−199
25	1.648 995	6 702	317	25	1.498 115	−5 746	−196
26	1.655 697	7 019	324	26	1.492 368	−5 942	−192
27	1.662 716	7 343	332	27	1.486 427	−6 134	−189
28	1.670 059	7 675	340	28	1.480 293	−6 323	−185
29	1.677 735	8 015	349	29	1.473 970	−6 508	−181
30	1.685 750	8 364	358	30	1.467 462	−6 689	−177
31	1.694 114	8 722	367	31	1.460 774	−6 866	−173
32	1.702 836	9 089	377	32	1.453 908	−7 039	−168
33	1.711 925	9 466	388	33	1.446 869	−7 207	−164
34	1.721 391	9 854	400	34	1.439 662	−7 371	−159
35	1.731 245	10 254	412	35	1.432 291	−7 531	−155
36	1.741 499	10 666	425	36	1.424 760	−7 685	−150
37	1.752 165	11 091	439	37	1.417 075	−7 835	−145
38	1.763 256	11 530	453	38	1.409 240	−7 980	−140
39	1.774 786	11 983	469	39	1.401 260	−8 120	−134
40	1.786 770	12 452	486	40	1.393 140	−8 254	−129
41	1.799 222	12 938	504	41	1.384 886	−8 382	−123
42	1.812 160	13 442	523	42	1.376 504	−8 505	−117
43	1.825 602	13 965	543	43	1.367 999	−8 622	−111
44	1.839 567	14 508	565	44	1.359 377	−8 733	−105
45	1.854 075	15 073	588	45	1.350 644	−8 838	− 98

TABLE XII—Continued

	F	Δ_1	Δ_2		E	Δ_1	Δ_2
45°	1.854 075	15 073	588	45°	1.350 644	−8 838	−98
46	1.869 148	15 661	613	46	1.341 806	−8 936	−92
47	1.884 809	16 274	640	47	1.332 870	−9 028	−85
48	1.901 083	16 914	669	48	1.323 842	−9 113	−78
49	1.917 997	17 584	700	49	1.314 729	−9 190	−71
50	1.935 581	18 284	735	50	1.305 539	−9 261	−63
51	1.953 865	19 017	770	51	1.296 278	−9 324	−56
52	1.972 882	19 787	809	52	1.286 954	−9 380	−48
53	1.992 670	20 597	852	53	1.277 574	−9 427	−40
54	2.013 266	21 449	898	54	1.268 147	−9 467	−31
55	2.034 715	22 347	949	55	1.258 680	−9 498	−22
56	2.057 062	23 296	1 004	56	1.249 182	−9 520	−14
57	2.080 358	24 300	1 064	57	1.239 661	−9 534	− 4
58	2.104 658	25 364	1 130	58	1.230 127	−9 538	+ 5
59	2.130 021	26 494	1 203	59	1.220 589	−9 533	+15
60	2.156 516	27 698	1 284	60	1.211 056	−9 518	+25
61	2.184 213	28 982	1 373	61	1.201 538	−9 492	36
62	2.213 195	30 355	1 472	62	1.192 046	−9 457	47
63	2.243 549	31 827	1 583	63	1.182 589	−9 410	58
64	2.275 376	33 410	1 708	64	1.173 180	−9 351	70
65	2.308 787	35 118	1 848	65	1.163 828	−9 281	82
66	2.343 905	36 965	2 006	66	1.154 547	−9 199	95
67	2.380 870	38 971	2 186	67	1.145 348	−9 104	109
68	2.419 842	41 158	2 393	68	1.136 244	−8 995	123
69	2.460 999	43 551	2 631	69	1.127 250	−8 872	138
70	2.504 550	46 181	2 907	70	1.118 378	−8 734	153
71	2.550 731	49 088	3 230	71	1.109 643	−8 581	169
72	2.599 820	52 318	3 611	72	1.101 062	−8 412	187
73	2.652 138	55 930	4 066	73	1.092 650	−8 225	205
74	2.708 068	59 996	4 614	74	1.084 425	−8 020	224
75	2.768 063	64 609	5 283	75	1.076 405	−7 796	245
76	2.832 673	69 892	6 112	76	1.068 610	−7 550	268
77	2.902 565	76 004	7 156	77	1.061 059	−7 282	292
78	2.978 569	83 160	8 497	78	1.053 777	−6 990	318
79	3.061 729	91 657	10 261	79	1.046 786	−6 672	347
80	3.153 385	101 918	12 647	80	1.040 114	−6 325	379
81	3.255 303	114 565	15 989	81	1.033 789	−5 946	415
82	3.369 868	130 554	20 879	82	1.027 844	−5 531	455
83	3.500 422	151 433	28 453	83	1.022 313	−5 076	502
84	3.651 856	179 886	41 130	84	1.017 237	−4 573	558
85	3.831 742	221 016	64 880	85	1.012 664	−4 016	626
86	4.052 758	285 896	118 167	86	1.008 648	−3 389	715
87	4.338 654	404 063	288 129	87	1.005 259	−2 675	842
88	4.742 717	692 193		88	1.002 584	−1 832	1081
89	5.434 910			89	1.000 752	− 752	
90				90	1.000 000		

TABLE XIII

Values of log F and log E

(See Note, p. 213)

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
45°0	0.2681 2722	3 4688	105	0.1305 4086	2 8279	52
45.1	0.2684 7411	3 4793	105	0.1302 5807	2 8331	52
45.2	0.2688 2204	3 4898	105	0.1299 7476	2 8383	52
45.3	0.2691 7102	3 5004	106	0.1296 9094	2 8434	52
45.4	0.2695 2106	3 5110	106	0.1294 0659	2 8486	51
45.5	0.2698 7216	3 5216	106	0.1291 2174	2 8537	51
45.6	0.2702 2431	3 5322	106	0.1288 3636	2 8589	51
45.7	0.2705 7753	3 5428	107	0.1285 5048	2 8640	51
45.8	0.2709 3181	3 5535	107	0.1282 6408	2 8691	51
45.9	0.2712 8716	3 5642	107	0.1279 7717	2 8742	51
46.0	0.2716 4358	3 5749	108	0.1276 8975	2 8793	51
46.1	0.2720 0108	3 5857	108	0.1274 0182	2 8844	51
46.2	0.2723 5965	3 5965	108	0.1271 1338	2 8894	50
46.3	0.2727 1930	3 6073	108	0.1268 2444	2 8945	50
46.4	0.2730 8003	3 6181	109	0.1265 3499	2 8995	50
46.5	0.2734 4184	3 6290	109	0.1262 4504	2 9045	50
46.6	0.2738 0474	3 6399	109	0.1259 5459	2 9095	50
46.7	0.2741 6873	3 6508	110	0.1256 6364	2 9145	50
46.8	0.2745 3381	3 6618	110	0.1253 7218	2 9195	50
46.9	0.2748 9999	3 6728	110	0.1250 8023	2 9245	50
47.0	0.2752 6727	3 6838	110	0.1247 8778	2 9295	49
47.1	0.2756 3565	3 6948	111	0.1244 9483	2 9344	49
47.2	0.2760 0513	3 7059	111	0.1242 0139	2 9393	49
47.3	0.2763 7572	3 7170	111	0.1239 0746	2 9443	49
47.4	0.2767 4741	3 7281	112	0.1236 1303	2 9492	49
47.5	0.2771 2023	3 7393	112	0.1233 1811	2 9541	49
47.6	0.2774 9415	3 7505	112	0.1230 2271	2 9589	49
47.7	0.2778 6920	3 7617	112	0.1227 2681	2 9638	49
47.8	0.2782 4537	3 7729	113	0.1224 3043	2 9687	48
47.9	0.2786 2266	3 7842	113	0.1221 3357	2 9735	48
48.0	0.2790 0109	3 7955	113	0.1218 3622	2 9783	48
48.1	0.2793 8064	3 8069	114	0.1215 3838	2 9831	48
48.2	0.2797 6133	3 8183	114	0.1212 4007	2 9879	48
48.3	0.2801 4315	3 8297	114	0.1209 4128	2 9927	48
48.4	0.2805 2612	3 8411	115	0.1206 4201	2 9975	48
48.5	0.2809 1023	3 8526	115	0.1203 4226	3 0022	47
48.6	0.2812 9548	3 8641	115	0.1200 4204	3 0070	47
48.7	0.2816 8189	3 8756	116	0.1197 4134	3 0117	47
48.8	0.2820 6945	3 8872	116	0.1194 4017	3 0164	47
48.9	0.2824 5817	3 8988	116	0.1191 3854	3 0211	47
49.0	0.2828 4805	3 9104	117	0.1188 3643	3 0258	47
49.1	0.2832 3909	3 9221	117	0.1185 3385	3 0304	46
49.2	0.2836 3130	3 9338	117	0.1182 3081	3 0351	46
49.3	0.2840 2467	3 9455	118	0.1179 2730	3 0397	46
49.4	0.2844 1923	3 9573	118	0.1176 2333	3 0443	46
49.5	0.2848 1495	3 9691	118	0.1173 1890	3 0489	46
49.6	0.2852 1186	3 9809	119	0.1170 1401	3 0535	46
49.7	0.2856 0996	3 9928	119	0.1167 0866	3 0581	46
49.8	0.2860 0924	4 0047	119	0.1164 0286	3 0626	45
49.9	0.2864 0971	4 0167	120	0.1160 9660	3 0671	45
50.0	0.2868 1137	4 0286	120	0.1157 8988	3 0717	45

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
50°0	0.2868 1137	4 0286	120	0.1157 8988	3 0717	45
50.1	0.2872 1424	4 0406	121	0.1154 8271	3 0762	45
50.2	0.2876 1830	4 0527	121	0.1151 7510	3 0807	45
50.3	0.2880 2357	4 0648	121	0.1148 6703	3 0851	45
50.4	0.2884 3005	4 0769	122	0.1145 5852	3 0896	44
50.5	0.2888 3774	4 0891	122	0.1142 4956	3 0940	44
50.6	0.2892 4665	4 1013	122	0.1139 4016	3 0985	44
50.7	0.2896 5677	4 1135	123	0.1136 3032	3 1028	44
50.8	0.2900 6812	4 1258	123	0.1133 2003	3 1072	44
50.9	0.2904 8070	4 1381	123	0.1130 0931	3 1116	43
51.0	0.2908 9451	4 1504	124	0.1126 9815	3 1159	43
51.1	0.2913 0955	4 1628	124	0.1123 8656	3 1203	43
51.2	0.2917 2584	4 1753	125	0.1120 7453	3 1246	43
51.3	0.2921 4336	4 1877	125	0.1117 6207	3 1289	43
51.4	0.2925 6214	4 2002	125	0.1114 4919	3 1332	43
51.5	0.2929 8216	4 2128	126	0.1111 3587	3 1374	42
51.6	0.2934 0344	4 2254	126	0.1108 2213	3 1417	42
51.7	0.2938 2597	4 2380	127	0.1105 0796	3 1459	42
51.8	0.2942 4977	4 2506	127	0.1101 9337	3 1501	42
51.9	0.2946 7483	4 2634	127	0.1098 7836	3 1543	42
52.0	0.2951 0117	4 2761	128	0.1095 6294	3 1584	41
52.1	0.2955 2878	4 2889	128	0.1092 4709	3 1626	41
52.2	0.2959 5767	4 3017	129	0.1089 3083	3 1667	41
52.3	0.2963 8784	4 3146	129	0.1086 1416	3 1708	41
52.4	0.2968 1930	4 3275	130	0.1082 9707	3 1749	41
52.5	0.2972 5205	4 3405	130	0.1079 7958	3 1790	41
52.6	0.2976 8610	4 3535	130	0.1076 6168	3 1831	40
52.7	0.2981 2144	4 3665	131	0.1073 4338	3 1871	40
52.8	0.2985 5810	4 3796	131	0.1070 2467	3 1911	40
52.9	0.2989 9606	4 3927	132	0.1067 0556	3 1951	40
53.0	0.2994 3533	4 4059	132	0.1063 8605	3 1991	40
53.1	0.2998 7592	4 4191	133	0.1060 6614	3 2030	39
53.2	0.3003 1783	4 4324	133	0.1057 4584	3 2070	39
53.3	0.3007 6107	4 4457	134	0.1054 2514	3 2109	39
53.4	0.3012 0564	4 4591	134	0.1051 0406	3 2148	39
53.5	0.3016 5155	4 4725	134	0.1047 8258	3 2186	38
53.6	0.3020 9880	4 4859	135	0.1044 6072	3 2225	38
53.7	0.3025 4739	4 4994	135	0 1041 3847	3 2263	38
53.8	0.3029 9733	4 5130	136	0.1038 1584	3 2301	38
53.9	0.3034 4863	4 5265	136	0.1034 9283	3 2339	38
54.0	0.3039 0128	4 5402	137	0.1031 6944	3 2377	37
54.1	0.3043 5530	4 5539	137	0.1028 4567	3 2414	37
54.2	0.3048 1069	4 5676	138	0.1025 2153	3 2451	37
54.3	0.3052 6745	4 5814	138	0.1021 9702	3 2488	37
54.4	0.3057 2559	4 5952	139	0.1018 7214	3 2525	37
54.5	0.3061 8511	4 6091	139	0.1015 4689	3 2562	36
54.6	0.3066 4602	4 6230	140	0.1012 2127	3 2598	36
54.7	0.3071 0833	4 6370	140	0.1008 9529	3 2634	36
54.8	0.3075 7203	4 6511	141	0.1005 6895	3 2670	36
54.9	0.3080 3714	4 6652	141	0.1002 4226	3 2705	35
55.0	0.3085 0365	4 6793	142	0.0999 1520	3 2741	35

TABLE XIII—Continued.

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
55°0	0.3085 0365	4 6793	142	0.0999 1520	3 2741	35
55.1	0.3089 7158	4 6935	142	0.0995 8779	3 2776	35
55.2	0.3094 4093	4 7077	143	0.0992 6003	3 2811	35
55.3	0.3099 1170	4 7220	143	0.0989 3193	3 2846	34
55.4	0.3103 8391	4 7364	144	0.0986 0347	3 2880	34
55.5	0.3108 5754	4 7508	145	0.0982 7467	3 2914	34
55.6	0.3113 3262	4 7652	145	0.0979 4553	3 2948	34
55.7	0.3118 0915	4 7798	146	0.0976 1605	3 2982	33
55.8	0.3122 8712	4 7943	146	0.0972 8623	3 3015	33
55.9	0.3127 6655	4 8089	147	0.0969 5607	3 3049	33
56.0	0.3132 4745	4 8236	147	0.0966 2559	3 3082	33
56.1	0.3137 2981	4 8384	148	0.0962 9477	3 3114	32
56.2	0.3142 1365	4 8532	149	0.0959 6363	3 3147	32
56.3	0.3146 9896	4 8680	149	0.0956 3216	3 3179	32
56.4	0.3151 8577	4 8829	150	0.0953 0037	3 3211	32
56.5	0.3156 7406	4 8979	150	0.0949 6826	3 3243	31
56.6	0.3161 6385	4 9129	151	0.0946 3583	3 3274	31
56.7	0.3166 5514	4 9280	151	0.0943 0309	3 3305	31
56.8	0.3171 4794	4 9432	152	0.0939 7003	3 3336	31
56.9	0.3176 4226	4 9584	153	0.0936 3667	3 3367	30
57.0	0.3181 3809	4 9736	153	0.0933 0300	3 3397	30
57.1	0.3186 3545	4 9890	154	0.0929 6903	3 3428	30
57.2	0.3191 3435	5 0044	155	0.0926 3475	3 3457	30
57.3	0.3196 3479	5 0198	155	0.0923 0018	3 3487	29
57.4	0.3201 3677	5 0353	156	0.0919 6531	3 3516	29
57.5	0.3206 4030	5 0509	156	0.0916 3014	3 3545	29
57.6	0.3211 4539	5 0666	157	0.0912 9469	3 3574	28
57.7	0.3216 5204	5 0823	158	0.0909 5895	3 3603	28
57.8	0.3221 6027	5 0980	158	0.0906 2292	3 3631	28
57.9	0.3226 7008	5 1139	159	0.0902 8662	3 3659	28
58.0	0.3231 8146	5 1298	160	0.0899 5003	3 3686	27
58.1	0.3236 9444	5 1458	160	0.0896 1317	3 3714	27
58.2	0.3242 0902	5 1618	161	0.0892 7603	3 3741	27
58.3	0.3247 2520	5 1779	162	0.0889 3862	3 3767	26
58.4	0.3252 4299	5 1941	162	0.0886 0095	3 3794	26
58.5	0.3257 6240	5 2104	163	0.0882 6301	3 3820	26
58.6	0.3262 8344	5 2267	164	0.0879 2481	3 3846	26
58.7	0.3268 0611	5 2431	165	0.0875 8635	3 3871	25
58.8	0.3273 3041	5 2595	165	0.0872 4764	3 3897	25
58.9	0.3278 5637	5 2761	166	0.0869 0867	3 3922	25
59.0	0.3283 8397	5 2927	167	0.0865 6945	3 3946	24
59.1	0.3289 1324	5 3094	168	0.0862 2999	3 3971	24
59.2	0.3294 4418	5 3261	168	0.0858 9028	3 3995	24
59.3	0.3299 7679	5 3429	169	0.0855 5033	3 4018	23
59.4	0.3305 1108	5 3598	170	0.0852 1015	3 4042	23
59.5	0.3310 4707	5 3768	171	0.0848 6973	3 4065	23
59.6	0.3315 8475	5 3939	171	0.0845 2908	3 4088	22
59.7	0.3321 2414	5 4110	172	0.0841 8820	3 4110	22
59.8	0.3326 6524	5 4282	173	0.0838 4710	3 4132	22
59.9	0.3332 0806	5 4455	174	0.0835 0578	3 4154	21
60.0	0.3337 5261	5 4629	175	0.0831 6424	3 4176	21

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
60°0	0.3337 5261	5 4629	175	0.0831 6424	3 4176	21
60.1	0.3342 9890	5 4803	175	0.0828 2248	3 4197	21
60.2	0.3348 4694	5 4979	176	0.0824 8051	3 4217	20
60.3	0.3353 9673	5 5155	177	0.0821 3834	3 4238	20
60.4	0.3359 4827	5 5332	178	0.0817 9596	3 4258	20
60.5	0.3365 0159	5 5510	179	0.0814 5338	3 4278	19
60.6	0.3370 5669	5 5688	179	0.0811 1060	3 4297	19
60.7	0.3376 1357	5 5868	180	0.0807 6763	3 4316	19
60.8	0.3381 7225	5 6048	181	0.0804 2446	3 4335	18
60.9	0.3387 3274	5 6229	182	0.0800 8111	3 4354	18
61.0	0.3392 9503	5 6412	183	0.0797 3758	3 4372	18
61.1	0.3398 5915	5 6595	184	0.0793 9386	3 4389	17
61.2	0.3404 2509	5 6778	185	0.0790 4997	3 4407	17
61.3	0.3409 9288	5 6963	186	0.0787 0590	3 4424	17
61.4	0.3415 6251	5 7149	187	0.0783 6167	3 4440	16
61.5	0.3421 3400	5 7336	188	0.0780 1727	3 4456	16
61.6	0.3427 0735	5 7523	188	0.0776 7270	3 4472	15
61.7	0.3432 8258	5 7712	189	0.0773 2798	3 4488	15
61.8	0.3438 5970	5 7901	190	0.0769 8310	3 4503	15
61.9	0.3444 3871	5 8091	191	0.0766 3807	3 4518	14
62.0	0.3450 1962	5 8283	192	0.0762 9290	3 4532	14
62.1	0.3456 0245	5 8475	193	0.0759 4758	3 4546	14
62.2	0.3461 8720	5 8668	194	0.0756 0212	3 4560	13
62.3	0.3467 7388	5 8863	195	0.0752 5652	3 4573	13
62.4	0.3473 6250	5 9058	196	1.0749 1079	3 4586	12
62.5	0.3479 5308	5 9254	197	0.0745 6494	3 4598	12
62.6	0.3485 4562	5 9451	198	0.0742 1895	3 4610	12
62.7	0.3491 4014	5 9650	199	0.0738 7285	3 4622	11
62.8	0.3497 3664	5 9849	200	0.0735 2664	3 4633	11
62.9	0.3503 3513	6 0050	202	0.0731 8030	3 4644	10
63.0	0.3509 3563	6 0251	203	0.0728 3387	3 4654	10
63.1	0.3515 3814	6 0454	204	0.0724 8732	3 4664	10
63.2	0.3521 4268	6 0658	205	0.0721 4068	3 4674	9
63.3	0.3527 4925	6 0862	206	0.0717 9394	3 4683	9
63.4	0.3533 5787	6 1068	207	0.0714 4711	3 4692	8
63.5	0.3539 6856	6 1275	208	0.0711 0019	3 4700	8
63.6	0.3545 8131	6 1483	209	0.0707 5319	3 4708	8
63.7	0.3551 9614	6 1693	210	0.0704 0610	3 4716	7
63.8	0.3558 1307	6 1903	212	0.0700 5895	3 4723	7
63.9	0.3564 3211	6 2115	213	0.0697 1172	3 4729	6
64.0	0.3570 5325	6 2328	214	0.0693 6442	3 4736	6
64.1	0.3576 7653	6 2542	215	0.0690 1706	3 4741	5
64.2	0.3583 0195	6 2757	216	0.0686 6965	3 4747	5
64.3	0.3589 2952	6 2974	218	0.0683 2218	3 4752	4
64.4	0.3595 5926	6 3191	219	0.0679 7466	3 4756	4
64.5	0.3601 9117	6 3410	220	0.0676 2710	3 4760	4
64.6	0.3608 2527	6 3630	221	0.0672 7950	3 4764	3
64.7	0.3614 6158	6 3852	223	0.0669 3186	3 4767	3
64.8	0.3621 0009	6 4075	224	0.0665 8420	3 4769	2
64.9	0.3627 4084	6 4299	225	0.0662 3650	3 4772	2
65.0	0.3633 8383	6 4524	227	0.0658 8879	3 4773	1

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
65°0	0.3633 8383	6 4524	227	0.0658 8879	3 4773	1
65.1	0.3640 2907	6 4751	228	0.0655 4106	3 4774	1
65.2	0.3646 7658	6 4979	229	0.0651 9331	3 4775	+0
65.3	0.3653 2637	6 5209	231	0.0648 4556	3 4775	−0
65.4	0.3659 7846	6 5439	232	0.0644 9781	3 4775	1
65.5	0.3666 3286	6 5672	234	0.0641 5005	3 4775	1
65.6	0.3672 8957	6 5905	235	0.0638 0231	3 4773	2
65.7	0.3679 4863	6 6141	237	0.0634 5457	3 4772	2
65.8	0.3686 1003	6 6377	238	0.0631 0686	3 4769	3
65.9	0.3692 7380	6 6615	239	0.0627 5916	3 4767	3
66.0	0.3699 3995	6 6855	241	0.0624 1150	3 4764	4
66.1	0.3706 0850	6 7096	242	0.0620 6386	3 4760	4
66.2	0.3712 7946	6 7338	244	0.0617 1626	3 4756	5
66.3	0.3719 5284	6 7582	246	0.0613 6870	3 4751	5
66.4	0.3726 2866	6 7828	247	0.0610 2119	3 4746	6
66.5	0.3733 0694	6 8075	249	0.0606 7373	3 4740	6
66.6	0.3739 8768	6 8324	250	0.0603 2633	3 4734	7
66.7	0.3746 7092	6 8574	252	0.0599 7899	3 4727	7
66.8	0.3753 5666	6 8826	254	0.0596 3172	3 4720	8
66.9	0.3760 4492	6 9080	255	0.0592 8453	3 4712	8
67.0	0.3767 3572	6 9335	257	0.0589 3741	3 4703	9
67.1	0.3774 2907	6 9592	259	0.0585 9037	3 4695	9
67.2	0.3781 2499	6 9851	260	0.0582 4343	3 4685	10
67.3	0.3788 2349	7 0111	262	0.0578 9658	3 4675	11
67.4	0.3795 2460	7 0373	264	0.0575 4983	3 4664	11
67.5	0.3802 2833	7 0637	266	0.0572 0318	3 4653	12
67.6	0.3809 3471	7 0903	268	0.0568 5665	3 4642	12
67.7	0.3816 4373	7 1170	269	0.0565 1023	3 4629	13
67.8	0.3823 5544	7 1440	271	0.0561 6394	3 4617	13
67.9	0.3830 6984	7 1711	273	0.0558 1777	3 4603	14
68.0	0.3837 8695	7 1984	275	0.0554 7174	3 4589	15
68.1	0.3845 0679	7 2259	277	0.0551 2585	3 4575	15
68.2	0.3852 2938	7 2536	279	0.0547 8011	3 4559	16
68.3	0.3859 5475	7 2815	281	0.0544 3451	3 4544	16
68.4	0.3866 8290	7 3096	283	0.0540 8908	3 4527	17
68.5	0.3874 1386	7 3379	285	0.0537 4380	3 4510	18
68.6	0.3881 4765	7 3664	287	0.0533 9870	3 4493	18
68.7	0.3888 8429	7 3951	289	0.0530 5377	3 4475	19
68.8	0.3896 2380	7 4240	291	0.0527 0903	3 4456	19
68.9	0.3903 6620	7 4531	293	0.0523 6447	3 4436	20
69.0	0.3911 1152	7 4825	296	0.0520 2010	3 4416	21
69.1	0.3918 5977	7 5120	298	0.0516 7594	3 4396	21
69.2	0.3926 1097	7 5418	300	0.0513 3198	3 4375	22
69.3	0.3933 6515	7 5718	302	0.0509 8824	3 4353	23
69.4	0.3941 2234	7 6020	305	0.0506 4471	3 4330	23
69.5	0.3948 8254	7 6325	307	0.0503 0141	3 4307	24
69.6	0.3956 4579	7 6632	309	0.0499 5834	3 4283	24
69.7	0.3964 1211	7 6941	312	0.0496 1551	3 4259	25
69.8	0.3971 8152	7 7253	314	0.0492 7292	3 4233	26
69.9	0.3979 5405	7 7567	317	0.0489 3059	3 4208	26
70.0	0.3987 2972	7 7883	319	0.0485 8851	3 4181	27

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
70°0	0.3987 2972	7 7883	319	0.0485 8851	3 4181	27
70.1	0.3995 0855	7 8202	322	0.0482 4670	3 4154	28
70.2	0.4002 9058	7 8524	324	0.0479 0516	3 4126	29
70.3	0.4010 7582	7 8848	327	0.0475 6390	3 4098	29
70.4	0.4018 6430	7 9175	329	0.0472 2292	3 4068	30
70.5	0.4026 5605	7 9504	332	0.0468 8224	3 4039	31
70.6	0.4034 5109	7 9836	335	0.0465 4185	3 4008	31
70.7	0.4042 4945	8 0171	337	0.0462 0177	3 3977	32
70.8	0.4050 5116	8 0508	340	0.0458 6201	3 3945	33
70.9	0.4058 5625	8 0849	343	0.0455 2256	3 3912	33
71.0	0.4066 6474	8 1192	346	0.0451 8344	3 3879	34
71.1	0.4074 7666	8 1538	349	0.0448 4465	3 3844	35
71.2	0.4082 9204	8 1887	352	0.0445 0621	3 3810	36
71.3	0.4091 1090	8 2239	355	0.0441 6812	3 3774	36
71.4	0.4099 3329	8 2594	358	0.0438 3038	3 3738	37
71.5	0.4107 5923	8 2952	361	0.0434 9300	3 3700	38
71.6	0.4115 8875	8 3313	364	0.0431 5600	3 3663	39
71.7	0.4124 2187	8 3677	367	0.0428 1937	3 3624	39
71.8	0.4132 5864	8 4044	371	0.0424 8313	3 3585	40
71.9	0.4140 9909	8 4415	374	0.0421 4729	3 3544	41
72.0	0.4149 4324	8 4789	377	0.0418 1184	3 3504	42
72.1	0.4157 9112	8 5166	381	0.0414 7681	3 3462	42
72.2	0.4166 4279	8 5547	384	0.0411 4219	3 3419	43
72.3	0.4174 9826	8 5931	388	0.0408 0799	3 3376	44
72.4	0.4183 5757	8 6319	391	0.0404 7423	3 3332	45
72.5	0.4192 2076	8 6710	395	0.0401 4091	3 3287	46
72.6	0.4200 8786	8 7105	399	0.0398 0804	3 3241	46
72.7	0.4209 5891	8 7503	402	0.0394 7563	3 3195	47
72.8	0.4218 3394	8 7906	406	0.0391 4368	3 3148	48
72.9	0.4227 1300	8 8312	410	0.0388 1220	3 3099	49
73.0	0.4235 9612	8 8722	414	0.0384 8121	3 3050	50
73.1	0.4244 8334	8 9136	418	0.0381 5070	3 3001	51
73.2	0.4253 7470	8 9554	422	0.0378 2070	3 2950	52
73.3	0.4262 7023	8 9976	426	0.0374 9120	3 2898	52
73.4	0.4271 6999	9 0402	430	0.0371 6221	3 2846	53
73.5	0.4280 7401	9 0832	435	0.0368 3375	3 2793	54
73.6	0.4289 8233	9 1267	439	0.0365 0582	3 2739	55
73.7	0.4298 9499	9 1706	443	0.0361 7843	3 2684	56
73.8	0.4308 1205	9 2149	448	0.0358 5160	3 2628	57
73.9	0.4317 3354	9 2597	452	0.0355 2532	3 2571	58
74.0	0.4326 5950	9 3049	457	0.0351 9961	3 2513	59
74.1	0.4335 9000	9 3506	462	0.0348 7448	3 2455	60
74.2	0.4345 2506	9 3968	467	0.0345 4993	3 2395	60
74.3	0.4354 6474	9 4435	472	0.0342 2598	3 2335	61
74.4	0.4364 0909	9 4906	477	0.0339 0263	3 2273	62
74.5	0.4373 5815	9 5583	482	0.0335 7989	3 2211	63
74.6	0.4383 1198	9 5865	487	0.0332 5778	3 2148	64
74.7	0.4392 7063	9 6352	492	0.0329 3630	3 2084	65
74.8	0.4402 3414	9 6844	498	0.0326 1546	3 2019	66
74.9	0.4412 0258	9 7341	503	0.0322 9528	3 1952	67
75.0	0.4421 7599	9 7844	509	0.0319 7575	3 1885	68

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
75°0	0.4421 7599	9 7844	509	0.0319 7575	3 1885	68
75.1	0.4431 5444	9 8353	514	0.0316 5690	3 1817	69
75.2	0.4441 3797	9 8867	520	0.0313 3872	3 1748	70
75.3	0.4451 2664	9 9387	526	0.0310 2124	3 1678	71
75.4	0.4461 2051	9 9913	532	0.0307 0446	3 1607	72
75.5	0.4471 1965	10 0446	538	0.0303 8839	3 1535	73
75.6	0.4481 2410	10 0984	544	0.0300 7304	3 1462	74
75.7	0.4491 3394	10 1528	551	0.0297 5842	3 1388	75
75.8	0.4501 4922	10 2079	557	0.0294 4454	3 1313	76
75.9	0.4511 7001	10 2637	564	0.0291 3141	3 1237	77
76.0	0.4521 9638	10 3201	571	0.0288 1904	3 1159	78
76.1	0.4532 2839	10 3771	578	0.0285 0745	3 1081	79
76.2	0.4542 6610	10 4349	585	0.0281 9664	3 1002	80
76.3	0.4553 0959	10 4934	592	0.0278 8663	3 0921	82
76.4	0.4563 5893	10 5526	599	0.0275 7742	3 0839	83
76.5	0.4574 1419	10 6126	607	0.0272 6902	3 0757	84
76.6	0.4584 7545	10 6733	615	0.0269 6145	3 0673	85
76.7	0.4595 4278	10 7347	622	0.0266 5472	3 0588	86
76.8	0.4606 1625	10 7970	630	0.0263 4884	3 0502	87
76.9	0.4616 9594	10 8600	639	0.0260 4382	3 0415	88
77.0	0.4627 8195	10 9239	647	0.0257 3967	3 0327	89
77.1	0.4638 7433	10 9886	656	0.0254 3640	3 0237	91
77.2	0.4649 7319	11 0541	664	0.0251 3403	3 0147	92
77.3	0.4660 7860	11 1206	673	0.0248 3257	3 0055	93
77.4	0.4671 9066	11 1879	682	0.0245 3202	2 9962	94
77.5	0.4683 0945	11 2561	692	0.0242 3240	2 9868	95
77.6	0.4694 3506	11 3253	701	0.0239 3372	2 9772	97
77.7	0.4705 6760	11 3954	711	0.0236 3600	2 9676	98
77.8	0.4717 0714	11 4665	721	0.0233 3925	2 9578	99
77.9	0.4728 5379	11 5386	731	0.0230 4347	2 9479	100
78.0	0.4740 0766	11 6118	742	0.0227 4868	2 9378	102
78.1	0.4751 6884	11 6860	753	0.0224 5490	2 9277	103
78.2	0.4763 3743	11 7612	764	0.0221 6213	2 9174	104
78.3	0.4775 1355	11 8376	775	0.0218 7039	2 9070	105
78.4	0.4786 9731	11 9150	786	0.0215 7969	2 8964	107
78.5	0.4798 8881	11 9937	798	0.0212 9005	2 8858	108
78.6	0.4810 8818	12 0735	810	0.0210 0148	2 8750	109
78.7	0.4822 9553	12 1545	823	0.0207 1398	2 8640	111
78.8	0.4835 1098	12 2368	835	0.0204 2758	2 8529	112
78.9	0.4847 3466	12 3203	848	0.0201 4229	2 8417	113
79.0	0.4859 6669	12 4052	862	0.0198 5811	2 8304	115
79.1	0.4872 0721	12 4914	876	0.0195 7507	2 8189	116
79.2	0.4884 5635	12 5789	890	0.0192 9318	2 8073	118
79.3	0.4897 1424	12 6679	904	0.0190 1246	2 7955	119
79.4	0.4909 8103	12 7583	919	0.0187 3291	2 7836	120
79.5	0.4922 5687	12 8503	934	0.0184 5454	2 7716	122
79.6	0.4935 4189	12 9437	950	0.0181 7739	2 7594	123
79.7	0.4948 3626	13 0387	966	0.0179 0145	2 7470	125
79.8	0.4961 4013	13 1353	983	0.0176 2675	2 7345	126
79.9	0.4974 5367	13 2336	1000	0.0173 5330	2 7219	128
80.0	0.4987 7703	13 3336	1018	0.0170 8111	2 7091	129

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
80°0	0.4987 7703	13 3336	1018	0.0170 8111	2 7091	129
80.1	0.5001 1040	13 4354	1036	0.0168 1020	2 6962	131
80.2	0.5014 5394	13 5390	1054	0.0165 4058	2 6831	132
80.3	0.5028 0783	13 6444	1073	0.0162 7227	2 6698	134
80.4	0.5041 7227	13 7517	1093	0.0160 0529	2 6564	136
80.5	0.5055 4744	13 8610	1113	0.0157 3965	2 6429	137
80.6	0.5069 3354	13 9724	1134	0.0154 7536	2 6291	139
80.7	0.5083 3078	14 0858	1156	0.0152 1245	2 6153	140
80.8	0.5097 3936	14 2014	1178	0.0149 5092	2 6012	142
80.9	0.5111 5949	14 3192	1201	0.0146 9080	2 5870	144
81.0	0.5125 9141	14 4393	1225	0.0144 3210	2 5726	145
81.1	0.5140 3534	14 5617	1249	0.0141 7484	2 5581	147
81.2	0.5154 9151	14 6867	1274	0.0139 1903	2 5433	149
81.3	0.5169 6018	14 8141	1300	0.0136 6470	2 5285	151
81.4	0.5184 4159	14 9441	1327	0.0134 1185	2 5134	152
81.5	0.5199 3600	15 0769	1355	0.0131 6052	2 4981	154
81.6	0.5214 4369	15 2124	1384	0.0129 1070	2 4827	156
81.7	0.5229 6493	15 3508	1414	0.0126 6243	2 4671	158
81.8	0.5245 0001	15 4922	1445	0.0124 1572	2 4513	160
81.9	0.5260 4923	15 6366	1477	0.0121 7058	2 4354	162
82.0	0.5276 1289	15 7843	1510	0.0119 2704	2 4192	163
82.1	0.5291 9132	15 9352	1544	0.0116 8512	2 4029	165
82.2	0.5307 8485	16 0896	1579	0.0114 4483	2 3863	167
82.3	0.5323 9381	16 2476	1616	0.0112 0620	2 3696	169
82.4	0.5340 1857	16 4092	1655	0.0109 6924	2 3527	171
82.5	0.5356 5949	16 5747	1694	0.0107 3397	2 3356	173
82.6	0.5373 1696	16 7441	1736	0.0105 0041	2 3183	175
82.7	0.5389 9137	16 9177	1779	0.0102 6859	2 3007	177
82.8	0.5406 8313	17 0955	1823	0.0100 3851	2 2830	179
82.9	0.5423 9268	17 2778	1870	0.0098 1021	2 2651	181
83.0	0.5441 2047	17 4648	1918	0.0095 8371	2 2469	184
83.1	0.5458 6695	17 6566	1968	0.0093 5902	2 2285	186
83.2	0.5476 3260	17 8534	2021	0.0091 3616	2 2100	188
83.3	0.5494 1795	18 0555	2076	0.0089 1517	2 1912	190
83.4	0.5512 2350	18 2631	2133	0.0086 9605	2 1721	193
83.5	0.5530 4980	18 4764	2193	0.0084 7884	2 1529	195
83.6	0.5548 9744	18 6956	2255	0.0082 6355	2 1334	197
83.7	0.5567 6700	18 9211	2320	0.0080 5021	2 1137	199
83.8	0.5586 5912	19 1532	2389	0.0078 3884	2 0937	202
83.9	0.5605 7443	19 3921	2460	0.0076 2947	2 0735	204
84.0	0.5625 1364	19 6381	2535	0.0074 2211	2 0531	207
84.1	0.5644 7745	19 8916	2614	0.0072 1680	2 0324	209
84.2	0.5664 6661	20 1531	2697	0.0070 1356	2 0115	212
84.3	0.5684 8192	20 4228	2784	0.0068 1241	1 9903	214
84.4	0.5705 2420	20 7012	2875	0.0066 1338	1 9689	217
84.5	0.5725 9431	20 9887	2972	0.0064 1649	1 9472	220
84.6	0.5746 9318	21 2859	3073	0.0062 2177	1 9252	222
84.7	0.5768 2177	21 5932	3180	0.0060 2925	1 9029	225
84.8	0.5789 8109	21 9112	3293	0.0058 3896	1 8804	228
84.9	0.5811 7221	22 2405	3413	0.0056 5092	1 8576	231
85.0	0.5833 9626	22 5818	3539	0.0054 6516	1 8345	234

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
85°0	0.5833 9626	22 5818	3539	0.0054 6516	1 8345	234
85.1	0.5856 5444	22 9357	3673	0.0052 8171	1 8111	237
85.2	0.5879 4801	23 3031	3816	0.0051 0060	1 7874	240
85.3	0.5902 7832	23 6846	3967	0.0049 2185	1 7634	243
85.4	0.5926 4679	24 0813	4127	0.0047 4551	1 7391	246
85.5	0.5950 5492	24 4940	4299	0.0045 7160	1 7145	249
85.6	0.5975 0432	24 9239	4481	0.0044 0015	1 6896	253
85.7	0.5999 9671	25 3720	4676	0.0042 3119	1 6643	256
85.8	0.6025 3391	25 8396	4885	0.0040 6476	1 6387	260
85.9	0.6051 1788	26 3281	5109	0.0039 0089	1 6127	263
86.0	0.6077 5069	26 8390	5349	0.0037 3962	1 5864	267
86.1	0.6104 3459	27 3739	5607	0.0035 8097	1 5598	270
86.2	0.6131 7198	27 9346	5886	0.0034 2499	1 5327	274
86.3	0.6159 6543	28 5231	6186	0.0032 7172	1 5053	278
86.4	0.6188 1775	29 1418	6512	0.0031 2118	1 4775	282
86.5	0.6217 3193	29 7929	6865	0.0029 7343	1 4493	286
86.6	0.6247 1122	30 4794	7248	0.0028 2850	1 4207	290
86.7	0.6277 5916	31 2042	7667	0.0026 8642	1 3917	295
86.8	0.6308 7958	31 9709	8124	0.0025 4725	1 3622	299
86.9	0.6340 7668	32 7834	8626	0.0024 1103	1 3323	304
87.0	0.6373 5501	33 6459	9177	0.0022 7779	1 3020	308
87.1	0.6407 1961	34 5636	9785	0.0021 4759	1 2712	313
87.2	0.6441 7597	35 5422	10459	0.0020 2048	1 2398	318
87.3	0.6477 3019	36 5881	11208	0.0018 9649	1 2080	324
87.4	0.6513 8900	37 7089	12043	0.0017 7569	1 1757	329
87.5	0.6551 5989	38 9132	12980	0.0016 5813	1 1428	335
87.6	0.6590 5121	40 2112	14035	0.0015 4385	1 1093	340
87.7	0.6630 7233	41 6147	15230	0.0014 3292	1 0753	347
87.8	0.6672 3380	43 1377	16590	0.0013 2540	1 0406	353
87.9	0.6715 4757	44 7967	18149	0.0012 2134	1 0053	360
88.0	0.6760 2724	46 6116	19948	0.0011 2081	9693	367
88.1	0.6806 8840	48 6064	22040	0.0010 2387	9327	374
88.2	0.6855 4904	50 8104	24492	0.0009 3060	8953	382
88.3	0.6906 3009	53 2597	27396	0.0008 4107	8571	390
88.4	0.6959 5605	55 9993	30870	0.0007 5536	8181	399
88.5	0.7015 5598	59 0862	35077	0.0006 7355	7782	408
88.6	0.7074 6460	62 5940	40245	0.0005 9573	7374	418
88.7	0.7137 2400	66 6184	46693	0.0005 2199	6956	429
88.8	0.7203 8584	71 2878	54895	0.0004 5242	6527	441
88.9	0.7275 1462	76 7773	65561	0.0003 8715	6087	453
89.0	0.7351 9234	83 3334	79812	0.0003 2628	5633	467
89.1	0.7435 2568	91 3146	99496	0.0002 6995	5166	483
89.2	0.7526 5714	101 2642	127847	0.0002 1829	4683	501
89.3	0.7627 8356	114 0489	170975	0.0001 7146	4181	522
89.4	0.7741 8844	131 1464	241655	0.0001 2965	3660	546
89.5	0.7873 0308	155 3119	370693	0.0000 9305	3114	576
89.6	0.8028 3427	192 3813	650756	0.0000 6192	2538	615
89.7	0.8220 7240	257 4569	1501510	0.0000 3654	1923	670
89.8	0.8478 1809	407 6079		0.0000 1731	1253	774
89.9	0.8885 7889			0.0000 0479	479	
90.0	Inf.			0.0000 0000		

The preceding table of logarithms of the elliptic integrals of the first and second kinds is taken from Legendre's *Traité des Fonctions Elliptiques*, volume 2, Table I. The values from 45° to 90° are given for intervals of 0°.1. The values from 0° to 45°, which are comparatively seldom required, have been omitted. For formula and table to be used in interpolation, see page 214.

TABLE XIV

Binominal Coefficients for Interpolation by Differences

k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3	
	K_2	K_3		K_2	K_3		K_2	K_3		K_2	K_3
0.01	−0.005	+0.003	0.26	−0.096	+0.056	0.51	−0.125	+0.062	0.76	−0.091	+0.038
.02	− .010	+ .006	.27	− .099	+ .057	.52	− .125	+ .062	.77	− .089	+ .036
.03	− .015	+ .010	.28	− .101	+ .058	.53	− .125	+ .061	.78	− .086	+ .035
.04	− .019	+ .013	.29	− .103	+ .059	.54	− .124	+ .060	.79	− .083	+ .033
.05	− .024	+ .015	.30	− .105	+ .060	.55	− .124	+ .060	.80	− .080	+ .032
.06	− .028	+ .018	.31	− .107	+ .060	.56	− .124	+ .059	.81	− .077	+ .031
.07	− .033	+ .021	.32	− .109	+ .061	.57	− .123	+ .058	.82	− .074	+ .029
.08	− .037	+ .024	.33	− .111	+ .062	.58	− .122	+ .058	.83	− .071	+ .028
.09	− .041	+ .026	.34	− .112	+ .062	.59	− .121	+ .057	.84	− .067	+ .026
.10	− .045	+ .028	.35	− .114	+ .063	.60	− .120	+ .056	.85	− .064	+ .024
.11	− .049	+ .031	.36	− .115	+ .063	.61	− .119	+ .055	.86	− .060	+ .023
.12	− .053	+ .033	.37	− .117	+ .063	.62	− .118	+ .054	.87	− .057	+ .021
.13	− .057	+ .035	.38	− .118	+ .064	.63	− .117	+ .053	.88	− .053	+ .020
.14	− .060	+ .037	.39	− .119	+ .064	.64	− .115	+ .052	.89	− .049	+ .018
.15	− .064	+ .039	.40	− .120	+ .064	.65	− .114	+ .051	.90	− .045	+ .016
.16	− .067	+ .041	.41	− .121	+ .064	.66	− .112	+ .050	.91	− .041	+ .015
.17	− .071	+ .043	.42	− .122	+ .064	.67	− .111	+ .049	.92	− .037	+ .013
.18	− .074	+ .045	.43	− .123	+ .064	.68	− .109	+ .048	.93	− .033	+ .012
.19	− .077	+ .046	.44	− .123	+ .064	.69	− .107	+ .047	.94	− .028	+ .010
.20	− .080	+ .048	.45	− .124	+ .064	.70	− .105	+ .045	.95	− .024	+ .008
.21	− .083	+ .049	.46	− .124	+ .064	.71	− .103	+ .044	.96	− .019	+ .007
.22	− .086	+ .051	.47	− .125	+ .064	.72	− .101	+ .043	.97	− .015	+ .005
.23	− .089	+ .052	.48	− .125	+ .063	.73	− .099	+ .042	.98	− .010	+ .003
.24	− .091	+ .053	.49	− .125	+ .063	.74	− .096	+ .040	.99	− .005	+ .002
.25	− .094	+ .055	.50	− .125	+ .063	.75	− .094	+ .039	1.00	− .000	+ .000

INTERPOLATION FORMULA

$$f(a+h)=f(a)+k\triangle_1+\frac{k(k-1)}{2\,!}\triangle_2+\frac{k(k-1)(k-2)}{3\,!}\triangle_3$$
$$+\frac{k(k-1)(k-2)(k-3)}{4\,!}\triangle_4+\cdots\cdot \qquad (a)$$

$$\text{or, } f(a+h)=f(a)+k\triangle_1+K_2\triangle_2+K_3\triangle_3+\cdots\cdot \qquad (b)$$

where the constants K_2 and K_3 are given in the above table as functions of k and

$$k=\frac{h}{\delta}$$

where h is the remainder above the value of a for which the function is given in the table, and δ is the increment of a in the table.

ILLUSTRATION

To find the value of log F for $49°\ 15'\ 36''=49°.260$

For $49°.2$ $\quad \log F=0.2836\ 3130=f(a)$

$h=.06,\quad \delta=0.1 \quad k=0.6$

From Table XIV, $\quad K_2=-.120$

$K_3=+.056$

From Table XIII,

$\triangle_1=39338$

$\triangle_2=\ \ 117$

$\triangle_3=\ \ 1$

Substituting these values of K_2, K_3, $\triangle_1$, $\triangle_2$, $\triangle_3$ in formula (b) above we have as the value of log F for the given angle

$$\log F=0.2836\ 3130+0.0002\ 3603-0.0000\ 0014=0.2838\ 6719.$$

TABLE XV

Values of the Quantities $q-\frac{l}{2}$ *or* $q_1-\frac{l_1}{2}$ *and* $Log_{10}\,(1+\epsilon)$ *with Argument* q *or* q_1

$$q-\frac{l}{2}=2\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+\cdots\cdot$$

$$\epsilon=3q^4-4q^6+9q^8-12q^{10}+\cdots\cdot$$

$$q_1-\frac{l_1}{2}=2\left(\frac{l_1}{2}\right)^5+15\left(\frac{l_1}{2}\right)^9+\cdots\cdot$$

(For use with Formulas (8), (9), (45), (76), (77), and (78))

q or q_1	$q-\frac{l}{2}$ or $q_1-\frac{l_1}{2}$	Δ	ϵ	Δ	$Log_{10}\,(1+\epsilon)$	Δ
0.020	0.000 00001	0	0.000 00048	22	0.000 00021	9
.022	.000 00001	1	.000 00070	29	.000 00030	13
.024	.000 00002	0	.000 00099	38	.000 00043	16
.026	.000 00002	1	.000 00137	47	.000 00059	21
.028	.000 00003	2	.000 00184	59	.000 00080	25
0.030	0.000 00005	2	0.000 00243	71	0.000 00105	31
.032	.000 00007	2	.000 00314	86	.000 00136	38
.034	.000 00009	3	.000 00400	103	.000 00174	44
.036	.000 00012	4	.000 00503	121	.000 00218	53
.038	.000 00016	5	.000 00624	142	.000 00271	61
0.040	0.000 00021	5	0.000 00766	165	0.000 00332	72
.042	.000 00026	7	.000 00931	191	.000 00404	83
.044	.000 00033	8	.000 01122	217	.000 00487	94
.046	.000 00041	10	.000 01339	249	.000 00581	109
.048	.000 00051	12	.000 01588	280	.000 00690	122
0.050	0.000 00063	13	0.000 01868	318	0.000 00812	138
.052	.000 00076	16	.000 02186	355	.000 00950	154
.054	.000 00092	18	.000 02541	397	.000 01104	172
.056	.000 00110	21	.000 02938	442	.000 01276	192
.058	.000 00131	25	.000 03380	490	.000 01468	213
0.060	0.000 00156	27	0.000 03870	540	0.000 01681	234
.062	.000 00183	32	.000 04410	596	.000 01915	259
.064	.000 00215	36	.000 05006	654	.000 02174	283
.066	.000 00251	40	.000 05660	715	.000 02457	312
.068	.000 00291	45	.000 06375	781	.000 02769	339
0.070	0.000 00336	51	0.000 07156	851	0.000 03108	369
.072	.000 00387	57	.000 08007	924	.000 03477	401
.074	.000 00444	63	.000 08931	1002	.000 03878	436
.076	.000 00507	70	.000 09933	1083	.000 04314	470
.078	.000 00577	78	.000 11016	1169	.000 04784	509

TABLE XV—Continued

q or q_1	$q-\frac{1}{2}$ or $q_1-\frac{1}{2}$	Δ	ε	Δ	$\text{Log}_{10}(1+\varepsilon)$	Δ
0.080	0.000 00655	86	0.000 12185	1259	0.000 05293	545
.082	.000 00741	95	.000 13444	1354	.000 05838	588
.084	.000 00836	105	.000 14798	1453	.000 06426	631
.086	.000 00941	114	.000 16251	1557	.000 07057	676
.088	.000 01055	126	.000 17808	1666	.000 07733	724
0.090	0.000 01181	137	0.000 19474	1779	0.000 08457	772
.092	.000 01318	150	.000 21253	1899	.000 09229	825
.094	.000 01468	162	.000 23152	2022	.000 10054	878
.096	.000 01630	177	.000 25174	2150	.000 10932	937
.098	.000 01807	193	.000 27324	2285	.000 11869	988
0.100	0.000 02000	102	0.000 29609	1194	0.000 12857	519
.101	.000 02102	106	.000 30803	1230	.000 13376	533
.102	.000 02208	110	.000 32033	1266	.000 13909	550
.103	.000 02318	115	.000 33299	1303	.000 14459	566
.104	.000 02433	119	.000 34602	1340	.000 15025	582
0.105	0.000 02552	123	0.000 35942	1379	0.000 15607	598
.106	.000 02675	129	.000 37321	1410	.000 16205	616
.107	.000 02804	134	.000 38731	1465	.000 16821	632
.108	.000 02938	138	.000 40196	1498	.000 17453	651
.109	.000 03076	144	.000 41694	1539	.000 18104	668
0.110	0.000 03220	149	0.000 43233	1581	0.000 18772	686
.111	.000 03369	154	.000 44814	1624	.000 19458	705
.112	.000 03523	160	.000 46438	1667	.000 20163	724
.113	.000 03683	166	.000 48105	1711	.000 20887	742
.114	.000 03849	172	.000 49816	1756	.000 21629	762
0.115	0.000 04021	178	0.000 51572	1802	0.000 22391	783
.116	.000 04199	184	.000 53374	1848	.000 23174	802
.117	.000 04383	191	.000 55222	1895	.000 23976	823
.118	.000 04574	196	.000 57117	1943	.000 24799	843
.119	.000 04770	204	.000 59060	1992	.000 25642	865
0.120	0.000 04974	210	0.000 61052	2041	0.000 26507	885
.121	.000 05184	218	.000 63093	2091	.000 27392	908
.122	.000 05402	226	.000 65184	2143	.000 28300	930
.123	.000 05628	232	.000 67327	2195	.000 29230	953
.124	.000 05860	240	.000 69522	2247	.000 30183	975
0.125	0.000 06100	248	0.000 71769	2301	0.000 31158	998
.126	.000 06348	255	.000 74070	2355	.000 32156	1022
.127	.000 06603	265	.000 76425	2410	.000 33178	1046
.128	.000 06868	272	.000 78835	2466	.000 34224	1071
.129	.000 07140	280	.000 81301	2523	.000 35295	1094

TABLE XV—Continued

q or q_1	$q-\frac{1}{2}$ or $q_1-\frac{l_1}{2}$	Δ	ε	Δ	$\text{Log}_{10}(1+\varepsilon)$	Δ
0.130	0.000 07420	290	0.000 83824	2581	0.000 36389	1120
.131	.000 07710	299	.000 86405	2639	.000 37509	1145
.132	.000 08009	308	.000 89044	2698	.000 38654	1171
.133	.000 08317	317	.000 91742	2759	.000 39825	1196
.134	.000 08634	327	.000 94501	2820	.000 41021	1224
0.135	0.000 08961	336	0.000 97321	2881	0.000 42245	1251
.136	.000 09297	347	.001 00202	2945	.000 43496	1277
.137	.000 09644	357	.001 03147	3012	.000 44773	1305
.138	.000 10001	367	.001 06155	3073	.000 46078	1333
.139	.000 10368	378	.001 09228	3138	.000 47411	1362
0.140	0.000 10746	389	0.001 12366	3204	0.000 48773	1389
.141	.000 11135	401	.001 15570	3272	.000 50162	1420
.142	.000 11536	411	.001 18842	3339	.000 51582	1448
.143	.000 11947	423	.001 22181	3409	.000 53030	1479
.144	.000 12370	435	.001 25590	3479	.000 54509	1509
0.145	0.000 12805	448	0.001 29069	3549	0.000 56018	1539
.146	.000 13253	459	.001 32618	3621	.000 57557	1571
.147	.000 13712	473	.001 36239	3694	.000 59128	1602
.148	.000 14185	485	.001 39933	3768	.000 60730	1634
.149	.000 14670	498	.001 43701	3842	.000 62364	1666
0.150	0.000 15168		0.001 47543		0.000 64030	

Tables XV and XVI are reproduced from Nagaoka's paper; see footnote, page 12.

TABLE XVI

Values of e_1 and $-e_1'$ with Argument q_1

$$e_1 = 32\,q_1^3 - 40\,q_1^4 + 48\,q_1^5 - \cdots$$
$$-e_1' = 8\,q_1^2 - e_1.$$

(For use with Formulas (9) and (9a))

q_1	e_1	Δ	$-e_1'$	Δ
0.0100	0.000 03160	93	0.000 76840	1499
.0099	.000 03067	92	.000 75341	1484
.0098	.000 02975	89	.000 73857	1471
.0097	.000 02886	89	.000 72386	1455
.0096	.000 02797	86	.000 70931	1442
0.0095	0.000 02711	84	0.000 69489	1428
.0094	.000 02627	83	.000 68061	1413
.0093	.000 02544	81	.000 66648	1399
.0092	.000 02463	78	.000 65249	1386
.0091	.000 02385	78	.000 63863	1370
0.0090	0.000 02307	76	0.000 62493	1356
.0089	.000 02231	74	.000 61137	1342
.0088	.000 02157	73	.000 59795	1327
.0087	.000 02084	71	.000 58468	1313
.0086	.000 02013	69	.000 57155	1299
0.0085	0.000 01944	67	0.000 55856	1285
.0084	.000 01877	67	.000 54571	1269
.0083	.000 01810	64	.000 53302	1256
.0082	.000 01746	62	.000 52046	1242
.0081	.000 01684	62	.000 50804	1226
0.0080	0.000 01622	60	0.000 49578	1212
.0079	.000 01562	59	.000 48366	1197
.0078	.000 01503	56	.000 47169	1184
.0077	.000 01447	55	.000 45985	1169
.0076	.000 01392	55	.000 44816	1153
0.0075	0.000 01337	52	0.000 43663	1140
.0074	.000 01285	51	.000 42523	1125
.0073	.000 01234	51	.000 41398	1109
.0072	.000 01183	48	.000 40289	1096
.0071	.000 01135	47	.000 39193	1081
0.0070	0.000 01088	46	0.000 38112	1066
.0069	.000 01042	45	.000 37046	1051
.0068	.000 00997	43	.000 35995	1037
.0067	.000 00954	42	.000 34958	1022
.0066	.000 00912	40	.000 33936	1008
0.0065	0.000 00872	40	0.000 32928	992
.0064	.000 00832	38	.000 31936	978
.0063	.000 00794	37	.000 30958	963
.0062	.000 00757	37	.000 29995	947
.0061	.000 00720	34	.000 29048	934

TABLE XVI—Continued

q_1	e_1	Δ	$-e_1'$	Δ
0.0060	0.000 00686	34	0.000 28114	918
.0059	.000 00652	33	.000 27196	903
.0058	.000 00619	30	.000 26293	890
.0057	.000 00589	31	.000 25403	873
.0056	.000 00558	30	.000 24530	858
0.0055	0.000 00528	27	0.000 23672	845
.0054	.000 00501	28	.000 22827	828
.0053	.000 00473	26	.000 21999	814
.0052	.000 00447	26	.000 21185	798
.0051	.000 00421	24	.000 20387	784
0.0050	0.000 00397	23	0.000 19603	769
.0049	.000 00374	22	.000 18834	754
.0048	.000 00352	22	.000 18080	738
.0047	.000 00330	21	.000 17342	723
.0046	.000 00309	19	.000 16619	709
0.0045	0.000 00290	18	0.000 15910	694
.0044	.000 00272	19	.000 15216	677
.0043	.000 00253	17	.000 14539	663
.0042	.000 00236	16	.000 13876	648
.0041	.000 00220	16	.000 13228	632
0.0040	0.000 00204	15	0.000 12596	617
.0039	.000 00189	14	.000 11979	602
.0038	.000 00175	14	.000 11377	586
.0037	.000 00161	13	.000 10791	571
.0036	.000 00148	12	.000 10220	556
0.0035	0.000 00136	11	0.000 09664	541
.0034	.000 00125	11	.000 09123	525
.0033	.000 00114	9	.000 08598	511
.0032	.000 00105	10	.000 08087	494
.0031	.000 00095	9	.000 07593	479
0.0030	0.000 00086	8	0.000 07114	464
.0029	.000 00078	8	.000 06650	448
.0028	.000 00070	7	.000 06202	433
.0027	.000 00063	7	.000 05769	417
.0026	.000 00056	6	.000 05352	402
0.0025	0.000 00050	6	0.000 04950	386
.0024	.000 00044	5	.000 04564	371
.0023	.000 00039	5	.000 04193	355
.0022	.000 00034	4	.000 03838	340
.0021	.000 00030	4	.000 03498	324
0.0020	0.000 00026	4	0.000 03174	308
.0019	.000 00022	3	.000 02866	293
.0018	.000 00019	3	.000 02573	277
.0017	.000 00016	3	.000 02296	261
.0016	.000 00013	2	.000 02035	246

TABLE XVI—Continued

q_1	e_1	Δ	$-e_1'$	Δ
0.0015	0.000 00011	2	0.000 01789	230
.0014	.000 00009	2	.000 01559	214
.0013	.000 00007	1	.000 01345	199
.0012	.000 00006	2	.000 01146	182
.0011	.000 00004	1	.000 00964	167
0.0010	0.000 00003	1	0.000 00797	151
.0009	.000 00002	0	.000 00646	136
.0008	.000 00002	1	.000 00510	119
.0007	.000 00001	0	.000 00391	104
.0006	.000 00001	1	.000 00287	87
0.0005	0.000 00000		0.000 00200	72
.0004	.000 00000		.000 00128	56
.0003	.000 00000		.000 00072	40
.0002	.000 00000		.000 00032	24
.0001	.000 00000		.000 00008	

TABLE XVII

Coefficients of the Hypergeometric Series in Formula (18)

Series	a_1	a_2	a_3
$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$	0.069 4444	0.035 5260	0.023 8485
$F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$	−0.097 2222	−0.047 0358	−0.031 0523
$F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$	0.300 9259	0.177 6300	0.126 0562
$F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$	0.356 4814	0.216 3645	0.155 2615

TABLE XVIII

Showing the Location and Magnitude of the Positive and Negative Maxima and the Positions of the Roots of the Coefficients in Gray's and Searle and Airey's Formulas

(For use in Formulas (40), (43), and (56))

$\frac{x}{A}$	X_2	$\frac{x}{A}$	X_4	$\frac{x}{A}$	X_6	$\frac{x}{A}$	X_8
0	3.0000	0	2.5000	0	2.1875	0	1.9688
0.8660	0	0.531	0	0.3898	0	0.3000	0
∞	−∞	1.118	−3.750	0.6961	− 1.8273	0.5162	− 1.2177
		1.489	0	0.9203	0	0.687	0
		∞	+∞	1.737	+30.69	1.1521	+ 7.364
				2.063	0	1.268	0
				∞	−∞	2.309	−570.97
						2.613	0
						∞	+∞

$\frac{x}{A}$	X_{10}	$\frac{x}{A}$	X_{12}	$\frac{x}{A}$	X_{14}
0	1.8407	0	1.6758	0	1.5710
0.2575	0	0.2193	0	0.1936	0
0.4145	− 0.924	0.3466	− 0.756	0.2992	− 0.6446
0.5520	0	0.4629	0	0.4010	0
0.8006	+ 3.428	0.6475	+ 2.000	0.5460	+ 1.396
0.9386	0	0.7627	0	0.6439	0
1.413	−60.80	1.052	− 18.20	0.8515	− 8.166
1.589	0	1.145	0	0.9559	0
2.862	+18892	1.734	+836.1	1.289	+154.6
3.158	0	1.902	0	1.414	0
∞	−∞	3.406	−963500	2.044	−16993
		3.618	0	2.207	0
		∞	+∞	3.950	+70850000
				4.226	0
				∞	−∞

The function X_{2n} has n roots, between which values it makes oscillations of ever-increasing amplitude, and for values of $\frac{x}{A}$ greater than the largest root the function increases rapidly without limit. The functions L_{2n} have the same form as X_{2n}, $\frac{l}{a}$ being the variable instead of $\frac{x}{A}$.

TABLE XIX

Values of Coefficients in Gray's and Searle and Airey's Formulas

(For use in Formulas (40), (43), and (56))

$\frac{x}{A}$	X_2	X_4	X_6	X_8	X_{10}	X_{12}	X_{14}
0.0	+ 3.000	+ 2.500	+ 2.188	+ 1.969	+ 1.841	+ 1.676	+ 1.571
0.1	2.960	2.400	2.015	1.712	1.494	1.234	+ 1.032
0.2	2.840	2.106	1.521	1.017	+ 0.618	+ 0.216	− 0.073
0.3	2.640	1.632	+ 0.780	+ 0.090	− 0.355	− 0.635	− 0.645
0.4	2.360	1.002	− 0.090	− 0.764	− 0.874	− 0.596	− 0.0098
0.5	2.000	+ 0.250	− 0.938	− 1.203	− 0.526	+ 0.483	+ 1.208
0.6	1.560	− 0.580	− 1.577	− 0.909	+ 0.760	+ 1.793	+ 1.000
0.7	1.040	− 1.438	− 1.814	+ 0.228	2.467	+ 1.662	− 3.175
0.8	+ 0.440	− 2.262	− 1.452	+ 2.207	3.423	− 1.733	− 7.231
0.9	− 0.240	− 2.976	− 0.335	+ 4.606	+ 1.924	− 8.748	− 6.811
1.0	− 1.000	− 3.500	+ 1.688	+ 6.719	− 3.878	− 16.46	+ 10.48
1.1	− 1.840	− 3.750	4.673	7.240			
1.2	− 2.760	− 3.606	8.589	+ 4.509	− 31.72	+ 22.27	+119.6
1.3	− 3.760	− 2.976	13.28	− 3.595			
1.4	− 4.840	− 1.734	18.44	− 19.49	− 60.66		+ 29.1
1.5	− 6.000	+ 0.250	+ 23.56	− 46.24	− 48.96	+765.7	−486.5
1.6	− 7.240	+ 3.114	27.90	− 89.42			
1.7	− 8.560	7.008	30.46	− 137.4	+151.0	+818.1	
1.8	− 9.960	12.09	29.83	− 205.2			
1.9	−11.44	18.53	24.50	− 285.9		+ 21.8	
2.0	−13.00	+ 26.50	+ 12.19	− 375.0	+1591	− 1969	−16740
2.1	−14.64	36.55	− 9.64	− 464.9			
2.2	−16.36	47.80	− 44.09	− 538.3	4059	−14090	−1840
2.3	−18.16	62.54	− 104.9	− 570.9			
2.4	−20.04	77.61	− 166.3	− 535.5			
2.5	−22.00	+ 96.24	− 263.4	− 386.7	10908	−80050	
2.6	−24.04	117.6	− 390.4	− 64.4			+ 658400
2.7	−26.16	142.2	− 559.0	+ 505.9			
2.8	−28.36	169.0	− 801.2	1433	18390	−222400	
2.9	−30.64	201.3	−1039.1	2833			
3.0	−33.00	+ 236.5	−1370.3	+4869	+15797	−509200	19132000
3.25	−38.25	+ 343.1	−2553	+14118			
3.5	−46.00	+ 480.2	−4414	+33030	−146970	−893400	33670000
3.75	−53.25	+ 653.1	−7215	+68410		+265600	
4.0	−61.00	+ 866.5	−11286	+130400	-1.229×10^{6}	$+6.625\times10^{6}$	59080000
4.25	−69.25						−16530000
4.5	−77.00	+1440.3	−24956	+399000	-5.683×10^{6}	$+5.972\times10^{7}$	-4.172×10^{8}
5.0	−97.00	+2252.5	−49810	+1038700	-2.007×10^{7}	$+3.463\times10^{8}$	-4.855×10^{9}

This table used in conjunction with the preceding should make it possible to investigate the convergence of Gray's or Searle and Airey's formula in any given case. It will also facilitate calculations by these formulas when $\frac{x}{A}$ has one of the values included in the table. This table gives also the values of the L_{2n} coefficients if $\frac{l}{a}$ be taken as argument in place of $\frac{x}{A}$.

TABLE XX

Nagaoka's Table of Values of the Correction Factor for the Ends K, as a Function of the Angle $\Theta = \tan^{-1} \frac{2a}{b}$

(For use in Formula (75))

θ	K	Δ_1	Δ_2	θ	K	Δ_1	Δ_2
0°	1.000 000	− 7370	+ 72	45°	0.688 423	− 7659	− 95
1	0.992 630	− 7298	67	46	.680 764	− 7754	− 102
2	.985 332	− 7231	63	47	.673 010	− 7856	− 108
3	.978 101	− 7168	60	48	.665 154	− 7964	− 115
4	.970 933	− 7109	56	49	.657 190	− 8079	− 120
5	0.963 825	− 7053	+ 52	50	0.649 111	− 8199	− 128
6	.956 771	− 7001	47	51	.640 912	− 8327	− 136
7	.949 770	− 6955	44	52	.632 585	− 8463	− 142
8	.942 815	− 6910	40	53	.624 122	− 8605	− 152
9	.935 906	− 6870	37	54	.615 517	− 8757	− 160
10	0.929 036	− 6833	+ 34	55	0.606 760	− 8917	− 169
11	.922 203	− 6799	30	56	.597 843	− 9086	− 179
12	.915 404	− 6769	27	57	.588 757	− 9265	− 190
13	.908 635	− 6742	24	58	.579 492	− 9455	− 200
14	.901 893	− 6718	19	59	.570 037	− 9655	− 214
15	0.895 175	− 6699	+ 18	60	0.560 382	− 9869	− 226
16	.888 476	− 6681	14	61	.550 513	−10095	− 239
17	.881 795	− 6667	10	62	.540 418	−10334	− 256
18	.875 128	− 6657	8	63	.530 084	−10590	− 270
19	.868 471	− 6649	4	64	.519 494	−10860	− 288
20	0.861 822	− 6645	+ 2	65	0.508 634	−11148	− 308
21	.855 177	− 6643	− 2	66	.497 486	−11456	− 328
22	.848 534	− 6645	− 5	67	.486 030	−11784	− 351
23	.841 889	− 6650	− 9	68	.474 246	−12135	− 376
24	.835 239	− 6659	− 10	69	.462 111	−12511	− 403
25	0.828 580	− 6669	− 16	70	0.449 600	−12914	− 435
26	.821 911	− 6685	− 17	71	.436 686	−13349	− 467
27	.815 226	− 6702	− 21	72	.423 337	−13816	− 506
28	.808 524	− 6723	− 24	72	.409 521	−14322	− 549
29	.801 801	− 6747	− 28	74	.395 199	−14871	− 597
30	0.795 054	− 6775	− 32	75	0.380 328	−15468	− 653
31	.788 279	− 6807	− 34	76	.364 860	−16121	− 717
32	.781 472	− 6841	− 39	77	.348 739	−16838	− 791
33	.774 631	− 6880	− 41	78	.331 901	−17629	− 881
34	.767 751	− 6921	− 46	79	.314 272	−18510	− 985
35	0.760 830	− 6967	− 50	80	0.295 762	−19495	− 1116
36	.753 863	− 7017	− 54	81	.276 267	−20611	− 1275
37	.746 846	− 7071	− 57	82	.255 656	−21886	− 1484
38	.739 775	− 7128	− 61	83	.233 770	−23370	− 1758
39	.732 647	− 7189	− 67	84	.210 400	−25128	− 2144
40	0.725 458	− 7256	− 71	85	0.185 272	−27272	− 2725
41	.718 202	− 7327	− 75	86	.158 000	−29997	− 3707
42	.710 875	− 7402	− 81	87	.128 003	−33704	− 5760
43	.703 473	− 7483	− 84	88	.094 299	−39464	−15371
44	.695 990	− 7567	− 92	89	.054 835	−54835	

TABLE XXI

Nagaoka's Table of Values of the End Correction K as Function of the Ratio $\frac{\text{Diameter}}{\text{Length}}$

For use in Formula (75))

$\frac{\text{Diameter}}{\text{Length}}$	K	Δ_1	Δ_2	$\frac{\text{Diameter}}{\text{Length}}$	K	Δ_1	Δ_2
0.00	1.000 000	−4231	+24	0.45	0.833 723	−3160	+21
.01	.995 769	−4207	26	.46	.830 563	−3139	22
.02	.991 562	−4181	24	.47	.827 424	−3117	21
.03	.987 381	−4157	25	.48	.824 307	−3096	21
.04	.983 224	−4132	25	.49	.821 211	−3075	21
0.05	0.979 092	−4107	+25	0.50	0.818 136	−3054	+21
.06	.974 985	−4082	26	.51	.815 082	−3033	21
.07	.970 903	−4056	24	.52	.812 049	−3012	21
.08	.966 847	−4032	24	.53	.809 037	−2991	20
.09	.962 815	−4008	26	.54	.806 046	−2971	21
0.10	0.958 807	−3982	+25	0.55	0.803 075	−2950	+20
.11	.954 825	−3957	24	.56	.800 125	−2930	20
.12	.950 868	−3933	23	.57	.797 195	−2910	20
.13	.946 935	−3910	26	.58	.794 285	−2890	20
.14	.943 025	−3884	27	.59	.791 395	−2870	20
0.15	0.939 141	−3857	+23	0.60	0.788 525	−2850	+19
.16	.935 284	−3834	23	.61	.785 675	−2831	19
.17	.931 450	−3811	26	.62	.782 844	−2812	20
.18	.927 639	−3785	24	.63	.780 032	−2792	19
.19	.923 854	−3761	24	.64	.777 240	−2773	19
0.20	0.920 093	−3737	+24	0.65	0.774 467	−2754	+19
.21	.916 356	−3713	24	.66	.771 713	−2735	19
.22	.912 643	−3689	25	.67	.768 978	−2716	19
.23	.908 954	−3664	23	.68	.766 262	−2697	18
.24	.905 290	−3641	25	.69	.763 565	−2679	18
0.25	0.901 649	−3616	+23	0.70	0.760 886	−2661	+18
.26	.898 033	−3593	24	.71	.758 225	−2643	19
.27	.894 440	−3569	23	.72	.755 582	−2624	17
.28	.890 871	−3546	24	.73	.752 958	−2607	18
.29	.887 325	−3522	24	.74	.750 351	−2589	18
0.30	0.883 803	−3498	+22	0.75	0.747 762	−2571	+17
.31	.880 305	−3476	24	.76	.745 191	−2554	17
.32	.876 829	−3452	23	.77	.742 637	−2537	18
.33	.873 377	−3429	23	.78	.740 100	−2519	17
.34	.869 948	−3406	22	.79	.737 581	−2502	16
0.35	0.866 542	−3384	+24	0.80	0.735 079	−2486	+19
.36	.863 158	−3360	22	.81	.732 593	−2467	16
.37	.859 799	−3338	23	.82	.730 126	−2451	16
.38	.856 461	−3315	22	.83	.727 675	−2435	16
.39	.853 146	−3293	23	.84	.725 240	−2419	17
0.40	0.849 853	−3270	+22	0.85	0.722 821	−2402	+16
.41	.846 583	−3248	23	.86	.720 419	−2386	16
.42	.843 335	−3225	21	.87	.718 033	−2370	15
.43	.840 110	−3204	21	.88	.715 663	−2355	16
.44	.836 906	−3183	23	.89	.713 308	−2339	17

TABLE XXI—Continued

Diameter/Length	K	Δ_1	Δ_2		Diameter/Length	K	Δ_1	Δ_2	Δ_3
0.90	0.710 969	−2322	+ 14		2.50	0.471 865	−9292	+ 405	
.91	.708 647	−2308	16		2.60	.462 573	−8887	378	
.92	.706 339	−2292	15		2.70	.453 686	−8509	355	
.93	.704 047	−2277	16		2.80	.445 177	−8154	330	
.94	.701 770	−2261	14		2.90	.437 023	−7824	312	
0.95	0.699 509	−2247	+ 15		3.00	0.429 199	−7512	+ 293	
.96	.697 262	−2232	15		3.10	.421 687	−7219	275	
.97	.695 030	−2217	15		3.20	.414 468	−6944	260	
.98	.692 813	−2202	14		3.30	.407 524	−6684	245	
.99	.690 611	−2188	14		3.40	.400 840	−6439	230	
1.00	0.688 423	−10726	+ 344		3.50	0.394 401	−6209	+ 220	
1.05	.677 697	−10382	330		3.60	.388 192	−5989	207	
1.10	.667 315	−10052	316		3.70	.382 203	−5782	195	
1.15	.657 263	−9736	303		3.80	.376 421	−5587	186	
1.20	.647 527	−9433	290		3.90	.370 834	−5401	174	
1.25	0.638 094	−9143	+ 278		4.00	0.365 433	−5227	+ 168	
1.30	.628 951	−8865	266		4.10	.360 206	−5059	161	
1.35	.620 086	−8599	255		4.20	.355 147	−4898	152	
1.40	.611 487	−8343	244		4.30	.350 249	−4746	141	
1.45	.603 144	−8099	236		4.40	.345 503	−4605	138	
1.50	0.595 045	−7863	+ 224		4.50	0.340 898	−4467	+ 134	
1.55	.587 182	−7639	215		4.60	.336 431	−4333	125	
1.60	.579 543	−7424	208		4.70	.332 098	−4208	118	
1.65	.572 119	−7216	198		4.80	.327 890	−4090	115	
1.70	.564 903	−7018	190		4.90	.323 800	−3975	102	
1.75	0.557 885	−6828	+ 184		5.00	0.319 825	−18321	+2227	−397
1.80	.551 057	−6644	176		5.50	.301 504	−16094	1830	−306
1.85	.544 413	−6468	170		6.00	.285 410	−14264	1524	−241
1.90	.537 945	−6298	161		6.50	.271 146	−12740	1283	−193
1.95	.531 647	−6137	154		7.00	.258 406	−11457	1090	−153
2.00	0.525 510	−11809	+ 580		7.50	0.246 949	−10367	+ 937	−127
2.10	.513 701	−11229	539		8.00	.236 582	−9430	810	−104
2.20	.502 472	−10690	499		8.50	.227 152	−8620	706	− 86
2.30	.491 782	−10191	465		9.00	.218 532	−7914	620	
2.40	.481 591	−9726	434		9.50	.210 618	−7294		
					10.00	0.203 324			

In the last part of this table several errors in the fifth and sixth places of decimals have been corrected.

TABLE XXII

Functions for Calculating Resistance and Inductance of Straight, Cylindrical Wires with Varying Frequency (sec. 10)

x	$\frac{W}{Y}$	Δ_1	Δ_2	$\frac{x}{2}\frac{W}{Y}$	Δ_1	Δ_2	$\frac{Z}{Y}$	Δ_1	Δ_2	$\frac{4}{x}\frac{Z}{Y}$	Δ_1	Δ_2
0.0	∞	....	...	1.00000	0	+ 1	0.	+2500	...	1.00000		
.1	20.00000	....	...	1.00000	+ 1	+ 2	0.02500	2500	...	1.00000	0	− 2
.2	10.00020	....	...	1.00001	3	+ 6	0.05000	2500	− 1	1.00000	− 2	− 3
.3	6.66693	....	...	1.00004	9	+ 10	0.07500	2499	0	0.99998	− 5	− 5
.4	5.00065	....	...	1.00013	19	+ 16	0.09999	2499	− 2	0.99993	− 10	− 8
0.5	4.00128	....	...	1.00032	+ 35	+ 22	0.12498	+2497	− 3	0.99984	− 18	−11
.6	3.33557	....	...	1.00067	57	+ 31	0.14995	2494	− 4	0.99966	− 29	−14
.7	2.86069	....	...	1.00124	88	+ 40	0.17489	2490	− 7	0.99937	− 43	−20
.8	2.50530	....	...	1.00212	128	+ 51	0.19979	2483	− 10	0.99894	− 64	−25
.9	2.22978	....	...	1.00340	179	+ 60	0.22462	2473	− 12	0.99830	− 89	−31
1.0	2.01038	....	...	1.00519	+ 239	+ 74	0.24935	+2461	− 18	0.99741	− 120	−36
1.1	1.83196	....	...	1.00758	313	+ 86	0.27396	2443	− 21	0.99621	− 156	−43
1.2	1.68451	....	...	1-01071	399	100	0.29839	2422	− 27	0.99465	− 199	−50
1.3	1.56108	....	...	1.01470	499	114	0.32261	2395	− 34	0.99266	− 249	−57
1.4	1.45670	....	...	1.01969	613	128	0.34656	2361	− 41	0.99017	− 306	−63
1.5	1.36776	....	...	1.02582	+ 741	141	0.37017	+2320	− 48	0.98711	− 369	−69
1.6	1.29154	....	...	1.03323	882	153	0.39337	2272	− 56	0.98342	− 438	−76
1.7	1.22594	....	...	1.04205	1035	165	0.41609	2216	− 64	0.97904	− 514	−81
1.8	1.16934	....	...	1.05240	1200	176	0.43825	2152	− 72	0.97390	− 595	−86
1.9	1.12042	....	...	1.06440	1376	183	0.45977	2080	− 81	0.96795	− 681	−89
2.0	1.07816	−3649	+505	1.07816	1559	192	0.48057	+1999	− 90	0.96113	− 770	−92
2.1	1.04167	−3144	442	1.09375	1751	192	0.50056	1909	− 96	0.95343	− 862	−92
2.2	1.01023	2702	387	1.11126	1943	195	0.51965	1813	−102	0.94482	− 954	−91
2.3	0.98321	2315	339	1.13069	2138	192	0.53778	1711	−108	0.93527	−1045	−90
2.4	0.96006	1976	297	1.15207	2330	188	0.55489	1603	−113	0.92482	−1135	−86
2.5	0.94030	−1679	+256	1.17538	2518	179	0.57092	+1490	−115	0.91347	−1221	−81
2.6	0.92351	1423	224	1.20056	2697	170	0.58582	1375	−116	0.90126	−1301	−73
2.7	0.90928	1199	190	1.22753	2867	157	0.59957	1259	−116	0.88825	−1374	−65
2.8	0.89729	1009	162	1.25620	3024	142	0.61216	1143	−114	0.87451	−1439	−56
2.9	0.88720	847	136	1.28644	3166	128	0.62359	1029	−111	0.86012	−1495	−47
3.0	0.87873	− 711	+114	1.31809	+3293	+109	0.63388	+ 918	−106	0.84517	−1542	−36
3.1	0.87162	597	92	1.35102	3402	+ 93	0.64306	812	−100	0.82975	−1578	−25
3.2	0.86565	505	75	1.38504	3495	+ 76	0.65118	712	− 93	0.81397	−1603	−16
3.3	0.86060	430	58	1.41999	3571	+ 61	0.65830	619	− 87	0.79794	−1619	− 6
3.4	0.85630	372	46	1.45570	3632	+ 44	0.66449	532	− 78	0.78175	−1625	+ 4
3.5	0.85258	− 326	+ 36	1.49202	+3677	+ 31	0.66981	454	− 70	0.76550	−1621	+11
3.6	0.84932	290	24	1.52879	3708	+ 19	0.67436	384	− 61	0.74929	−1610	+20
3.7	0.84642	266	18	1.56587	3727	+ 10	0.67820	323	− 55	0.73320	−1590	26
3.8	0.84376	248	13	1.60314	3737	− 1	0.68143	268	− 47	0.71729	−1564	31
3.9	0.84128	235	8	1.64051	3736	− 7	0.68411	221	− 40	0.70165	−1533	36

TABLE XXII—Continued

x	$\frac{W}{Y}$	Δ_1	Δ_2	$\frac{x}{2}\frac{W}{Y}$	Δ_1	Δ_2	$\frac{Z}{Y}$	Δ_1	Δ_2	$\frac{4}{x}\frac{Z}{Y}$	Δ_1	Δ_2
4.0	0.83893	− 227	+ 5	1.67787	+3729	− 12	0.68632	+ 181	− 33	0.68632	−1497	+39
4.1	0.83666	222	3	1.71516	3717	− 17	0.68813	148	− 28	0.67135	−1458	43
4.2	0.83444	219	+ 1	1.75233	3700	− 19	0.68961	120	− 23	0.65677	−1415	43
4.3	0.83225	218	0	1.78933	3681	− 20	0.69082	97	− 17	0.64262	−1372	45
4.4	0.83007	218	0	1.82614	3661	− 22	0.69179	80	− 15	0.62890	−1327	45
4.5	0.82789	− 218	+ 1	1.86275	+3639	− 20	0.69259	+ 65	− 12	0.61563	−1282	+45
4.6	0.82571	217	+ 1	1.89914	3619	− 21	0.69324	53	− 8	0.60281	−1237	45
4.7	0.82354	216	+ 1	1.93533	3598	− 19	0.69377	45	− 6	0.59044	−1192	43
4.8	0.82138	215	+ 1	1.97131	3579	− 17	0.69422	39	− 4	0.57852	−1149	43
4.9	0.81923	− 214	+ 2	2.00710	+3562	− 15	0.69461	35	− 3	0.56703	−1106	42
5.0	0.81709	−419	+14	2.04272	+ 7081	−45	0.69496	+62	−6	0.55597	−2091	+151
5.2	0.81290	405	18	2.11353	7036	−31	0.69558	56	−1	0.53506	1940	138
5.4	0.80885	387	20	2.18389	7005	−19	0.69614	55	0	0.51566	1802	124
5.6	0.80498	367	22	2.25393	6987	− 8	0.69669	55	+1	0.49764	1678	112
5.8	0.80131	345	23	2.32380	6979	0	0.69725	56	−1	0.48086	1566	101
6.0	0.79786	−322	+21	2.39359	+ 6979	+ 4	0.69781	+55	−1	0.46521	−1465	+ 91
6.2	0.79464	301	21	2.46338	6983	8	0.69836	54	−2	0.45056	1374	82
6.4	0.79163	280	19	2.53321	6992	8	0.69891	52	−3	0.43682	1292	74
6.6	0.78883	261	17	2.60313	6999	8	0.69942	49	−3	0.42389	1218	68
6.8	0.78621	244	16	2.67312	7007	8	0.69991	46	−4	0.41171	1150	62
7.0	0.78377	−228	+13	2.74319	+ 7015	+ 6	0.70037	+42	−3	0.40021	−1088	+ 57
7.2	0.78149	215	13	2.81334	7021	5	0.70080	39	4	0.38933	1031	52
7.4	0.77934	202	12	2.88355	7026	5	0.70118	35	3	0.37902	979	49
7.6	0.77731	190	11	2.95380	7031	3	0.70154	32	3	0.36923	930	45
7.8	0.77541	179	9	3.02411	7034	+ 1	0.70185	29	−2	0.35992	885	42
8.0	0.77361	−170	+ 8	3.09445	+ 7035	+ 3	0.70214	+27	−2	0.35107	− 843	+ 39
8.2	0.77191	162	8	3.16480	7038	+ 1	0.70241	25	−2	0.34263	804	36
8.4	0.77028	154	7	3.23518	7039	+ 1	0.70265	23	−2	0.33460	768	34
8.6	0.76874	147	7	3.30557	7040	+ 1	0.70288	20	−1	0.32692	734	32
8.8	0.76727	140	6	3.37597	7041	+ 1	0.70308	19	−1	0.31958	702	30
9.0	0.76586	−134	+ 6	3.44638	+ 7042	+ 1	0.70327	+18	−2	0.31257	− 672	+ 28
9.2	0.76452	128	5	3.51680	7043	− 1	0.70345	16	1	0.30585	644	26
9.4	0.76324	123	6	3.58723	7043	+ 2	0.70362	15	1	0.29941	617	25
9.6	0.76201	117	4	3.65766	7045	+ 4	0.70377	14	1	0.29324	593	23
9.8	0.76084	−113	+4	3.72812	7046	+ 2	0.70391	+13	−1	0.28731	− 569	+ 21
10.0	0.75971	−261	+24	3.79857	+17620	+ 3	0.70405	+30	−4	0.28162	−1330	+120
10.5	0.75710	237	21	3.97477	17623	4	0.70435	26	4	0.26832	1210	104
11.0	0.75473	216	19	4.15100	17627	4	0.70461	22	3	0.25622	1106	91
11.5	0.75257	197	16	4.32727	17631	4	0.70483	19	2	0.24516	1015	80
12.0	0.75060	181	15	4.50358	17635	3	0.70503	17	−2	0.23501	935	71
12.5	0.74879	−166	+13	4.67993	+17638	+ 3	0.70520	+15	−1	0.22567	− 863	+ 64
13.0	0.74712	154	11	4.85631	17641	2	0.70535	14	2	0.21703	800	57
13.5	0.74559	142	10	5.03272	17643	2	0.70549	12	1	0.20903	743	51
14.0	0.74416	132	9	5.20915	17645	3	0.70561	11	2	0.20160	692	46
14.5	0.74284	−123	+ 8	5.38560	17648	+ 2	0.70572	+ 9	−1	0.19468	− 646	+ 40

TABLE XXII—Continued

x	$\frac{W}{Y}$	Δ_1	Δ_2	$\frac{x}{2}\frac{W}{Y}$	Δ_1	Δ_2	$\frac{Z}{Y}$	Δ_1	Δ_2	$\frac{4}{x}\frac{Z}{Y}$	Δ_1	Δ_2
15.0	0.74161	−222	+27	5.56208	+35301	+ 6	0.70581	+16	−3	0.18822	−1172	+137
16.0	0.73939	195	22	5.91509	35307	5	0.70597	13	2	0.17649	1035	114
17.0	0.73743	173	19	6.26817	35312	5	0.70611	11	1	0.16614	921	97
18.0	0.73570	154	16	6.62129	35317	4	0.70622	10	2	0.15694	824	82
19.0	0.73416	139	13	6.97446	35321	3	0.70632	8	−1	0.14870	742	70
20.0	0.73277	−125	+11	7.32767	+35324	+ 3	0.70640	7	−1	0.14128	− 672	+ 61
21.0	0.73151	114	10	7.68091	35327	2	0.70646	6	1	0.13456	611	53
22.0	0.73038	104	9	8.03418	35329	2	0.70652	5	0	0.12846	558	46
23.0	0.72935	95	8	8.38748	35331	2	0.70657	5	1	0.12288	511	41
24.0	0.72840	87	7	8.74079	35333	2	0.70662	4	−1	0.11777	470	36
25.0	0.72753	− 80	+ 5	9.09412	+35335	+ 2	0.70666	+3	0	0.11307	− 434	+ 32
26.0	0.72673	−143	+20	9.44748	+70674	+ 5	0.70669	6	−1	0.10872	− 776	+103
28.0	0.72530	123	15	10.15422	70679	+ 4	0.70675	5	−1	0.10096	672	84
30.0	0.72407	−108	+13	10.86101	+70683	+ 3	0.70680	+4	−1	0.09424	− 589	+ 69
32.0	0.72299	95	11	11.56785	70686	3	0.70684	3	..	0.08835	519	58
34.0	0.72204	84	9	12.27471	70689	2	0.70687	2	..	0.08316	462	49
36.0	0.72120	75	8	12.98160	70691	2	0.70689	2	..	0.07854	413	41
38.0	0.72045	68	7	13.68852	70693	2	0.70691	2	..	0.07441	372	35
40.0	0.71977	− 61	+ 6	14.39545	+ 70695	1	0.70693	+2	..	0.07069	− 336	+ 30
42.0	0.71916	55	5	15.10240	70696	2	0.70695	1	..	0.06733	306	27
44.0	0.71861	50	4	15.80936	70698	1	0.70696	2	..	0.06427	279	23
46.0	0.71810	46	4	16.51634	70699	0	0.70698	1	..	0.06148	256	20
48.0	0.71764	− 43	+ 3	17.22333	+ 70699	+ 1	0.70699	+1	..	0.05892	− 236	+ 17
50.0	0.71721	−170	+49	17.93032	+353509	+13	0.70700	+3	..	0.05656	− 942	+269
60.0	0.71551	121	30	21.46541	353522	8	0.70703	2	..	0.04713	673	168
70.0	0.71430	91	20	25.00063	353530	5	0.70705	1	..	0.04040	505	112
80.0	0.71340	70	+14	28.53593	353535	+ 3	0.70706	1	..	0.03535	393	79
90.0	0.71270	− 56	..	32.07127	353538	..	0.70707	+1	..	0.03142	− 314	+ 55
100.0	0.71213			35.60666			0.70708			0.02828		
∞	0.70711			∞			0.70711			0.		

TABLE XXIII

Values of Limiting Change of Inductance with the Frequency

$\frac{2l}{\rho}$	Single Wire			$\frac{d}{\rho}$	Parallel Wires			$\frac{2a}{\rho}$	Circular Rings		
	$\left(\frac{\Delta L}{L}\right)_{x=\infty}$	Δ_1	Δ_2		$\left(\frac{\Delta L}{L}\right)_{x=\infty}$	Δ_1	Δ_2		$\left(\frac{\Delta L}{L}\right)_{x=\infty}$	Δ_1	Δ_2
50	0.07906			5	0.13445	−1201	+ 342				
				6	.12244	859	206				
100	0.06485	−988	+538	7	.11385	653	137	100	0.08756	−1710	+987
200	5497	450	173	8	.10732	516	84	200	7046	723	894
300	5047	277	82	9	.10216	− 432		300	6323	429	134
400	4770	195	47					400	5894	295	75
500	4575	−148	+ 30	10	0.09784	−2078	+1219	500	5600	− 220	+ 47
				20	7706	859	359				
600	0.04427	−118	+ 21	30	6847	500	160	600	0.05380	− 173	+ 32
700	4309	97	15	40	6347	340	88	700	5207	141	23
800	4213	82	11	50	6007	− 252	+ 55	800	5066	118	17
900	4131	− 71	+ 9					900	4948	− 101	+ 13
				60	0.05755	− 197	+ 37				
1000	0.04060	−411	+207	70	5557	160	26	1800	0.04847	− 574	+297
2000	3649	204	72	80	5397	134	20	2000	4273	277	101
3000	3445	131	36	90	5263	− 114	+ 15	3000	3996	176	50
4000	3314	95	22					4000	3820	126	29
5000	3219	− 74	+ 14	100	0.05149	− 643	+ 336	5000	3694	− 97	+ 19
				200	4506	307	113				
6000	0.03145	− 60	+ 10	300	4199	194	56	6000	0.03597	− 78	+ 13
7000	3085	50	7	400	4005	138	32	7000	3519	65	10
8000	3035	43	6	500	3867	− 106	+ 21	8000	3454	55	8
9000	2992	− 37	+ 4					9000	3399	− 48	+ 6
				600	0.03761	− 85	+ 14				
10000	0.02955	−224	+108	700	3676	71	11	10000	0.03351	− 285	+140
20000	2731	116	40	800	3605	60	8	20000	3066	145	50
30000	2615	76	20	900	3545	− 52	+ 6	30000	2921	95	25
40000	2539	56	12					40000	2826	70	16
50000	2483	− 44	+ 8	1000	0.03493	− 309	+ 154	50000	2756	− 54	+ 10
				2000	3183	155	53				
60000	0.02438	− 36	+ 6	3000	3028	102	27	60000	0.02702	− 44	+ 7
70000	2402	30	4	4000	2926	75	17	70000	2658	37	5
80000	2372	26	3	5000	2851	− 58	+ 11	80000	2621	32	4
90000	2346	− 23	+ 2					90000	2589	− 28	+ 3
				6000	0.02793	− 47	+ 7				
100000	0.02323			7000	2746	40	6	100000	0.02561		
1000000	1913			8000	2706	34	4	1000000	2072		
				9000	2672	− 30	+ 3				
				10000	2643						

TABLE XXIV

Values of the Argument x_0 for Copper Wires 1 mm Radius and Conductivity 5.811×10^{-4} c g. s. Units

f cycles per second	x_0	λ meters	f cycles per second	x_0	λ meters
100	0.2142	.....	50000	4.790	6000
200	.3029	.....	60000	5.247	5000
300	.3710	.....	70000	5.667	4286
400	.4284	.....	80000	6.058	3750
500	.4790	.....	90000	6.426	3333
600	0.5247	.....	100000	6.774	3000
700	.5667	.....	150000	8.296	2000
800	.6058	.....	200000	9.579	1500
900	.6426	.....	250000	10.710	1200
1000	.6774	.....	300000	11.732	1000
2000	0.9579	.....	333333	12.367	900
3000	1.1732	.....	375000	13.117	800
4000	1.3547	.....	428570	14.023	700
5000	1.5146	.....	500000	15.146	600
			600000	16.592	500
6000	1.6592	.....	700000	17.921	429
7000	1.7921	.....	750000	18.550	400
8000	1.9158	.....	800000	19.158	375
9000	2.0321	.....	900000	20.321	333
.....	.....	.....	1000000	21.42	300
10000	2.142	30000	1500000	26.23	200
15000	2.623	20000	3000000	37.10	100
20000	3.029	15000			
30000	3.710	10000			
40000	4.284	7500			

INDEX

[Italicized page numbers refer to examples illustrating the formulas. Proper names are also italicized.]

A

Absolute formulas: Mutual inductance of coaxial circles, 6, 7, 9, *20*, *21*, *23;* mutual inductance of concentric, coaxial solenoids, 64, 73, *78*, *91;* mutual inductance of coaxial solenoids, 64, 69, 73, *89;* self-inductance of solenoids, 117, 118, *129*, *132*, *133*, Table IV; mutual inductance of circle and solenoid, 99, 100, *103*, *106*, *107*, *108*, *109;* inductance of rectangle of rectangular section, 155.
Absolute invariant 17.
Adjacent conductors. *See* Linear conductors.
Airey. *See Searle* and *Airey*.
Ampere balance. *See* Current balance.
Amplitude of incomplete elliptic integrals, 64, 100.
Annular area, geometric mean distance of, 168, 169; geometric mean distance of point to area, 169.
Annulus. *See* Annular area.
Approximate formulas; inductance of solenoid of more than one layer, *Cohen*, 140, *150;* inductance of circular ring, *Kirchhoff*, 110, *114;* mutual inductance of coaxial solenoids of equal length, 55, *78;* inductance of coil of rectangular cross section, *Maxwell*, 135, *143;* *Perry* 126, *143;* mutual inductance of coaxial circles, *Wiedemann*, footnote 13.
Arithmetical mean distance, 171, 172.
Arithmetical mean square distance, 171, 172.
Asymptotic formulas for *W*, *Y*, and *Z*, 176.
Attraction of coils. *See* Current balance.
Ayrton and *Jones* current balance, *104*.

B

Bar. *See* Rectangular bar.
Ber and *bei* functions, 174, 175.
Bessel functions, 15, 174.
Bláthy, inductance of a ring, 113.
Breadth, equivalent of coil 38, *47*, *48*.
Bromwich, 16, 111, 113.

C

Campbell, extension of *Jones's* formula, 100, *109;* form for standard of mutual inductance, *108*.
Choice of formulas: Mutual inductance of coaxial circles, 19; mutual inductance of coaxial coils of rectangular section, 43; mutual inductance of coaxial solenoids, 73, *83*, *84*, *86;* inductance of solenoids, 125; inductance of coils of rectangular cross section, 142.
Circles. *See* Coaxial circles, Circular areas, Circle and coaxial solenoid.
Circle and coaxial solenoid, mutual inductance of, 98-110.
Circular areas: Geometric mean distance of, 167; geometric mean distance of a point from, 169; geometric mean distance of two, 170; arithmetical mean distance of, 171; arithmetical mean square distance of, 171.

Circular coils of rectangular cross section: Inductance of, 135–150; choice of formulas, 142; mutual inductance of coaxial, 33–52; choice of formulas, 42.
Coaxial circles: Formulas for the inductance of, 6–32; choice of formulas, 19; summary of formulas, 19.
Coaxial coils of rectangular cross section: Formulas for the mutual inductance of, 33–52; choice of formulas, 42.
Coaxial solenoids: Formulas for the mutual inductance of, 52–98; case where coils are not concentric, 59, 64, 73; choice of formulas, 73, *83*, *84*, *86*.
Coefficients: In *Stefan's* formula (90), 196; in hypergeometrical series of *Mathy* formula (18), 220; in formulas (40), (43), and (56), 221, 222.
Coffin, Absolute formula for mutual inductance of equal circles, 14, 15, *30;* coaxial solenoids, 73; inductance of solenoid, 117, *129*; derivation of *Lorenz's* formula, 118.
Cohen, absolute formula for the mutual inductance of coaxial solenoids 64, 69, 73, *78*, *79*, *91;* correction of *Wien's* formula, 111; approximate formula for the inductance of solenoids of several layers, 140, *150*.
Complementary modulus, 8, 10, 11.
Complete elliptic integrals. *See* Elliptic integrals.
Concentric coaxial solenoids. *See* Coaxial solenoids.
Concentric conductors, inductance of, 158; with high frequency, 179.
Conductors. *See* Concentric conductors, linear conductors, rings, tubes, tapes, etc.
Constant of *Lorenz* apparatus. *See Lorenz* apparatus.
Constants. *See* Tables of constants, etc.
Correction, of current sheet formulas for inductance of solenoids for windiug of round wire, 122, *127;* for unequal distribution of current over cross section of coil, 140–142, *147–149;* of simple expression for toroidal coil, 125.
Correction factor. *See* End correction, correction, etc.
Cross section. *See* Coaxial coils of rectangular cross section, Circular coils of rectangular cross section, Equal coils of rectangular cross section, Square cross section.
Current balance, of *Ayrton* and *Jones*, *104*, *106;* of National Physical Laboratory, *107*.
Current sheets, 76, *97*, 119.
Cylindrical wire. *See* Straight cylindrical wire.
Cylindrical coils. *See* Solenoids, Coaxial solenoids.
Cylindrical conductors. *See* Linear conductors.

D

Decrease of inductance with the frequency: General considerations, 172; of straight cylindrical wires, 173, 174, 177, 179, *183–185;* of two parallel cylindrical wires, 180, 181, *185*, *186;* of a circular ring, 181, 182, *186*, *187*.
Differential coefficients, 34, 39.
Dimensions of equivalent current sheet 76, *97*, 119.
Disk in *Lorenz* apparatus. *See Lorenz* apparatus.
Distribution of current. *See* Correction for unequal, High frequency formulas, Increase of resistance, Decrease of inductance.
Dynamometers: *Gray*, 60, *86; Ayrton* and *Jones*, *104*, *106*.

E

Eddy currents. *See* High frequency formulas, Decrease of inductance, Increase of resistance.
Ellipse. *See* Elliptical area.
Elliptical area, geometric mean distance of, 167.
Elliptic integrals. *See also* Incomplete elliptic integrals, mutual inductance formulas involving 6, 7, 9, 64, 68, 71, 98, 99, 100; inductance of solenoids, 118; series expansions for complete, 8, 9; tables of, as functions of tan γ, 193; *Legendre's* Tables, 202–212.
End correction: For mutual inductance of equal concentric solenoids, 55; for self-inductance of solenoids, 119–121, *130–132*, 223–225.
End effect. *See* End correction.

Equal circles, mutual inductance of 14, *28*, *30*, 18.
Equal coils of rectangular cross section, mutual inductance of, 33, 39, 40, *44*, *45*, *47*, *49–51*.
Equal parallel rectangles. *See* Parallel rectangles.
Equal squares. *See* Squares, Geometric mean distance.
Equal solenoids, mutual inductance of, 69–71, *94–97*.
Equal radii. *See* Equal circles, Equal coils of rectangular cross section, Equal solenoids, Solenoids of equal radii.
Equal cross section. *See* Equal coils of rectangular cross section.
Equivalent circles. *See* Equivalent filaments.
Equivalent breadth. *See* Equivalent filaments.
Equivalent radius. *See* Equivalent filaments.
Equivalent length. *See* Dimensions of equivalent current sheet.
Equivalent filaments, 38, 39, *47*, *48*.
Errors in *Rowland's* and *Rayleigh's* formulas, 36, 37.
Exact formulas. *See* Absolute formulas.
Examples illustrating the formulas: For mutual inductance of coaxial circles, *20–32;* mutual inductance of coaxial coils of rectangular cross section, *44–52;* mutual inductance of coaxial solenoids, *77–98;* mutual inductance of circle and solenoid, *103–110;* self-inductance of circular ring, *114–115;* self-inductance of solenoids, *126–135;* self-inductance of coils of rectangular cross section, *142–150;* self and mutual inductance of linear conductors, *159–166;* inductance and resistance with high frequency, *183–187*.
Extension of *Maxwell's* series formula for circles, 14, *30;* *Maxwell's* formula for equal, concentric solenoids, 53, *77–80;* *Ròiti's* formula for coaxial solenoids, 57–59, *80–86;* *Jones's* formula for circle and solenoid, 101, 102, *103–106*, *108*, *109;* *Rayleigh's* and *Nivens's* formula for the inductance of solenoids, 117, *129;* *Russell's* formula for *W*, *Y*, and *Z*, 176.

F

Filaments. *See* Equivalent filaments.
Force of attraction of coils. *See* Dynamometers, Current balances.
Formulas. *See* Absolute formulas, Approximate formulas, Correction formulas, Interpolation formulas.
Frequency. *See* High frequency formulas.
Fröhlich, inductance of toroidal coil, 125.
Functions. *See Bessel* functions, **p** function, *q* series, *Ber* and *bei* functions, *Ker* and *kei* functions, tables, coefficients, *W*, *Y*, and *Z*.

G

General term: In Wallis's formulas for *F* and *E*, 9; in formulas (5) and (6), 9, 10; in Havelock's formula for coaxial solenoids, 56; in Gray's and Searle and Airey's formulas, 63; in Russell's formula for coaxial solenoid, 68; in Lorenz's formula for circle and solenoid, 99; in the Webster-Havelock formula, 121; in Russell's formulas for *W*, *Y*, and *Z*, 176.
Geometric mean distance, 42, *52*, 166-170.
Glazebrook, 34.
Gray, mutual inductance of coaxial solenoids, 59, 60, *86-89;* dynamometer, 60, *86;* geometric mean distance formulas, 170.

H

Harmonics. *See* Zonal harmonics.
Havelock, mutual inductance of coaxial circles, 15, 16, *27*, *29*, *30;* mutual inductance of coaxial solenoids, 55, 56, 72, *78;* mutual inductance of short secondary on long primary, 68; self-inductance of solenoids (see *Webster*).
Heaviside, equal coaxial solenoids, 55, *78;* high frequency formulas, 173.
Hicks, inductance of a ring, 113.
High frequency formulas, 111, 172-187.

Himstedt, mutual inductance of solenoids, 72.
Hollow tube. *See* Tube.
Hypergeometric Series, 17, 56, Table XVII.

I

Illustration of formulas. *See* Examples for illustrating the formulas.
Incomplete elliptic integrals, 64, 67, 100.
Increase of resistance with the frequency, 172-174, 177, 178-183, *183-187*, Table XXII.
Infinite solenoid, inductance of, 116.
Insulation of wires, correction for, 138, 139, 140-142.
Integrals. *See* Elliptic integrals, Incomplete elliptic integrals.
Interpolation: In Tables XV and XVI, 12; in Table XXII, 177; formula for, 214.
Invariant. *See* Absolute invariant.

J

Jacobi, *q* series, 11, 12, 65-67, 120, 121; Theta functions, 66.
Jones, mutual inductance of circle and solenoid, 99, 102, *106; see also Ayrton, Campbell.*
Joubert. See Mascart.

K

Kelvin, resistance and inductance of straight wires at high frequency, 173, 174; *ber* and *bei* functions, 175.
Ker and *Kei* functions, 175.
Kirchhoff, inductance of a ring, 110, *114;* inductance of a solenoid, 118; summation formula for the inductance of a solenoid, 123; inductance of a square, 154; formula for mutual inductance of coaxial coils, 73.

L

Landen's transformation, 7.
Layers, coils of several. *See* Solenoids.
Legendre's tables, 6, 8, 10, *20*, *44*, *100*, Tables XII and XIII.
Limiting formulas for resistance and inductance of straight wires with the frequency, 177.
Line, geometric mean distance of, 167; geometrical mean distance of two lines, 168, 169, 170; arithmetical mean distance of, 171; arithmetical mean square distance of, 171.
Linear conductors, self and mutual inductance of, 150-159, *159-166;* mutual inductance of two conductors in the same straight line, 152, *161*.
Logarithms. *See* Natural logarithms.
Long coils, mutual inductance, choice of formulas, 73-77; self-inductance of, 116, 117, 120, *129*, *130-133*.
Lorenz, apparatus, 34, *103*, 107; mutual inductance of circle and solenoid, 98; inductance of solenoids, 117, 118, *129*, *130*, *132*, *133*.
Lyle, mutual inductance of coils of rectangular cross section, 38, *47*, *48*.

M

Martens, mutual inductance of rectangles, 156.
Mascart, 55.
Mathy, corrected formula of, for coaxial circles, 17, *31;* simple special formula derived from this, 18, *32*.
Maxwell, mutual inductance of coaxial circles, 6-8, 13, *20*, *21*, *28*, Table I; mutual inductance of equal coaxial, concentric solenoids, 53, *77*, *79*, *80;* inductance of a ring, 110, 111, *114*, *115;* inductance of coil of rectangular cross section, 135, 136, *142*, *143;* correction for distribution of current in round wires, 140.

McGill University, Lorenz apparatus of, *103*.
Method of obtaining dimension of coils. *See* Dimensions of equivalent current sheets.
Minchin, inductance of ring, 113.
Misprints in authorities quoted, 34, 55, 64, 100, *103*, 170.
Modified Radius. *See* Equivalent filaments.
Modulus. *See* Elliptic integrals, Incomplete integrals, Complementary modulus.
Multiple conductors, inductance of, 159, *165*.
Mutual inductance. *See* detailed headings, such as Coaxial circles, Coaxial solenoids, etc.
Mutual inductance by means of self-inductance formulas, 71, *95*, *96*.

N

Nagaoka, mutual inductance of coaxial circles, 11, 12, *26*, *27*, Table XV and XVI; mutual inductance of coaxial solenoids 64-67, 73, *89*, *91-93;* inductance of solenoids, 119-121, *130-132*.
Nasmyth, 87.
National Physical Laboratory, current balance of, *107*.
Natural logarithms of numbers from 1 to 100, Table XI.
Neumann, inductance of straight cylindrical wire, 151, *159;* mutual inductance of parallel rectangles, 156, *164*.
Nicholson, inductance and resistance of parallel wires at high frequency, 180.
Niven. *See Rayleigh*.
Noninductive Shunts, inductance of, 158.

O

Ohm, determination of, 34; *see also* Lorenz apparatus.
Olshausen, absolute formulas for mutual inductance of coaxial coils, 73.

P

$\wp$ function of Weierstrass, 17, 65.
Parallel bars. *See* Rectangular bars.
Parallel circles. *See* Coaxial circles.
Parallel lines. *See* Lines.
Parallel rectangles. *See* Rectangles.
Parallel squares. *See* Squares.
Parallel wires, mutual inductance of, 151, *160; see also* Return circuit.
Parameter, in *Olshausen's* formula, 73.
Permeability. *See* High frequency formulas, Linear conductors.
Perry, inductance of coil of rectangular cross section, 136, *143*.
Pitch of winding, 76, *97*, 119.
Plane, coaxial circles in the same, 14, 15, 18.

Q

q Series of Jacobi, 11, 65–67.
Quadratures, formula of, 34, 35, *45*, *46*, *90*.

R

Rayleigh (also *Rayleigh* and *Niven*), formula of quadratures, 34, 35, *45*, *46*, *90;* inductance of circular ring, 111, *114*, *115;* inductance of solenoids, 116, *126–129;* inductance and resistance of straight wires at high frequency, 173, 177.
Rectangles, geometric mean distance of, *see* Rectangular area; inductance of, 154, 155, *163*, *164;* mutual inductance of parallel rectangles, 155, *164*.
Rectangular areas, geometric mean distance of, 167, 168; geometrical mean distance of two, 170.

Rectangular bars, inductance of, 152, 153, *162;* mutual inductance of, 153, 154.
Rectangular cross section. *See* Coaxial coils of rectangular cross section, Equal coils of rectangular cross section.
Reduction from current sheet to winding of round wires, 122, *128*, Tables VII and VIII.
Resistance. *See* Increase of resistance with the frequency.
Return Circuit, inductance of parallel wire, 151, 152, *161;* same at high frequency, 180, 181, *185*, *186;* inductance with rectangular cross section, 154, *162*.
Ring, inductance of circular solid, 110–112, *114;* inductance with elliptical cross section, 113; inductance and resistance with high frequency, 181, 182, *186*.
Ròiti, Mutual inductance of coaxial solenoids, 57, 58, *80–84*, *87*.
Rosa, extension of Maxwell's formula for coaxial coils, 14, *30;* mutual inductance of coils of rectangular cross section, 39, *49*, *50;* *Rosa-Weinstein* formula for coaxial coils, 40, 41, *51*, 69–71, *94–97;* extension of *Searle and Airey's* formula 61–63, *82–86*, *94;* mutual inductance of circle and solenoid, 101, 102, *103*, *105*, *109;* correction for inductance of toroidal coil, 125, *134*, *135;* correction of current sheet formulas for winding of round wires, 122, *128–130*, Tables VII and VIII; correction for distribution of current in coil of rectangular cross section, 138, 139, 141, *144–150*, Tables IX and X; geometric mean distance, 168–170; arithmetical mean distance and arithmetical mean square distance, 171, 172.
Rowland, mutual inductance of coaxial coils, 33, 34, *44*, *45*.
Russell, mutual inductance of coaxial solenoids, 67, 68, 69, *83;* self-inductance of solenoids, 121; formulas for the functions *W*, *Y*, and *Z*, 175, 176; inductance and resistance of concentric main, 179.

S

Savidge, tables of functions for high frequency calculations, 175, 176.
Searle and Airey, mutual inductance of coaxial solenoids, 61–63, *82–86*, *94*.
Self-inductance, mutual inductance by means of, 41, *52*, 71, *95*, *96;* *see also* detailed headings as Solenoids, Circular coils of rectangular cross sections, etc.
Series formulas for *F* and *E*, 8, 9, *22*, *see* Hypergeometric Series, *W*, *Y*, and *Z* functions, and other detailed headings.
Short coils, inductance of short solenoids, 116, 120, *126*, *128*, *130*.
Short secondary on long primary, mutual inductance of, 68, 69, *93*, *94*.
Shunts. *See* Noninductive shunts.
Simple formula for certain coaxial circles, 18, *32*.
Single layer coils. *See* Solenoids, Circle and coaxial solenoids.
Solenoids, inductance of infinite, 116; inductance of, 116–126, *126–135;* inductance of solenoid of more than one layer 138–140, *150;* *see also* Coaxial solenoids.
Solid Ring. *See* Ring.
Squares, inductance of, 154, *162*, *164;* mutual inductance of parallel, 155, 156, *164;* geometric mean distance of, 167.
Square cross section, mutual inductance of coaxial coils of, 39, 41, *47*, *49*.
Stefan, mutual inductance of coaxial coils, 40; inductance of circular coil of rectangular cross section, 137, *144–147*, *149;* correction for distribution of current in round wires, 141.
Stirling's constants, 17, 19.
Straight cylindrical wire, inductance of, 150, 151, *159*, 160; inductance and resistance of, at high frequency, 173–179, *183–185*, Tables XXII, XXIII.
Strasser, inductance of single layer coil, 123, 124, *134*, *147*, Table V.
Strip. *See* Tape.
Subdivision, calculation of inductance by, 43, 74, *88–90*.
Summation formula for inductance of solenoid, 123, *127*.
Sumec, inductance of rectangle of rectangular cross section, 155.

T

Tables of Constants, 189-230. For detailed summary *see* Table of contents.
Tape, winding of 119; self and mutual inductance of, 156, 157, *165;* geometric mean distance of, 168, 169.

Taylor's theorem, 33.
Terezawa, inductance of ring, footnote 111.
Theta functions, 66.
Thick current sheet, 138, 139.
Thin strip. *See* Tape.
Thin-walled Tube. *See* Tube.
Thomson, J. J., inductance of ring of elliptical section, 113.
Toroidal coil, inductance of, 124, 125, *134*.
Tube, inductance of hollow, tubular ring, 112, 113, *115*; straight tube, *see* High frequency formulas.

U

Unequal circles. *See* Coaxial circles.
Unequal cross section, choice of formulas for mutual inductance of coils of, 43.
Uniform magnetic field, inductance of ring rotating in, 113, *114*.
Uniform distribution of current in cross section, 140, 141, *147–149*; *see also* High frequency formulas.
Uniform winding. *See* Dimensions of equivalent current sheets.

W

W, Y, and Z functions, definitions of, 174; expansions for, 176, 177; *see also* Table XXII.
Wallis, series expansions for elliptic integrals, 9.
Webster, inductance of a solenoid, 121, *132*.
Weierstrass, p functions, 17, 65.
Weinstein, mutual inductance of coaxial circles, 7, *25*; mutual inductance of coaxial coils, 40, 41, *51*, 71, *96*; inductance of coil of rectangular cross section, 137, *143*.
Wiedemann, mutual inductance of circles, footnote, 13.
Wien, inductance of ring, 111, 112; ring rotating in uniform magnetic field, 113, *114*.

Y

Y function. *See W*, *Y*, and *Z* functions.

Z

Z function. *See W*, *Y*, and *Z* functions.
Zonal harmonics, 59.

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110. A New Method of Determining the Focal Length of a Converging Lens . . . . . *Irwin G. Priest*
111. A New Method for the Absolute Measurement of Resistance . *Edward B. Rosa*
112. The Theory of Coupled Circuits . . . . . *Louis Cohen*
113. A Volt Scale for a Watts-per-candle Meter . . . . . *Herbert E Ives*
114. The Coefficient of Reflection of Electrical Waves at a Transition Point . . . . . *Louis Cohen*

115. A Tungsten Comparison Lamp in the Photometry of Carbon Lamps . . . . . *Herbert E. Ives and L. R. Woodhull*
116. The Determination of the Ratio of Transformation and of the Phase Relations in Transformers . . . . . *E. B. Rosa and M. G. Lloyd*
117. The Determination of the Magnetic Induction in Straight Bars . . . . . *Charles W. Burrows*
118. A Method for Constructing the Natural Scale of Pure Color . . *P. G. Nutting*
119. An Approximate Method for the Analysis of EMF. Waves . . *P. G. Agnew*
120. The Thermoelectric Properties of Tantalum and Tungsten . *W. W. Coblentz*
121. The Estimation of the Temperature of Copper by Means of Optical Pyrometers . . . . . *George K. Burgess assisted by J. F. Crowe*
122. The Resolving Power of Objectives . . . . . *P. G. Nutting*
123. The Theory of the Hampson Liquefier . . . . . *Edgar Buckingham*
124. Platinum Resistance Thermometry at High Temperatures . . . . . *C. W. Waidner and G. K. Burgess*
125. The Daylight Efficiency of Artificial Illuminants . . . . . *H. E. Ives*
126. Coupled Currents in which the Secondary has Distributed Inductance and Capacity . . . . . *Louis Cohen*
127. Effect of Phase of Harmonics upon Acoustic Quality . . . . . *M. G. Lloyd and P. G. Agnew*
128. White Light from the Mercury Arc and its Complementary . *Herbert E. Ives*
129. The Regulation of Potential Transformers and the Magnetizing Current . . . . . *M. G. Lloyd and P. G. Agnew*
130. The Determination of the Constants of Instrument Transformers . . . . . *P. G. Agnew and T. T. Fitch*
131. Selective Radiation from Various Solids, II . . . . . *W. W. Coblentz*
132. Luminous Efficiency of the Firefly . . *Herbert E. Ives and W. W. Coblentz*
133. Luminosity and Temperature . . . . . *P. G. Nutting*
134. A Theoretical and Experimental Study of the Vibration Galvanometer . . . . . *F. Wenner*
135. Specific Heat of Some Calcium Chloride Solutions between −35° C. and +20° C . . . . . *H. C. Dickinson, E. F. Mueller, and E. B. George*
136. On the Definition of the Ideal Gas . . . . . *Edgar Buckingham*
137. Mica Condensers as Standards of Capacity . . . . . *Harvey L. Curtis*
138. The Mutual Induction of Two Parallel Coaxial Circles in Terms of Hypergeometrical Series . . . . . *Frederick W. Grover*
139. A New Method for the Absolute Measurement of Electric Quantity . . . . . *Burton McCollum*
140. The Comparative Sensitiveness of Some Common Detectors of Electrical Oscillations . . . . . *Louis W. Austin*
141. Photometric Units and Nomenclature . . . . . *E. B. Rosa*
142. A Modified Method for the Determination of Relative Wave Lengths, especially adapted to the Establishment of Secondary Standards . . . . . *Irwin G. Priest*
143. Note on the Temperature Scale between 100 and 500° C. . . . . *C. W. Waidner and G. K. Burgess*
144. A New Form of Direct-Reading Candlepower Scale and Recording Device for Precision Photometers . . . . . *George W. Middlekauff*
145. A Device for Measuring the Torque of Electrical Instruments . . . . . *F. G. Agnew*
146. The Intensities of Some Hydrogen Argon and Helium Lines in Relation to Current and Pressure . . . *P. G. Nutting and Orin Tugman*
147. The Temperature Coefficient of Resistance of Copper . . . *J. H. Dellinger*
148. The Electrical Conductivity of Commercial Copper . . . . . *F. A. Wolff and J. H. Dellinger*
149. On the Constancy of the Sulphur Boiling Point . . . . . *C. W. Waidner and G. K. Burgess*
150. Note on Oscillatory Interference Bands and Some of their Practical Applications . . . . . *G. O. Squier and A. C. Crehore*

151. The Effect of Preliminary Heat Treatment upon the Drying of Clays . . . . . *A. V. Bleininger*
152. The Reflecting Power of Various Metals . . . . . *W. W. Coblentz*
153. The Action of Sunlight and Air upon Some Lubricating Oils . . *C. E. Waters*
154. The Visibility of Radiation. A Recalculation of Konig's Data . . *P. G. Nutting*
155. A Photometric Attachment for Spectroscopes . . . . . *P. G. Nutting*
156. Selective radiation from Various Substances, III . . . . . *W. W. Coblentz*
157. The Measurement of Electrical Oscillations in the Receiving Antenna . . . . . *L. W. Austin*
158. Some Experiments with Coupled-High-Frequency Circuits . . . *L. W. Austin*
159. Some Quantitative Experiments in Long Distance Radiotelegraphy . . . . . *L. W. Austin*
160. The Behavior of High-Boiling Oils on Heating in the Air . . . *C. E. Waters*
161. The Determination of Vanadium in Vanadium and Chrome-Vanadium Steels . . . . . *J. R. Cain*
162. On the Computation of the Constant C_2 of Planck's Equation by an Extension of Paschen's Method of Equal Ordinates . . . . . *Edgar Buckingham and J. H. Dellinger*
163. A Comparison of American Direct Current Switchboard Voltmeters and Ammeters . . . . . *T. T. Fitch and C. J. Huber*
164. Study of the Current Transformer with Particular Reference to Iron Loss . . . . . *P. G. Agnew*
165. Thermodynamics of Concentration Cells . . . . . *Henry S. Carhart*
166. The Capacity and Phase Difference of Paraffined Paper Condensers as Functions of Temperature and Frequency . . . . . *Frederick W. Grover*
167. The Steam Expansion Line on the Mollier Diagram and a Short Method of Finding the Reheat Factor . . . . . *Edgar Buckingham*
168. Radiometric Investigation of Water of Crystallization, Light Filters, and Standard Absorption Bands . . . . . *W. W. Coblentz*
169. Formulas and Tables for the Calculation of Mutual and Self Induction (2d edition, revised and enlarged) . . . . *E. B. Rosa and Frederick W. Grover*
170. The Correction for Emergent Stem of a Mercurial Thermometer . . . . . *Edgar Buckingham*
171. Thermometric Lag . . . . . *D. R. Harper*
172. Deflection Potentiometers for Current and Voltage Measurements . . . . . *H. B. Brooks*
173. Outline of Design of Deflection Potentiometers with Notes on the Design of Moving-Coil Glavanometers . . . . . *H. B. Brooks*
174. The Determination of Total Sulphur in India Rubber . . . . . *C. E. Waters and J. B. Tuttle*

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No. 1. Verification of Standards and Measuring Instruments.
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No. 28. The Determination of the Optical Properties of Materials.
No. 29. Announcement of a Change in the Value of the International Volt.
No. 30. Lime: Its Properties and Uses.

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International Metric System (chart).
Table of the Equivalents of Customary and Metric Weights and Measures.
The International Metric System of Weights and Measures. (Pamphlet.)
First Conference on the Weights and Measures of the United States.
Second Annual Conference on the Weights and Measures of the United States.
Fourth Annual Conference on the Weights and Measures of the United States.
Fifth Annual Conference on the Weights and Measures of the United States.

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DEPARTMENT OF COMMERCE AND LABOR

Vol. 8 No. 2

BULLETIN

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

JUNE 15, 1912

CONTENTS

Page

The Correction for Emergent Stem of the Mercurial Thermometer . . . . . *Edgar Buckingham* 239

A Determination of the International Ampere in Absolute Measure . . . . . *E. B. Rosa, N. E. Dorsey, and J. M. Miller* 269

Deflection Potentiometers for Current and Voltage Measurements . . . . . *H. B. Brooks* 395

Outline of Design of Deflection Potentiometers, with Notes on the Design of Moving-Coil Galvanometers . . . . . *H. B. Brooks* 410

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		Page
The Correction for Emergent Stem of the Mercurial Thermometer . . .	*Edgar Buckingham*	239
A Determination of the International Ampere in Absolute Measure . . .	*E. B. Rosa, N. E. Dorsey, and J. M. Miller*	269
Deflection Potentiometers for Current and Voltage Measurements . . .	*H. B. Brooks*	395
Outline of Design of Deflection Potentiometers, with Notes on the Design of Moving-Coil Galvanometers . . .	*H. B. Brooks*	419

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THE CORRECTION FOR EMERGENT STEM OF THE MERCURIAL THERMOMETER

By Edgar Buckingham

CONTENTS

	Page
1. The standard scale	239
2. The mercury-in-glass scale	241
3. The correction for emergent stem	243
4. The use of an auxiliary stem	245
5. The "fadenthermometer" and its use	249
6. The coefficient of relative expansion; numerical values	254
7. The stem error of the fadenthermometer; Mahlke's method of setting	257
8. On the selection of a fadenthermometer	259
9. Relative merits of the auxiliary stem and the fadenthermometer	260
Notes:	
Note I.—The present status of the standard gas scale	262
Note II.—On the use of the term "mean temperature"	264
Note III.—The apparent expansion of mercury in glass	265

Two important methods for determining the "correction for emergent stem" to be applied to the reading of a mercurial thermometer when the thermometer is not used with "total immersion" consist, first, in the use of an auxiliary stem as described by Guillaume [1]; and, second, in the use of the "fadenthermometer" or thread thermometer as described by Mahlke.[2] The original accounts of these two methods are not available in English and neither of them has been discussed very fully. In view of this and of the further fact that the standard of precision of mercurial as of other thermometry has continued to advance since the original publications, it seems worth while to treat the subject in some detail. For various suggestions as to both form and substance the writer is much indebted to other members of the heat division, and especially to Dr. H. C. Dickinson.

1. THE STANDARD SCALE

It is universally considered desirable that statements of temperature should be made in terms of the scale of the ideal-gas thermometer as an ultimate standard. This scale, also called the "thermodynamic scale," is at present represented by a series of

[1] Zeitschrift für Instrumentenkunde, 12, p. 69; 1892. 13, p. 155; 1893.

[2] Ibid., 13, p. 58; 1893.

numerical values assigned to certain reproducible "fixed points" at which primary standard determinations of temperature have been made by using gas thermometers filled with helium, hydrogen, nitrogen, or air. Such a gas thermometer gives *directly* only readings in terms of its own individual scale, but if the properties of the gas have been sufficiently investigated, corrections may be applied which convert these readings into values of the temperature on the ideal-gas scale. Usually, though not always, these corrections are less than the uncertainties of even the best determinations of temperature by the gas thermometer.

On account of the difficulties of accurate work with the gas thermometer, some discrepancies still exist in the values obtained for definite fixed points, and the corrections for reducing to the ideal-gas scale are also somewhat in doubt, though this latter uncertainty is relatively unimportant. The series of numbers to be adopted as the most probable values, on the ideal-gas scale, of the temperatures of any set of fixed points, is therefore to a small extent a matter of opinion.

The series of fixed points and the numerical values for their temperatures adopted by the authorities of a given standardizing laboratory constitute the foundation of the standard scale of that laboratory and provide, so to speak, a certain number of base points. The definition of the scale is completed by stating the means used for interpolation between the base points in determining intermediate temperatures.

When a thermometric instrument for such interpolation has been standardized at two or more base points, it becomes a secondary standard for the given range. For determining temperatures between any two adjacent base points, various secondary standards, dependent for their action on entirely different physical principles, may often be used, and each form of instrument may have its particular advantages and disadvantages of precision, convenience, etc., for a given piece of experimental work. It is therefore usual to have each interval between the base points covered by two or more secondary standards of different nature. The greater the number of different methods of interpolation, and the simpler the means adequate to attaining a satisfactory agreement of the instruments in defining a single

scale over the interval in question, the greater is the confidence usually felt that the scale so defined would agree with an interpolation by a primary standard gas thermometer. It is by no means certain that this confidence is always justified.

The standard scale of another laboratory may be slightly different because of the adoption of other fixed points or of other numerical values as most probable, and also by the adoption of other methods of interpolation. But if the definitions of the scales are clear in these respects, each standard scale is definite and the relation between two such scales, if not immediately evident, may be found by simple experiments capable of a precision far surpassing the accuracy with which the standard scales represent the ideal-gas scale.

Up to about 1,650° C, there are, with one or two exceptions, no very serious divergences of opinion as to the base points, and the various methods used for interpolation also agree closely; so that the expression "standard gas scale" may be regarded as sufficiently precise in meaning for our present purposes, even though we feel sure that a slightly closer approximation to the ultimate standard scale of the ideal gas will be attained in the future.[3]

2. THE MERCURY-IN-GLASS SCALE

Let us consider a mercurial thermometer which has a capillary stem of exactly uniform cross section. Let the glass be such that its changes of volume with temperature are reversible and show neither frictional nor viscous hysteresis. Let a mark be placed on the stem exactly at the point where the end of the mercury column stands when the mercury and its surrounding glass are at the temperature of the ice point, and let another mark be similarly placed for the steam point. The external as well as the internal pressures on the bulb must be the same when the two marks are made. Let the stem be provided with a scale of equal parts[4] upon which these marks are at 0° and 100°, respectively.

[3] See Note I at the end of this paper.

[4] The scale divisions are to be of equal length when the glass on which the graduations are made is all at the same temperature.

Then wherever the end of the mercury column may stand, if the whole of the mercury and the surrounding glass are at the same temperature, and if the internal and external pressures on the walls of the bulb are the same as when the 0° and 100° marks were made, the reading on the scale is *by definition* the numerical value of the temperature, on the centigrade scale of this thermometer.

The ideal thermometer just described is, like the ideal gas, nonexistent. The actual mercurial thermometer is a more or less approximate representation of it, and by the application of suitable corrections, the readings of the actual thermometer may be reduced to those of the ideal thermometer at the same uniform temperature. These corrections are as follows: (*a*) The scale correction to allow for errors of graduation; (*b*) the calibration correction to allow for nonuniformity of bore; (*c*) the external pressure correction to allow for the effect on the volume of the bulb, of variations in the outside pressure; (*d*) the internal pressure correction to allow for similar effects due to variations of pressure within; (*e*) the zero correction to allow for the fact that at the temperature of the ice point the reading is not exactly at the zero mark of the scale; (*f*) the fundamental-interval correction to allow for the fact that the difference of the readings at the temperatures of the ice and steam points is not exactly 100° of the scale.

If the thermometer has been suitably constructed, all these corrections may be determined by experiment. By the determination of the corrections, the thermometer is converted into a primary standard thermometer which gives, after the application to its readings of these several corrections, numerical values of temperature in terms of the mercury-in-glass scale of this particular thermometer.[5] It is not our purpose here to discuss these various corrections, but merely to call attention to the fact that by means of them, we may find the readings of an ideally perfect thermometer made of the same glass as the actual thermometer, so that we may dismiss the corrections from further consideration and refer at once to the ideal thermometer as if it did exist, and to its scale of temperature as something definite.

[5] These corrections have been discussed by Messrs. Waidner and Dickinson in this Bulletin, 8, p. 663, where full references will be found.

The indications of a mercury-in-glass thermometer depend on the relative expansion of mercury and glass; and since glass is not a precisely reproducible substance and different kinds of glass expand and contract differently, no two mercurial thermometers define precisely the same scale. Different thermometers, well made of as nearly as possible the same kind of good thermometer glass, do, however, define almost exactly the same scale, which is then known as the mercury-in-glass scale of that particular glass. A determination of temperature even by a primary standard mercurial thermometer has no precise meaning except in connection with one particular glass, and to give it any general significance the result must be expressed in terms of some scale which is better defined, or more easily reproducible. The scale actually used for this purpose is the standard gas scale described in section 1, and to reduce a primary standard determination of temperature on any mercury-in-glass scale to the standard gas scale, one further correction, the "gas-scale correction," is needed.

At ordinary temperatures, the gas-scale correction is comparatively small for the common thermometer glasses, but at high temperatures it may be very large. According to Mahlke's observations,[6] at 500° C of the standard gas scale, a mercurial thermometer made of Jena 59^{III} borosilicate glass reads 527°.8, so that the gas-scale correction is −27°.8.

Ordinarily, mercurial thermometers are not primary standards but are standardized by comparison with other thermometers. Thus, in any case, the corrections given by the test reduce the readings to some other scale than the mercury-in-glass scale of the thermometer being tested, and it is most convenient to make the reduction directly to the standard gas scale, the temperature of the bath in which the test is made being given by some instrument which has been standardized as a secondary interpolation instrument for the gas scale, as described in section 1.

3. THE CORRECTION FOR EMERGENT STEM

It was assumed in the preceding section that whenever the mercurial thermometer was read, the whole of the mercury and the surrounding glass were at a uniform temperature, for if this

[6] Zeitschrift für Instrumentenkunde, 15, p. 178; 1895.

is not the case the reading has by itself no definite meaning. In order, therefore, that the results of testing a mercurial thermometer should be capable of precise interpretation, the corrections are stated for "total immersion," i. e., they refer to readings taken when the whole of the thermometer, at least up to the end of the mercury column in the stem, is immersed or enclosed in a bath or space of uniform temperature.

In practice, mercurial thermometers are often used with only partial immersion, and if the temperature of the space into which the bulb of the thermometer is inserted is markedly different from that of the surroundings of the emergent stem, the temperature of a part of the stem and its enclosed mercury column will be different from that of the bulb and, furthermore, will not be uniform. If, for example, a high temperature is to be determined, the emergent stem will be colder, the mercury column shorter, and the reading lower than if the thermometer were totally immersed. To allow for this fact, a "correction for emergent stem," or more briefly a "stem correction," must be applied. This stem correction is of sensible importance more frequently than is sometimes recognized, and may in extreme cases amount to as much as 30° C or even more.

The temperature of the glass above the end of the mercury column has no direct influence on the reading. But mercurial thermometers for use above 200° or 250° C have to be filled above the mercury with nitrogen or other inert gas under pressure, to prevent the mercury from boiling, so that raising the whole of such a thermometer to a high temperature might, by increasing the internal pressure beyond that intended by the maker, cause an explosion and the destruction of the thermometer. In any high-temperature mercurial thermometer used with the gas space at a lower temperature than that to be measured, there is a troublesome tendency for the mercury to distill off and condense in the colder part of the gas space. It may be impossible to reunite the resulting drops of mercury with the column below, so that an error is produced equivalent to a permanent depression of the ice point. To reduce this distillation, the temperature of the meniscus should be kept as low as possible, so that high-temperature mercurial thermometers must, in general, be used with a portion of

the stem emergent and at a much lower temperature than that of the bulb.

In a large majority of cases it is needless to attempt to determine the stem correction closer than to about 5 per cent of itself. For frequently when the correction is large, which occurs usually only in work at high temperatures, such causes as unsteadiness of the temperature or failure of the thermometer to repeat its readings under identical conditions, make it impossible to be certain of the final result of the measurement of a temperature closer than to 5 per cent of the value of the stem correction, no matter how exactly this value itself may be known. We shall speak of such work as "ordinary work," in distinction from "work of high precision," in which a greater final certainly in the result is attainable and the highest possible accuracy is desired.

The magnitude of the stem correction can be found either directly or by calculation. Since it obviously depends on the length and temperature of the emergent stem, it can, in principle, be calculated from measurements of these quantities in connection with data on the relative expansion of mercury in the glass of the stem; but certain difficulties in both the theory and the practice of this method make it advisable, in most cases, to use the much simpler direct method described by Guillaume. This will be treated first.

4. THE USE OF AN AUXILIARY STEM

Let SB (Fig. 1) be the thermometer of which the stem correction is to be determined, immersed to the level h in a bath or space of uniform or nearly uniform temperature. Let the bulb B and the lower part of the stem up at least to the level d, have the uniform temperature $t°$, expressed in the standard gas scale. The upper part of the stem and enclosed mercury column ending in the meniscus at c, have a different temperature, lower let us say.

Let A be an auxiliary stem similar in material and construction to a portion of the working stem S and of approximately the same inside and outside dimensions. It is to be sealed at both ends and the space above the mercury is to be exhausted or filled with gas under pressure to correspond with the conditions in S. A is provided, in the vicinity of the meniscus, with a scale of equal parts, e. g., a millimeter scale. The length l of the mercury

column in A must be sufficient that when A is placed parallel and close to the working stem S and with its meniscus at the same level c, its lower end at d reaches into the region of uniform temperature. The magnitude of l is of little importance so long as it is great enough. It is assumed that both A and S are of sensibly uniform bore.

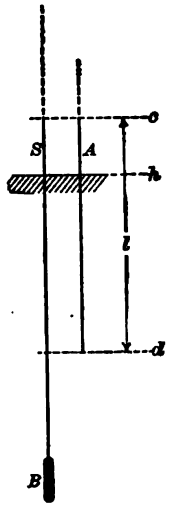

Fig. 1

When A and S are thus placed parallel and close together, the temperature at any given level may be assumed to be the same in both, this temperature being, on the average, lower than t° between c and d, if t° is above the temperature of the surroundings of the emergent stem. If A were now totally immersed and thus brought to the temperature t°, its meniscus would rise by a distance Δ in consequence of the rise in temperature of most or all of its elements. Since S is of the same glass as A, the same increase of length of the mercury column cd in S would occur if the temperature of each of its elements were also raised to t° by total immersion, while no change would occur in the volume of the mercury below d since that is already at the temperature t°. Hence the meniscus in S would rise by the same amount Δ, and Δ is the linear magnitude of the stem correction to be applied to the reading on the working stem S to reduce the reading to the condition of total immersion. The correction *in degrees* is then

given by the equation

$$K = n\Delta \tag{1}$$

in which n is the number of degrees of the scale of S, at the level c, included in one scale division of the auxiliary stem A. The problem is now to determine the value of Δ from observations on A, and for this purpose we need, in addition to the reading when in the position shown in the figure, the reading with total immersion at the temperature $t°$.

If the conditions are similar to those during the comparison of high-temperature thermometers in a well-stirred oil bath, the operation is very simple. After an observation of the position of its meniscus when at the level c, the auxiliary stem is lowered to a position of total immersion, left a short time (to be determined by trial) till it assumes the uniform temperature $t°$, then quickly raised just far enough to make another observation possible, and read immediately. The glass of the stem being thick and a poor conductor, the reading after raising changes but slowly, and if made at once may be taken as the reading for total immersion. The difference of the two readings is the value of Δ in terms of the scale of A.

Under the above conditions, when the space of uniform temperature is large enough to permit of total immersion of the auxiliary stem, the method just given is the simplest and most expeditious possible for finding the stem correction. If A satisfies the conditions of being made of the same glass as S, and of approximately the same dimensions, it requires no preliminary study. It is not necessary that the length of its divisions be known in millimeters. All that is needed for converting Δ into degrees of the working stem S is that the number n of degrees on S per scale division of A shall be known. This may be found, after removal from the bath, by using A as a scale with which to measure the degrees on S. Since the two scales are graduated on glass of the same kind, the fact that the comparison is made at room temperature instead of the higher temperature does not introduce any error. It is not necessary that the value of $t°$ be known, even approximately, so that the stem correction may be applied to the reading of S before any other corrections and even if the other

corrections—zero correction, reduction to gas scale, etc.—are unknown.

If the space of uniform temperature is not long enough for total immersion of the auxiliary stem, the foregoing method can not be used; and if the space, though long enough, is changing in temperature rather rapidly, the time required may be sufficient to introduce a sensible error—the reading after total immersion corresponding to a different temperature from that of the bulb B when the original reading was made. In either case, the difficulty may be surmounted by standardizing the scale of the auxiliary stem, once for all, by total immersion in baths of known temperatures so that its reading with total immersion at any temperature $t°$ may be found from a table of the results of the standardization. If we know the values of the relative expansion of mercury in the given glass, and the value, in scale divisions, of the length of the column at the ice point, the scale reading for total immersion at any other temperature, $t°$, may also be computed from that at the ice point, and an experimental standardization is then unnecessary. Since the value of $t°$, which is usually the quantity sought, is involved in this process, successive approximations may be needed. For both these reasons the method loses, in such cases, something of its primitive simplicity and directness.

The stipulation was made that A and S should be of sensibly uniform bore, though it will be noted that we have, in reality, used only the condition that the cross-sectional areas of the bores should have the same ratio at all levels, not that each should be constant. The condition was put in the simpler form because uniformity of bore is always the ideal aimed at, and the stems of well-made thermometers do in practice have nearly cylindrical bores. A complete treatment of the general case in which the cross-section ratio varies with the level will not be attempted because it would certainly not be worth while from a practical standpoint. Accurate determinations of temperature can not be obtained with poorly made mercurial thermometers, and it may safely be assumed that thermometers which are satisfactory in other respects will have stems of sufficiently uniform caliber that, when used with auxiliary stems of a similar degree of uniformity, the error introduced into the stem correction by assuming both stems to be exactly uniform, will not be serious.

5. THE "FADENTHERMOMETER" AND ITS USE

As noted above, the use of an auxiliary stem may require a preliminary standardization by total immersion. If we define the "mean temperature" of a column of mercury enclosed in glass as the uniform temperature at which the mercury would fill the tube between the same two marks on the glass, the auxiliary stem, after the standardization, may be regarded as a thermometer which indicates its own mean temperature [7] and therefore that of the adjacent portion *cd* of the working stem *S*. This mean temperature $f°$ is given in terms of the scale used in standardizing the auxiliary stem, which we assume to have been the gas scale.

As a thermometer, the simple auxiliary stem is very insensitive, for if the mercury column is 20 cm long it requires a change of of about 30° in the mean temperature to change the reading by 1 mm. The sensitiveness and precision of reading may evidently be increased by adding a still finer capillary stem in which the changes of level of the meniscus may be observed on a magnified scale, and by this modification Mahlke converts the simple auxiliary stem into a thermometer which differs from the more usual forms only in having a very elongated, thick-walled, cylindrical bulb of small total volume, and an unusually fine stem.

Like the simple auxiliary stem of which it is a development, the bulb of this "fadenthermometer" must be of similar construction and dimensions to the working stem with which it is to be used. The method of making sure that the temperature of the auxiliary instrument is the same at any level as that of the neighboring portion of the working stem, by having the two as nearly as possible geometrically similar and similarly placed, is thus retained. But the determination of the stem correction directly as a *length* Δ is abandoned, and a computed value of Δ, involving the length and mean temperature of the emergent stem, and the relative expansion [8] of mercury in glass is substituted for it. The fadenthermometer need not, therefore, be made of the same glass as

[7] See Note II at the end of the paper.
[8] See Note III at the end of the paper, also sec. 6

the working stem S.[9] As in the case of the auxiliary stem, the length of the bulb of the fadenthermometer is of minor importance if it is sufficient, though it should not be excessive, as will be shown in section 8.

Let SB (Fig. 2) be the thermometer for which the stem correction is desired, the meniscus being at the level c above the surface of the bath h, so that a length ch *at least* is at a different mean temperature from the bulb B and the lower part of the stem S.

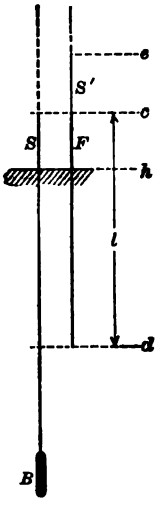

Fig. 2

As before, we may assume for the sake of concreteness that B is at a higher temperature than the emergent stem.

Let F be the bulb of the fadenthermometer set parallel and close to S with its upper end at the level c. Let S' be the very fine thread of mercury, ending at e, in the stem of the fadenthermometer. Its volume is small and we may, for the present, disregard the fact that its temperature is not quite the same as that of the bulb F. The length of the bulb of the fadenthermometer must be sufficient that its lower end, at the level d, is within the region of uniform temperature containing the bulb of the main thermometer. It may now be assumed that the mean temperature of F is the same as that of the length cd of the

[9] But see Note II at end of the paper.

working stem S, which includes all of S which is not at the same temperature as B.

Let the fadenthermometer have been standardized in terms of the gas scale by immersion in baths of uniform temperature so that its reading gives the mean temperature $f°$ of its bulb. Let $t°$ be the true temperature of B, also in terms of the gas scale. Let a_f be the total relative expansion of mereury in the glass of which the working stem is made, from $o°$ to $f°$, and let a_t be that from $o°$ to $t°$. Let l be the length in centimeters of the bulb of the fadenthermometer measured *at room temperature*. Let n be the number of degrees per centimeter on the scale of the working stem at the point where the reading is made, also measured when the stem is *at room temperature*. Then the value of the stem correction, K, is given in degrees by the equation [10]:

$$K = nl\frac{a_t - a_f}{1 + a_f} \tag{2}$$

If n, l, t, and f have been determined, it remains to find the values of a_t and a_f. For this purpose the relative expansion of mercury in the glass of the working stem must have been investigated and expressed in the form of a curve, a table, or an equation from which, knowing t and f, we find a_t and a_f.

This process may be modified by introducing the mean *coefficient* of relative expansion between $f°$ and $t°$—i. e., the average rate of increase of a, per degree, within this interval. This mean coefficient α is defined by the equation—

$$a_t - a_f = \alpha(t - f);$$

so that equation (2) may be put into the equivalent form—

$$K = \frac{nl\alpha(t - f)}{1 + f\alpha_f} \tag{3}$$

in which α_f is the mean coefficient of relative expansion between $o°$ and $f°$.

The value of $1 + a_f$ or $1 + f\alpha_f$ is always close to unity and may be computed with sufficient accuracy by setting $\alpha_f = 0.000\,16$, for any glass and any value of f. When $f \lesseqgtr 500°$, the value of α_f lies

[10] See Note III at end of the paper, where the deduction of this equation is given.

between 0.000 15 and 0.000 18, and using these as limits, we have the following table of values of $1+f\alpha_f$:

	$f=300°$	400°	500°
$1+0.000\ 15f=$	1.045	1.060	1.075
$1+0.000\ 16f=$	1.048	1.064	1.080
$1+0.000\ 17f=$	1.051	1.068	1.085
$1+0.000\ 18f=$	1.054	1.072	1.090

From this it appears that even at $f=500°$, the greatest possible error due to using the fixed value $\alpha_f=0.000\ 16$ is only 1 per cent, an amount which would invariably be negligible at such a high stem temperature. At lower values of f, the error is less, both because f is a factor of the error and because the true value of α_f is closer to 0.000 16. It is therefore always sufficiently exact to write equation (3) in the form

$$K=\frac{nl\alpha(t-f)}{1+0.000\ 16f} \tag{4}$$

For practical use this may be put into the more convenient form

$$K=nl\alpha(t-f)-A \tag{5}$$

where A is 1 per cent of K for each 60° in f. If, for example, $f=300°$, $A=5$ per cent of K so that in "ordinary work" A may be disregarded when f is less than 300°.

Since α depends on both t and f, it can not be represented exactly by a curve, a table with one argument, or an equation in one independent variable, so that it seems, at first sight, as if equations (3), (4), and (5) would be less convenient in practice than equation (2). In fact, however, to the degree of precision needed here, these equations are easier to use than equation (2), for the following reasons:

(*a*) The value of α does not vary very much, so that for many rough calculations it is sufficiently exact to use a constant value of α regardless of the values of f and t.

(*b*) The variation of α with f and t being slow, a comparatively small table with two arguments is sufficient to give the required value of α, by a single reading with no interpolation, or a very easy one, to an accuracy at least as high as that of the experimental data on the relative expansion.

(c) To the precision required in determining a stem correction this double-argument table may be replaced by one with the single argument $\frac{t+f}{2}$, which may also be made quite short without requiring any but the simplest interpolation.

We shall therefore use equations (3), (4), and (5) in preference to the apparently simpler but really less convenient equation (2). Numerical values of α will be considered in section 6, but we now proceed to the consideration of the quantities n and t, which appear in the second members of equations (3), (4), and (5).

The value of n may be readily found by measurement with a millimeter scale, and in a region of uniform graduation requires no further comment than that its percentage accuracy must be commensurate with that desired in K. High-temperature thermometers are often, however, graduated uniformly over intervals of 50° or 100° with abrupt changes in the length of the scale divisions at the even 50° or 100° marks, the change in n amounting sometimes to 6 or 7 per cent. If, as frequently happens in testing thermometers, the observed reading falls just below such a point while the correction carries the reading past it, the proper value of n will evidently be an appropriate weighted mean value. For ordinary work it is sufficiently exact to use the value of n for the interval in which the greater part of the correction falls, for this will never differ by more than 3 or 4 per cent from the correct mean value. When the highest accuracy is desired, the possibility of introducing an unnecessary error by neglecting this point should not be overlooked.

The temperature $t°$ in the equations is the temperature of the bulb B expressed, like f, in terms of the gas scale. Its value may be known, but more often it is the quantity sought and is to be found by applying the stem correction K to the observed reading. If t_1 is the reading after the application of all the known corrections, leaving only the stem correction outstanding, we have $t=t_1+K$, and may easily derive an exact equation for K in terms of the observed temperature t_1. But it is more convenient to proceed by successive approximations, the first consisting in the use of the observed t_1 instead of t in equation (4) or (5). If K is small,

the first approximation may suffice, but if not, a second is easily made. For example, in equation (5) let

$$n=8,\ l=19.\ \text{cm},\ \alpha=0.000\ 17,\ t_1=430°,\ f=200°.$$

We then have $nl\alpha=0.0258$ and $A=3.3$ per cent of K, so that the first approximation gives us

$$K=0.0258\ (430-200)-A=5.93-0.20=5.73,$$

whence the value of t is 435°73. In the second approximation A will have sensibly the same value as in the first, and we have

$$K=0.0258\ (435.73-200)-A=6.08-0.20=5.88,$$

whence the value of t is 435°88, or 0.15° higher than before, a difference which would usually but not always be negligible.

The error in t due to stopping at the first approximation K_1 is very nearly $\frac{K^2_1}{t_1-f}$, so that it is easy to see whether, in any given case, a second approximation is needed. Frequently it is, but a third is probably always superfluous.

6. THE COEFFICIENT OF RELATIVE EXPANSION; NUMERICAL VALUES

If all temperatures could be expressed in terms of the mercury-in-glass scale of the glass of which the working stem is made, the coefficient of relative expansion would be constant by definition and could be determined by a dilatometer experiment between the ice and steam points. But since, for reasons already given, we express all our temperatures in terms of the standard gas scale, the value of α is not constant, and experimental data on expansion are needed over the whole range $f°$ to $t°$. Our experimental knowledge of the values of α is limited to a very few glasses. Such data as available are, with one exception, quoted in Hovestadt's "Jena Glass," [11] where references to the original papers will be found.

The three most important thermometer glasses are the French "verre dur," Jena 16^{III} "normal glass," and Jena 59^{III} borosilicate glass. These and a few others have been very carefully studied between 0° and 100°, but at the higher temperatures we

[11] English translation by J. D. and A. Everett: Macmillan, 1902.

have data only on verre dur to 200°, 16^{III} to 300°, and 59^{III} to 500°. For these three glasses the values in the following table may be used in connection with equations (4) and (5); they are probably correct to somewhat better than 1 per cent:

Table of Mean Coefficients of Relative Expansion

Value of $\alpha \times 10^6$

$\frac{t+f}{2}$	Verre dur	16^{III}	59^{III}
50	158	158	164
100	158	158	164
150	158	158	165
200	159	159	167
225		160	169
250		161	171
275		162	173
300		164	175
325			177
350			179
375			181
400			183
425			186
450			189
475			193
500			198

The data used in computing the table were as follows: (*a*) The mean coefficients from 0° to 100° determined by Thiesen, Scheel, and Sell [12]; (*b*) the gas-scale corrections for 16^{III} given by Wiebe and Böttcher [13]; (*c*) data on 59^{III} obtained by Mahlke [14]; (*d*) the corrections of the verre dur scale to the hydrogen normal scale, given in the pamphlet of 1896, which is issued with the certificates for mercurial thermometers tested at the International Bureau of Weights and Measures.

[12] Zeitschrift für Instrumentenkunde, **16**, p. 55; 1896.
[13] Ibid., **10**, p. 245; 1890.
[14] Ibid., **15**, p. 171; 1895.

The values are given in each case up to the highest temperature for which experimental data are available, but it is evident that the use of the higher figures, when t and f are far apart and t therefore above the upper limit of the experiments, involves an extrapolation. Suppose, for example, that a thermometer with a stem of 16^{III} is being used at a temperature $t=450°$ while the stem temperature is $f=100°$. We then have $\frac{t+f}{2}=275°$ for which the table gives $\alpha=0.000\ 162$, but this value is quite uncertain, because we have no precise knowledge of how the glass behaves above 300°. On the other hand, if $t=300°$ and $f=250°$, we again have $\frac{t+f}{2}=275°$, but the value $\alpha=0.000\ 162$ may now be relied upon.

Since both verre dur and 16^{III} thermometers are used up to 450° C, it is evident that our data for determining the correction for emergent stem are very deficient, but the situation is not quite so bad as it seems. As regards verre dur, it may be said that the behavior of this glass bears a close resemblance to that of Jena 16^{III} in so many respects that the values of α given for 16^{III} will probably not be in error by 5 per cent, even at 300°, if applied to verre dur.

With respect to glasses which have not been investigated, we may make the following remarks: A comparison of the absolute expansion of mercury as determined by Callendar and Moss [15] with that of Jena 59^{III} glass, as determined by Holborn and Grüneisen,[16] in connection with Mahlke's [17] results on the relative expansion of mercury in 59^{III}, shows that for this glass the increase of α with temperature, denoting a departure of the total relative expansion from linearity, is accounted for mainly by the behavior of the mercury and only to a minor degree by that of the glass. And since the expansion of glass is only about one-tenth that of mercury, it seems probable that the change of α with temperature is not very different for different thermometer glasses, but is nearly parallel with the change for 59^{III}. The best we can do at present with stems of unknown composition or of glasses

[15] Phil. Trans. Roy. Soc., London, A 211, p. 1; 1911.
[16] Landolt and Börnstein, Tables, 3d ed., p. 201.
[17] Zeitschrift für Instrumentenkunde, 15, p. 171; 1895.

which have not been investigated is probably to use the values for 59^{III} rounded off to two significant figures. These values, thus applied, will probably not be in error by over 10 per cent, except at very high temperatures. This remark applies to the numerous thermometers which have bulbs of 16^{III}, but stems of some softer and less brittle glass. In all cases where the value of α is not known by direct experiment, it is useless to attempt great percentage accuracy in determining the stem correction, and the equation

$$K = n\,l\,\alpha\,(t-f),$$

with α from the table for 59^{III}, may be used with no further refinements.

7. THE STEM ERROR OF THE FADENTHERMOMETER; MAHLKE'S METHOD OF SETTING

We have stipulated that the fadenthermometer shall have been standardized so that its reading gives the mean temperature $f°$ of its bulb in terms of the gas scale. If this has been done, equations (2) to (5) determine the value of K; but the question arises whether such a standardization is generally possible, and if not, how the equations or the method of procedure are to be modified.

Let us first assume that the fadenthermometer has been standardized by total immersion. When it is in use in the position shown in Fig. 2, its stem S' is not at the same mean temperature as the bulb F, and its reading is subject to a secondary stem correction K' for the "emergent" stem of the fadenthermometer itself, which must be applied to the observed reading f_1 so as to give $f = f_1 + K'$ for use in the equations. The value of K' can be found in the same manner as that of K if the mean temperature of S' is known; and since it is evident that K' need be known only roughly, this temperature may be found sufficiently well by a thermometer placed with its bulb close [18] to the middle of S'.

Usually, and unless K is large, this secondary stem correction may be neglected and f_1, the reading of the fadenthermometer, may be identified with $f = f_1 + K'$, the true mean temperature of

[18] Guillaume suggests wrapping tin foil around the two.

the bulb F. This is not always permissible, however, as may be seen from the following example: Let $n=10$, $l=20^{cm}$, $\alpha=0.000\ 17$, $t=450°$, $f_1=300°$, as read without correction for the secondary stem error, and A therefore 5 per cent of K. Then we have, using the uncorrected value for f in equation (5)

$$K=10\times 20\times 0.000\ 17\times(450-300)-A=4.85 \text{ degrees.}$$

Now, suppose that the stem of the fadenthermometer is at a mean temperature of only 100°, that the length of the thread in its stem is 10^{cm}, and that there are 30 degrees per centimeter on its scale. Then its stem correction will be approximately

$$K'=30\times 10\times 0.000\ 16\times(300-100)=9°.6$$

where we have used α from 100° to 300° = 0.000 16, a sufficiently approximate value. Hence we have for the corrected value of f

$$f=f_1+K'=309°.6$$

and $A=5.2$ per cent of K. A recomputation of K now gives us

$$K=10\times 20\times 0.000\ 17\times(450-309.6)-A=4.53 \text{ degrees.}$$

In this case, therefore, neglecting the secondary stem correction would cause an error of over 0.3 degree in the primary stem correction K, an amount which may or may not be negligible under the given conditions.

Mahlke recommends a slightly modified method of setting the fadenthermometer, designed to eliminate this secondary stem error. The mercury in S' is always small in volume; its temperature also will seldom vary greatly between c and e; hence we shall make only a negligible error if we assume the mean temperature of S' to be the same as the temperature at its lower end c where it joins the bulb. If, then, we imagine the cylindrical bulb extended upward and the mercury now in S' run down without change of temperature into this extension; and if we next lower the whole fadenthermometer by a short distance m till this imaginary new position of the meniscus (instead of the end of the bulb F) is at the level c, the mean temperature of the mercury in the whole fadenthermometer will now be the same

as that of a length $(l+m)$ of the working stem S. Hence we shall have eliminated the secondary stem error if we use the observed reading of the fadenthermometer but use $(l+m)$ in place of l. Mahlke's actual procedure, therefore, is to estimate m from the easily computed relative cross sections of S' and F, and set the fadenthermometer so that the top of its bulb is below the level c by this small amount m, adding m to the measured length l of the bulb before making further computations for the primary correction K.

On account of the difficulty of construction, the transition from stem to bulb in the fadenthermometer is usually somewhat irregular and there is frequently a slight enlargement at the junction. The correct setting and the point which is to be considered as the end of the bulb, from which l is measured, are thus often somewhat uncertain, and it is doubtful whether Mahlke's method presents any real advantage over the simpler method of setting always at the same point and using a fixed value of l, applying the secondary stem correction in the few cases where it is worth while.

8. ON THE SELECTION OF A FADENTHERMOMETER

It sometimes happens that several fadenthermometers are available for determining a certain stem temperature. The following principles should then govern the selection of the one to be used. Since it is a fundamental assumption that the temperature distribution is the same along the fadenthermometer bulb as along the working stem, the more alike these are the better. If the fadenthermometer is very much larger than the working stem, especially of larger bore, it will read too high (when used for temperatures above those of the surroundings) on account of longitudinal conduction, so that very large diameters are undesirable. A fadenthermometer of the ordinary type must not be used with a working stem of the enclosed-scale or "einschluss" form nor vice versa, as the fundamental condition regarding temperature distribution would not be at all closely fulfilled in either case. These remarks are equally applicable to the choice of an auxiliary stem when several are available.

Of several fadenthermometers otherwise equally suitable, the shortest which will reach into the region of uniform temperature

is to be preferred. One reason for this is that the shorter the bulb F the lower will be its mean temperature and, therefore, the less the secondary correction for its emergent stem in which the mercury thread will be short. A second reason appears from equation (4). For the larger $(t-f)$, i. e., the smaller f is, the less is the error involved in using the observed temperature t_1 instead of the corrected temperature t, and the less, also, is the effect of any error in f. Hence with a suitably short fadenthermometer several approximations may be permissible, which would not be sufficient for the desired accuracy with a longer one.

The emergent stem of the fadenthermometer should be short, both in degrees and in absolute value. The stem is often made unnecessarily fine. Beside making the emergent thread long at high temperatures and thus liable to have a very different mean temperature from that of the bulb, the use of a very fine capillary involves difficulties of construction which result in an irregular joining of the bulb and stem. With a bulb 20 centimeters long, a stem of half the diameter of the bulb gives a scale of 7° or 8° per millimeter, which is quite open enough. If accurate work is to be done conveniently, a series of fadenthermometers should be available and that one should be used which will give a reading as low down in the stem as possible, thereby reducing the secondary stem correction.

9. RELATIVE MERITS OF THE AUXILIARY STEM AND THE FADENTHERMOMETER

It will have become evident, upon reading the preceding sections, that the theory of the use of the fadenthermometer is not altogether simple and that as an instrument of precision, it is open to certain practical objections, one difficulty arising from the fact that above 200° C, the coefficient of relative expansion has been investigated for only two glasses, one of which, Jena 16^{III}, is not entirely suitable for stems.

The fadenthermometer was evolved from the simple auxiliary stem for the sake of giving a more open scale. The openness of the scale has a certain practical advantage in that the reading requires less care and imposes less strain and fatigue on the observer; but when the stem of the fadenthermometer is as fine as it has often been made, the greater ease of reading the position

of the meniscus on the more open scale is far more than offset by the difficulty of finding the meniscus at all.

The ostensible object of introducing the fadenthermometer was the greater accuracy attainable by reducing the reading errors. It may be seen, however, upon considering the nature of the reading errors when the simple auxiliary stem is used, that the improvement in accuracy attained by the fadenthermometer is altogether illusory. With equally good graduation and illumination, the *linear magnitude* of the reading error is of the same order for the auxiliary stem as for the working stem. There is thus a possibility of adding together two similar reading errors. When the method of double reading, as described in section 4, is followed, we have, taking this second reading into account, the possibility of a still further increase in the sum of the reading errors, though this last possibility may be eliminated by standardizing the auxiliary stem once for all with sufficient care that the results may be regarded as exact relatively to the precision of a single reading, when the illumination in practical use is similar to that during the calibration. But in reality these reading errors are not important. If the temperature to be measured is steady, a number of readings may be taken and averaged, while if the temperature is unsteady or changing rapidly in one direction, it can not be determined with any great accuracy by a mercurial thermometer, no matter how exactly the individual readings may be made. Furthermore, at high temperatures even three times the probable accidental error of *reading* the position of the meniscus in either the working or the auxiliary stem usually falls well within the probable error of the final result on account of the failure of mercurial thermometers to repeat their readings exactly, at high temperatures.

For purposes of the highest attainable accuracy, where it is desirable to avoid all unnecessary errors, even small ones, the readings of the auxiliary stem, both during standardization and during the determination of a stem correction, may, if thought advisable, be made with a micrometer microscope. This presents no difficulty even when micrometric readings on the working stem would be very difficult,[19] because the meniscus in the auxiliary stem is so much more steady than that in the working stem.

[19] It is doubtful whether micrometric readings on the working stem are ever worth while.

It therefore appears that the use of the fadenthermometer presents no real advantage over the use of the auxiliary stem, while it is open to several objections. Not the least of these is the difficulty of obtaining satisfactory fadenthermometers, whereas old thermometer stems, easily convertible for use as auxiliary stems, are often embarrassingly numerous. The maker of any thermometer can easily supply a suitable auxiliary stem of the same glass and the glass need not have been studied as to its expansion.

There seems little doubt that the auxiliary stem method, which avoids several difficulties inherent in the use of the fadenthermometer is the best we have for work of high accuracy, as well as the most convenient in a large number of cases in which only ordinary accuracy is desired.

NOTES

NOTE I.—THE PRESENT STATUS OF THE STANDARD GAS SCALE

In adopting a set of values for the base points, the corrections for reducing to the ideal gas scale have, up to the present time, usually been ignored on account of their uncertainties and relative insignificance. When this is done, the values adopted must, in strictness, be regarded as referred to or expressed in terms of the individual scales of the particular gas thermometers used in the primary standard determinations of the separate points. And since no series of determinations has ever been made with a single instrument, or even with a single gas, over the whole range of temperatures accessible to the gas thermometer, it can not be said that there is any entirely consistent series of values for the base points except over limited ranges.

This appears at first sight to be a very unsatisfactory state of affairs, and in reality a good deal of haziness seems to exist as to the meaning of the term "standard gas scale." The *practical* indefiniteness of the term has, however, been of little or no importance in the past, because the differences between the different gas scales used or between any one of them and the ideal gas or thermodynamic scale have been less than the uncertainties of the determinations of temperature on the individual gas scales. For this reason the various gas scales have in most, though not all

cases, been practically indistinguishable from one another and from the ideal gas scale, and the term "gas scale," with no further qualification, has been a sufficient specification of the *scale*, although the same numerical value for the temperature of a given fixed point referred to that scale may not have been accepted as most probable by all authorities.

The art of gas thermometry has, however, now advanced to the point where the accuracy of the determinations is comparable with that of our knowledge of the differences between the gas scales used in different pieces of work and of the probable corrections to the ideal gas scale. This makes a more precise definition of the scale used in any case desirable, and makes a greater consistency of statement possible by reducing the different gas scales to a common standard. The corrections are often appreciable and though they are not known with a high percentage accuracy, there is no doubt that applying them brings, in general, a slightly greater degree of consistency into the series of base-point values as determined by different gas thermometers.

The authority responsible for the series of values to be used for the base points of the scale of a standardizing laboratory is therefore faced by the question whether it is well to abandon a familiar set of values which have been in use, perhaps for a number of years, in favor of a new set which is only slightly different from the old one and is liable to further modifications of the same order of magnitude as those now proposed, whenever, in the course of a few years, the individual gas thermometer determinations and the determinations of the corrections to the ideal gas scale are more accurate. Up to the present time this question has almost universally been answered in the negative. It seems likely, however, that within a very few years the change will be made and that all standardizing laboratories will use, for reference, the nearest possible approach to the ideal gas scale.

The success of the movement which has resulted in international agreement on the practical values to be used for the electrical units points clearly to a similar international agreement with regard to the standard scale of temperature. Such an agreement has already existed for many years as regards the use of the hydrogen "normal scale" of the International Bureau of Weights and

Measures, for temperatures between 0° and 100° C, and the time appears nearly ripe for a similar agreement over a wider range.

NOTE II.—ON THE USE OF THE TERM "MEAN TEMPERATURE"

The "integral temperature" of a thread of mercury might be defined as the value of the expression $\frac{\int t dv}{\int dv}$ taken over the length of the thread, where t is the temperature of the volume element dv. Since this value is determined solely by the geometrical distribution of the temperature, it is independent of the nature of the material of either the thread or the tube inclosing it. For two threads such as those in a working stem and an auxiliary stem or a fadenthermometer bulb used with it, the integral temperatures are nearly the same. If the longitudinal distribution of temperature is the same for both tubes, as is assumed in practice, and if the cross section ratio of the tubes is constant, the integral temperatures will be exactly the same. Such a relation of cross sections could evidently not subsist *exactly*, except with one particular distribution of temperature, unless the tubes were made of the same glass; but the error here is of a lower order of magnitude, and we may safely regard the integral temperatures as being identical for the working stem and the fadenthermometer bulb when properly placed.

The *mean* temperatures of the two threads, as defined in section 5, depend, however, on the relative expansion of mercury in glass, and will evidently not be quite identical for different kinds of glass, even though the integral temperatures be equal. Nevertheless, to the order of accuracy required in determining a stem correction, the difference is negligible and the mean temperatures may be identified when the integral temperatures have been made sensibly equal.

The complete investigation of the subject of mean temperature, its dependence on the nature of the glass, and its relation to integral temperature requires even for an ideally perfect tube an amount of precise and detailed reasoning quite out of proportion to the value of the result obtainable and will therefore not be touched upon here. The object of the present note is to forestall the possibility that the cautious reader may, upon recognizing the somewhat offhand way in which the term "mean temperature"

has been used in the text, be apprehensive that the difficulties of the subject have not been considered.

NOTE III.—THE APPARENT EXPANSION OF MERCURY IN GLASS

1. Let a glass bulb be exactly filled by v_0 cm³ of mercury at $o°$. Let the expansion of the glass from $o°$ to $t°$ be g, so that at $t°$ the internal volume of the bulb is $v_0(1+g)$. If m is the expansion of the mercury in the same range, and if the bulb with the mercury which exactly filled it at $o°$ be heated to $t°$, the volume of the mercury will increase to $v_0(1+m)$ cm³, and the volume $v_0(m-g)$ cm³ must run out, m being greater than g. Since 1 cm³ of glass increases to $(1+g)$ cm³ when heated from $o°$ to $t°$, if the volume of mercury $v_0(m-g)$ cm³ at $t°$ were measured, not absolutely but in a glass vessel graduated to read true cubic centimeters at $o°$, its volume would be only $v_0\frac{m-g}{1+g}$ of these "glass cubic centimeters." Hence the quantity

$$a=\frac{m-g}{1+g} \tag{1}$$

is the expansion of the mercury as it would *appear* to an observer who supposed that the glass did not expand at all; or it may also be regarded as the expansion of the mercury *relatively to the glass*, considered as of fixed volume. The quantity a is therefore known as the apparent or relative expansion of mercury in the given glass from $o°$ to $t°$.

2. Let the glass envelope consist of a capillary tube graduated with a scale of equal volumes, the volume of each scale division being ϕ_0 at $o°$ and $\phi_t=\phi_0\ (1+g)$ at $t°$. Let the mercury be in the form of a thread which fills N_0 divisions of the tube at $o°$, and N divisions at $t°$. The volume of the thread being $N_0\ \phi_0$ at $o°$, its volume at $t°$ is $N_0\ \phi_0\ (1+m)$, and the number of divisions occupied at $t°$ is

$$N=\frac{N_0\ \phi_0\ (1+m)}{\phi_t}=N_0\frac{1+m}{1+g} \tag{2}$$

and by equation (1) this may be written

$$N=N_0\ (1+a) \tag{3}$$

3. If a_f is the relative expansion of mercury in glass from $o°$ to $f°$, and a_t that from $o°$ to $t°$, a thread which occupies N_0 divisions

at $o°$ will occupy

$$N_f = N_0 (1 + a_f) \tag{4, a}$$

and

$$N_t = N_0 (1 + a_t) \tag{4, b}$$

divisions, at $f°$ and $t°$ respectively. Upon heating from $f°$ to $t°$ the number of divisions occupied will increase by the amount

$$K = N_t - N_f \tag{5}$$

By equations (4) this has the value

$$K = N_0 (a_t - a_f) \tag{6}$$

and if we eliminate N_0 by (4, a), it may also be written in the form

$$K = N_f \frac{a_t - a_f}{1 + a_f} \tag{7}$$

4. We now let the tube be of uniform bore, as is assumed of both the working stem and the fadenthermometer bulb, in the determination of a stem correction. The scale divisions are now of equal *length* when the tube is at an uniform temperature. Let λ be this length, in centimeters, measured when the glass is at room temperature. Let l be the length, in centimeters, of N_f scale division of the tube (the working stem), also measured with the glass at room temperature. We then have

$$N_f = \frac{l}{\lambda} \tag{8}$$

Let $n = \frac{1}{\lambda}$ be the number of scale divisions per centimeter, measured at room temperature. We then have, by equation (8)

$$N_f = n\, l \tag{9}$$

so that equation (7) reduces to

$$K = n\, l \frac{a_t - a_f}{1 + a_f} \tag{10}$$

5. Equation (10) is identical in form and practically identical in meaning with equation (2) of section 5. The actual working stem may have a nonuniform graduation, but the quantity N_f is then not the actual number of scale divisions of S opposite the fadenthermometer bulb, but the number there would be if the divisions were all of the same length as in the region of the meniscus where n is measured, and this number we get by taking the product of n and l as measured.

If the working stem and the fadenthermometer bulb are of the same glass, so that their mean linear coefficients of expansion are identical between the stem temperature $f°$ and the room temperature $r°$ at which the length measurements are made, the l which appears above in (10) is identical with the l which appears in equation (2) of section 5, and the two equations are identical in all respects. If the two glasses are different, there is a discrepancy between the l above, measured at room temperature *on the glass tube we have been considering which corresponds to the working stem*, and the l of equation (2), section 5, which is measured *on the fadenthermometer*, these two lengths being such that they would become exactly equal if the temperature of measurement were raised to $f°$.

This discrepancy is insignificant. The fractional error introduced by identifying these two different values of l is, very approximately,

$$\varepsilon = (f-r)\ (\beta_s - \beta_f)$$

in which β_s is the mean coefficient of linear expansion of the glass of the working stem between $r°$ and $f°$, and β_f the corresponding coefficient for the glass of the fadenthermometer. To take an extreme case, let $(f-r)=500°$ and $\beta_s-\beta_f=0.000\,01$. We then have $\varepsilon=0.005$ or one-half of 1 per cent—an error which is always negligible. In reality neither β_s nor β_f will ever be greater than 0.000 01 and their difference is not likely to be over 0.000 003 for thermometer glasses, so that the error is not likely ever to exceed two-tenths of 1 per cent. The above-mentioned discrepancy between the values of l is therefore always negligible and the deduction given for equation (10) of the present note is valid for equation (2) of section 5.

WASHINGTON, July 13, 1911.

A DETERMINATION OF THE INTERNATIONAL AMPERE IN ABSOLUTE MEASURE

By E. B. Rosa, N. E. Dorsey, and J. M. Miller

CONTENTS

	Page
I. Introduction:	
1. Relation of absolute to international units	270
2. Recent absolute measurements of current	272
3. Types of instruments available	277
4. The Rayleigh balance	278
II. Description of the balance and of the electrical connections:	
5. The physical balance	283
6. The fixed coils	284
7. The moving coils	287
8. The cooling system	288
9. The assembled current balance	291
10. The electrical connections	294
III. The ratio of the radii:	
11. Development of the method	297
12. The potentiometer method	300
13. The shunt method	304
14. The combination method	304
15. Adjustments of coils and needle	305
16. Correction terms	310
17. Sources of error	314
18. Results	317
IV. Theory of the balance and computation of constants:	
19. Theory of the Rayleigh balance	323
20. Method employed in the calculation of the constants	327
21. Distance for maximum force	328
22. Computation of force	330
23. Effect of errors in sectional dimensions	332
V. The measurement of current:	
24. General description	334
25. Manipulation of the balance	336
26. Adjustment of the coils	338
27. Insulation of the coils and electrostatic effect	342
28. Temperatures	343
29. Sum and difference of forces	345
30. Magnetic tests by balance	345

V. THE MEASUREMENT OF CURRENT—Continued. Page
31. Corrections to the observed force 348
32. The observed difference in the forces 352
33. Results 354
34. Discussion of results 361
35. Acceleration of gravity 363
36. The standard cells 364
37. The resistances 366
38. Final result in terms of the electrochemical equivalent of silver 367
APPENDIX:
1. Effect of humidity on the radii of the coils 368
2. The sectional dimensions of a coil 371
3. The effect of errors in the sectional dimensions of the coils 376
4. Relation between the observed and the true maximum force 380
5. Direct measurement of the coils 384
6. Table of log sin $\gamma[2F_\gamma-(1+\sec^2\gamma)E_\gamma]$ 392

I. INTRODUCTION

1. RELATION OF ABSOLUTE TO INTERNATIONAL UNITS

The three fundamental units in electrical measurements are the ohm, the ampere, and the volt. By means of Ohm's law any one can be fixed from the other two. Hence, at the London conference of 1908 it was agreed to define the international ohm in terms of the resistance of a specified column of mercury and the international ampere in terms of the amount of silver deposited per second in the silver voltameter, while the international volt is to be derived from these two independent units, being defined as the electromotive force which will cause an international ampere to flow through an international ohm.

The important distinction between the *ohm*, which is equal to 10^9 cgs units in the electromagnetic system, and the *international ohm*, which is the resistance of a certain column of mercury (and similarly for the ampere, volt, watt, etc.), was clearly drawn by the London conference. Heretofore, there has often been confusion in this respect, and the international ohm has sometimes been defined both as 10^9 cgs units and as the resistance of a certain column of mercury (and similarly for the other units).

The international ohm, ampere, volt, coulomb, and watt, therefore, all depend upon the two concrete standards, the mercury ohm and the silver voltameter, and not upon absolute measurements of resistance and current. However, it is very

important that we know the value of the small difference between the ohm and the international ohm and between the ampere and the international ampere, in order that we may, by applying the proper corrections, obtain absolute units of power, energy, etc., from the international electrical units. The *watt* is a rate of work equal to 10^7 ergs per second; the *international watt* is the rate of work when an international ampere flows through an international ohm. If we know accurately the correction to be applied to convert international ohms into ohms and international amperes into amperes, we can obtain, with very high precision, the rate of work in an electric circuit in *watts*, or the total work in joules or ergs. Owing to the close interrelation of all physical phenomena and the increasing accuracy demanded in physical measurements, it has become much more necessary to be able to do this now than formerly.

Inasmuch as precise electrical measurements are usually made in terms of standard resistances and standard cells, it would simplify the problem if a voltage instead of a current were directly determined in absolute measure. The direct determination of the volt with precision in absolute measure, however, presents very formidable difficulties and has never been made. It was largely for this reason that the ampere was chosen as the second independent unit instead of the volt. Many absolute measurements of the ampere and of the ohm have, however, been made since Gauss and Weber first showed in 1832 that a system of absolute units is possible. These determinations, having been made at different times and at different places, can be compared only by comparing the various concrete standards adopted at the different times. As there can be no concrete standard ampere, two expedients have been adopted for preserving the results of the absolute measurement of current. That most favored in the past is to determine the mass of silver deposited per second in a silver voltameter by a current of one ampere; the other, which in many cases in the past served as intermediary between the absolute determination and the silver voltameter measurement, is to determine the electromotive force of a certain standard cell in terms of a standard of resistance and the absolute ampere determined by the balance.

Until quite recently, the constancy and reproducibility of these concrete representations of the results scarcely surpassed 1 in 10 000, and the construction of the absolute instruments was such as to give an accuracy inferior to this. Indeed, previous to 1907 many thought that an accuracy of 1 in 10 000 was the ideal to be aimed at, and that its achievement would be a triumph.

In that year, however, appeared the report of the work on the absolute measurement of current by means of the new current weigher of the National Physical Laboratory.[1] This work, claiming an accuracy (exclusive of errors in the value of the acceleration of gravity) of 1 in 100 000, marks the beginning of a new epoch in the history of the absolute measurement of electrical quantities. No work of this kind begun after the publication of the above can be considered satisfactory unless it attempts to attain an accuracy considerably surpassing 1 in 10 000.

The extreme accuracy claimed for the work done at the National Physical Laboratory exceeds the reproducibility of the silver voltameter when used according to any specifications as yet officially adopted. It is somewhat better than the reproducibility of the standard cell, or of the international ohm from its specifications in terms of the resistance of a column of mercury. It thus gives promise of an early realization of the conditions under which our electrical standards can be determined in terms of the absolute units, with an accuracy equaling that with which we can trust our concrete standards, unless, indeed, as we hope may be the case, the constancy and reproducibility of our concrete electrical standards shall be appreciably improved in the near future.

2. RECENT ABSOLUTE MEASUREMENTS OF CURRENT

The results of the various absolute measurements of current made prior to the appearance of the paper of which we have been speaking, having been discussed in that and other papers and belonging to a distinctly different epoch, need not be considered here. This leaves but five determinations to be considered, all of which were begun prior to 1907.

(*a*) *Ayrton, Mather, and Smith*[2].—In the measurements made at the National Physical Laboratory, a beautifully constructed

[1] Ayrton, Mather, and Smith: Phil. Trans., 207A, pp. 463–544; 1908.
[2] See note 1.

current weigher, with single layer cylindrical coils wound upon marble forms, was used. The balance was a double instrument, symmetrical with reference to the plane through the central knife edge and normal to the beam. The entire instrument was carried by a massive stand of phosphor bronze, which had satisfactorily undergone tests for magnetic impurities. When in adjustment, each moving coil was situated so that its end planes coincided with the mean planes of two fixed coils of the same length, wound upon opposite ends of the same hollow marble cylinder. The axes of the moving coils were vertical, and coincided with the axes of the fixed coil. The constant of such a balance is determined from direct measurements of the linear dimensions of the coils. These measurements can be made for single-layer coils with much greater accuracy than is possible for multiple-layer coils. The accuracy of measurement required, however, to give the desired precision in the result is extraordinary, even for the most favorable case of single-layer coils. From data given in the paper, however, it appears probable that sufficient accuracy has been secured. All weighings were made within the first 30 minutes after putting on the current, because when the current was kept on for a longer period the behavior of the balance became erratic, on account of the heating due to the current. Working in this way, under rather rapidly changing temperature conditions, it would appear that the dimensions of the coils, and so the constant of the balance, might be subject to some uncertainty. The detection of a minute quantity of magnetic material in a large mass of nonmagnetic metal, such as the stand on which the balance and coils rest, is difficult; and one can not avoid thinking that the presence of this stand may possibly be a source of very slight error. A direct indication of the existence or nonexistence of such an error can be obtained by measuring the difference in the forces exerted upon either moving coil by the two fixed coils in which it is hung; then interchanging these fixed coils (turning the cylinder end for end) and repeating. If the stand is without effect, these two values will be the same. We have found such a procedure of great value in the work to be described in this paper.

This most elaborate and excellent piece of work gave for the value of the Weston normal cell, at 17° C, in terms of the inter-

national ohm, and the absolute ampere, the value 1.018 30, which corresponds to the value 1.018 19 at 20°.00 C. Somewhat more recent but as yet unpublished measurements, kindly communicated by the Director of the National Physical Laboratory, gave 1.018 18 as the value at 20°.00 C.

(*b*) *Janet, Laporte, and Jouaust*[3]—The next article upon this subject that appeared described the work done at the Laboratoire Central d'Électricité. The authors used a current balance of the Rayleigh type. The distance between the fixed coils was greater than that which would give the maximum force upon the moving coil, and hence the force varied less rapidly in the region occupied by the moving coil. This permitted a less exact placing of the moving coil, but requires that the distance between the mean planes of the fixed coils be determined with greater accuracy; an error in this case of 0.01 mm in this distance produces an error of 5 parts in 100 000. The radii of the coils were determined from the measurement of the length of the wire wound on them. As stated in a note appended to the article, and as verified since (Bull. Soc. Int. d'Élec.; 1910), the international ohm as used at the Laboratoire Central is 1 in 10 000 smaller than that used in England, Germany, and America. Further, in calculating the constant of their instrument, the authors assumed that the axial breadth of a coil was given by the total breadth of the channel in which it was wound; but that the radial depth was given by the distance from the *axis* of the wire in the bottom layer to the *axis* of the wire in the top layer. This assumed radial depth is evidently too small (p. 373). In a recent paper[4] the authors have given the result of a complete recomputation of the constant of their balance using the correct sectional dimensions. They now find for the value of the Weston normal cell at 20°.00 C., taking into account the difference between the French ohm and those of the other countries, the value 1.01836.

(*c*) *A. Guillet*—The next paper in order appears in the same volume as the preceding (pp. 535–561), and describes work carried out at the Sorbonne. This work was done by the method

[3] P. Janet, F. Laporte, and R. Jouaust: Bull. de. la Soc. Internat. des Électriciens (2), 8, pp. 459–522; 1908.

[4] Comptes rendus, 153, p. 718, Oct. 16, 1911.

suggested by Prof. G. Lippmann in 1906. Here again the instrument is of the attracted-coil type, but the multiple-layer fixed coils are placed very close together and the multiple-layer moving coils are but slightly smaller than the fixed coils. The constant of the instrument is not determined from its dimensions, but the mutual inductance of the fixed coils with respect to the moving coil, for various positions of the latter, is determined by direct comparison with absolute standards of mutual inductance. From these observations, an empirical equation, connecting the mutual inductance with the position of the moving coil, is determined; whence, by differentiation, we get the force for any given position of the coil. The difficulty in carrying out the method in practice is due to the rapid variation in the mutual inductance with the displacement of the moving coil. From the data given, it appears that the relative positions of the moving coil can not be determined with an accuracy exceeding about 1 micron. By using the mean of several readings carried out to 0.1μ, Guillet finds for the factor a in his notation the value $1.660\ 91 \times 10^7$. If these values for the relative positions of the coil be rounded off to the nearest micron (which involves an increase of 0.4μ in the first and third values and no change in the second value) we find for the constant a the value $1.660\ 58 \times 10^7$. Since a is proportional to the force, we see that this slight change will produce a change of about 1 in 10 000 in the current. The balance being double and symmetrical, there are two groupings of the currents that may be used; the constant was independently determined for each of these groupings, and thus two independent determinations were obtained. As only a single value was given for each grouping, it is impossible to determine the reproducibility of the observations. The value found for the Weston normal cell, at 20°.00 C, after applying the correction of 1 in 10 000 for the difference in the French ohm, was 1.018 12.

(*d*) *Pellat.*—In the same volume (pp. 573–633) there is still a third paper on this subject. In this paper Prof. Pellat gives a thorough discussion of the construction, the computation of the constant, and the result obtained by him with a radically different type of instrument. This consists of a long multiple-layer solenoid, placed with its axis horizontal, in the interior of which is

placed a small single-layer coil with its axis vertical. The center of the two coils coincide. The small coil is supported on a knife-edge, normal to the axes of the two coils. The torque which the current in the long solenoid exerts upon the small coil is balanced by means of weights applied to the end of an arm attached to the small coil. Thus it belongs to the type of instrument usually denoted by the term electrodynamometer, although the torque which balances the effect of the current is due to gravity instead of being produced by the torsion of a wire. The result obtained for the electromotive force of the Weston normal cell at 20°.00 C, corrected for the difference in the ohm, is 1.018 31 ± 0.000 15.

In all of the papers so far considered it is implied, if not explicitly stated, that the constant of the instrument has a zero temperature coefficient if all parts are constructed of the same material. Consequently little or no attention has been paid to the temperature of the instrument. That the coefficient is zero under these conditions is true, *provided that all parts of the instrument are at the same temperature.* But in practice this condition is never fulfilled; the coils are heated by the current, and in general they are not all heated to the same extent, and in the last case considered the balance arm, from which the weight is suspended, is undoubtedly at a different temperature from the coils. In the absence of the necessary data it is impossible to form an estimate of the errors that may thus be introduced; but in order to obtain the highest accuracy of which an instrument is capable it is very necessary to study carefully the temperatures of its various parts.

(*e*) *Haga.*—The most recent work on the subject is that by Prof. Haga, carried out at the University of Groningen.[5] He has measured the current in absolute units by means of a tangent galvanometer, and has thus based it upon the calculated constant of the galvanometer and the absolute determination of the horizontal component of the earth's magnetic field. Owing to continual variations in the latter, the method is exceedingly difficult, but Prof. Haga appears to have brought the work to a very satisfactory conclusion. He finds for the Weston normal cell in terms of the international ohm and the absolute ampere, as given

[5] Prof. H. Haga and J. Boerema: Konink. Akad. Wetensch. Amsterdam Proc., p. 587; 1910.

by his instruments, the value 1.0183_6 at 17°.00 C, or 1.0182_8 at 20°.00 C.

Collecting these values, we have

N. P. L.	1.018 18
L. C. E.	1.018 36
Guillet	1.018 12
Pellat	1.018 31
Haga	1.018 25
Mean	**1.018 24 semiabsolute volts.**[6]

Considering all the circumstances, this agreement is very striking and leaves no doubt that the value 1.0182 is correct to at least 1 in 10 000.

3. TYPES OF INSTRUMENTS AVAILABLE

In all absolute measurements of electric current a force or torque produced by the current is balanced against another force or torque, which in turn is determined in dynamical units. In choosing an instrument to be used in the absolute measurement of current the first thing to be considered is the nature of the auxiliary force or torque that is to be employed. There are three available methods: (*a*) Using a known magnetic field, (*b*) using an elastic deformation of some body, (*c*) using the gravitational attraction of the earth. The only magnetic field for which the value of the force or torque produced can be at all readily determined in dynamical units is that due to the earth; and, owing to the continual variations in the strength of this field, an accurate determination of its strength at any given time and place involves many difficulties and scarcely gives promise of an accuracy superior to 1 in 10 000. Though the recent determination by Haga, using this method (with a tangent galvanometer), gave excellent results, it would appear that any method based upon the direct measurement of a given magnetic field would involve many difficulties when extreme accuracy is desired.

Measurements based upon the second method involve a direct determination of the force of restitution of the deformed body at the time and under the exact conditions under which it was used

[6] See footnote to p. 362.

in the electrical measurements, and in addition are subject to some uncertainty introduced by the elastic after effect shown by all material bodies. While these difficulties are not insuperable, Dr. Guthe[7] and others have shown that they are quite formidable.

In measurements based upon the third method the evaluation of the force or torque can be reduced to the direct comparison of two masses and the determination of the acceleration of gravity. The first can be readily done with extreme precision; the second involves many difficulties, but these are largely offset by the facts that we have every reason to believe that in any region, geologically stable, the value of the acceleration of gravity remains unaltered over long epochs and that relative values of the acceleration of gravity at any two places can be readily determined with high precision. The latter fact enables the results obtained by this method at different times and places to be rendered strictly comparable with one another; the former renders the results obtained capable of correction at any future time when the absolute value of the acceleration of gravity shall have been determined to a higher accuracy than that with which it is now known.

For these reasons it appeared desirable that the work should be based on the third or gravitational method. Furthermore, it appeared on the whole desirable that the construction should be such that a force rather than a torque be measured. Accordingly a current weigher, or current balance, was constructed.

4. THE RAYLEIGH BALANCE

The particular type decided upon was that used by Lord Rayleigh in 1884, and since employed by Janet, Laporte, and Jouaust, at Paris. It consists essentially of a pair of multiple layer fixed coils placed coaxially, with their planes horizontal, and at such a distance apart that the vertical force which they exert for a given current upon a smaller coil placed coaxially with the fixed coils, and midway between them, either is a maximum or varies at a minimum rate for small vertical displacements. The smaller coil is suspended from one pan of a balance, being counterpoised by equal weights in the other pan. The force exerted upon it is counterpoised by additional weights placed on one pan or the other,

[7] This Bulletin, 2, p. 33; 1906.

according to the direction of the force. Preferably, however, the weight is equivalent to twice the force, which is the change produced by reversing the current in the fixed coils. The earth's magnetic field causes a torque which will not affect the measured force; local disturbances of the magnetic field would be eliminated by taking the mean of the two forces observed for currents in direct and in reversed directions through the fixed coils, the direction in the moving coil being the same in both cases. The most striking feature of the balance, theoretically, is the fact that the principal constant need not be determined by direct measurements upon the coils, but may be determined by an electrical method. This eliminates the difficulties incident to the direct measurement of the mean diameter of a multiple layer coil, and enables the constant to be determined with extreme precision and to be redetermined at any time.[8]

Rayleigh used a single balance, consisting of one pair of fixed coils and one moving coil. Janet, Laporte, and Jouaust used a double current balance, with two pairs of fixed coils and two moving coils, one suspended from each pan of a balance. There are both advantages and disadvantages in a double balance, and after careful consideration we decided to use a single balance. The force is doubled by a double balance, or for the same force one can use coils of smaller cross section. But, as some of our coils were much larger than those of Janet, Laporte, and Jouaust, it would have required a beam at least 75 cm long instead of 30 cm, as used by us. The balance would have been much more complicated, and harder to manipulate and adjust accurately, and we feel now, as we did four years ago when considering the question, that for work of the highest precision the simplification gained by using a single balance of this type more than overbalances the disadvantage of having a larger cross section of the fixed coils for the same force.

The coils of such a balance must have a large number of turns of wire in order to give a force large enough to be measured with precision. Lord Rayleigh's balance, as he used it, gave a change of force of about 1 g on reversal of the current, and Janet, Laporte,

[8] Janet, Laporte, and Jouaust, however, calculated the constant from the radii obtained by counting the turns and measuring the length of the wire as it was wound on the coils.

and Jouaust's about 4 g. The coils of our balance have been wound in such a way that with a current not exceeding 1 ampere the force (on reversal of the current) would be from 3 to 6 g. This is a large enough force to be measured with the necessary precision, if the conditions are favorable. It would have been possible to make the force 10 or 20 g, and that would have been favorable to the weighing, but there would have been a loss of accuracy in other directions. The resistance of the windings increases as the square of the number of turns for a given size and cross section of the coil. One soon reaches a limit in attempting to increase the force by increasing the number of turns. First, the heat developed should not be allowed to become excessive, as it would if the number of turns be greatly increased; second, the voltage required to overcome the resistance should not be large enough to give rise to serious electrostatic attractions due to differences of potential between the parts of the balance; and, third, the current must be maintained uniform to an extraordinary degree if very accurate measurements are to be made. This requires a special storage battery, used for no other purpose, and a ballast resistance of small temperature coefficient can be employed advantageously if the instrument resistance is small enough.

One can not use a small enough resistance, however, to make the heating effect negligible, without making the cross section of the coils too great or the force to be measured too small.

If the cross section of the winding of the coils is large, the correction for section will be relatively large, and it will be difficult to determine its value with sufficient precision. After careful study of the theory of the balance, we decided to use a square cross section of 2 by 2 cm for the large fixed coils (50 cm radius); 1.4 by 1.4 cm for the small fixed coils (40 cm radius); and 1 by 1 cm for the moving coils (20 and 25 cm radius).

To reduce the error due to the heating of the fixed coils, a system of water cooling was designed to carry away the heat as fast as generated, so that a state of equilibrium could be attained, and weighings could be made deliberately for an indefinite period with the dimensions and temperatures of the coils constant. There is, of course, a difference in temperature between the wire of the windings and the brass form on which it is wound, and

hence a careful study had to be made of the effect of the load. The ratio of the radii of the fixed and moving coils was determined with water flowing through the cooling system precisely as when the balance was used in the weighings, so that the heating effect of the current was known with great accuracy and could be perfectly controlled.

The moving coil can not be water-cooled. The heat generated in it is made small by using a coil of low resistance. The number of watts varied from 0.6 to 1.2, heating the coil from 1.7 to 3.3 degrees centigrade above the surrounding temperature. Even this slight increase of temperature sets up convection currents which are affected by any changes in the temperature of the fixed coils or of the instrument case. To protect the moving coil from outside influences, and to permit the convection currents to become constant so that their lifting force on the moving coil should be sufficiently constant to be eliminated by successive weighings with the current in fixed coils reversed, the moving coil was inclosed in a water jacket, through which water at a constant temperature was passed. This gives almost ideal conditions. The fixed coils being held at a definite known temperature, and the moving coil being suspended in a constant-temperature chamber and carrying a constant current, a constant circulation of air is set up in this chamber, which removes the heat from the moving coil and exerts a constant lifting force upon the moving coil. This force is completely eliminated by the method of weighing.

In order to obtain checks upon the work and a final result of greater weight, several pairs of fixed coils have been used and several different moving coils, with two different radii both in the fixed and in the moving coils.

The details of the construction of the balance are given in Section II, the measurements of the ratio of the radii are given in Section III, the theory of the instrument and calculation of the constants are given in Section IV, and the tests of the balance and the measurement of the current are given in Section V. It is intended to give in this section only a general outline of the main features of the balance, and reasons for choosing this type of balance. Some of the advantages of the balance as we have used it may now be stated briefly, by way of a summary of what precedes.

(*a*) The only direct length measurements that have to be made are those of the sectional dimensions of the coils, and these are used only in the calculation of correction terms. The principal term in the constant of the instrument depends upon the ratios of the radii of the fixed coils to that of the moving coil, and these ratios may be obtained by an electrical method, based upon that described by Bosscha and used by Lord Rayleigh. This method is capable of giving results of extraordinary precision.

(*b*) The coils being compact and readily replaced in the instrument, a number of coils can be constructed and used interchangeably, thus giving not only great flexibility to the instrument, but also a number of independent determinations, thus affording a better indication of the magnitude of the errors in the work.

(*c*) A feature which we believe to be of great importance in a balance which is to be used at intervals throughout a long term of years, is the ease with which the ratio of the radii of the coils, and, consequently, the constant of the balance, can be redetermined from time to time, as a check upon the constancy of the coils. This has proved to be of great importance in our instrument.

(*d*) The balance lends itself readily to the water-cooling of its fixed coils, so that their temperature can be controlled and maintained constant, not only in the current weighings but also in the determinations of the ratio of the radii, permitting a more exact determination of the constant of the balance and more accurate weighings. The moving coil can also be readily inclosed in a constant-temperature chamber, so that the convection currents in the air, by means of which the heat generated in the moving coil is carried away, can become steady, and the lifting force of these air currents will be constant enough to be successfully eliminated by the method of weighing.

Other types of balances undoubtedly possess some of these features, and perhaps others of importance not possessed by this balance; but it was because the Rayleigh balance if improved in the way indicated would possess so many good qualities that we chose it in 1906 when the work was first taken up.

The results obtained (given on pp. 357–363) have more than justified our expectations, although it has required a very large amount of time and labor to carry out the work.

II. DESCRIPTION OF THE BALANCE AND OF THE ELECTRICAL CONNECTIONS

5. THE PHYSICAL BALANCE

A 2-kg precision balance by Rueprecht, with a 30-cm beam, was modified in the shop of this Bureau so as to adapt it to this work. Certain magnetic portions of the balance case and of the balance itself were replaced by brass or by phosphor bronze. Each piece of the balance, as finally constructed, except the knife-edges and the blocks holding the agate planes, was tested by a very sensitive astatic magnetometer, and all were found to be satisfactorily nonmagnetic. As used, the knife-edges were always at least 70 cm above the upper coil, and, during the latter portion of the work, they were 100 cm above it. It was proved by actual tests that a much larger mass of steel at this distance produced no appreciable effect upon the force. For further discussion of the magnetic tests of the balance, see pages 345–348.

A mirror was mounted on the beam over the central knife-edge, and the deflections of the balance were read by means of a telescope and vertical scale about two and a half meters distant. With this arrangement, 1-mm scale deflection corresponds to a difference in the force of 0.36 mg, and to a displacement of the moving coil of 0.025 mm. The time for a single swing of the balance with 1-kg load (the weight of the moving coil and suspension system) is 15 seconds.

The weights employed were of platinum, and were compared at intervals with the standards of this Bureau. They have remained constant throughout the work. By a simple device, the weights could be placed on the pan, or removed, without opening the balance case. A correction for the buoyancy of the air was, of course, applied.

The moving coil was suspended from the lower end of a tube passing through the center of the right pan, and the weighings were accomplished by the addition or the removal of weights from this pan. Consequently, neither the ratio of the lever arms nor the flexure of the beam enters into the discussion of the results.

6. THE FIXED COILS

Three pair of fixed coils were used. All were wound bifilarly, with enamel-insulated wire, upon brass forms having sections as shown in Figs. 1 and 2. Care was taken to make the faces of the forms normal to the axis of the form. Two wires were wound side by side in 36 layers of 18 double turns in each layer in the larger coils, and 28 layers of 14 double turns in each layer in the smaller coils.

There are several advantages in a bifilar winding: (*a*) It enables the insulation resistance from one wire to the other to be measured, and any leakage from one turn to the next can be detected by a simple test. (*b*) It permits the coils to be joined in series or parallel, and so permits different currents to be used with the same heating effect. It is also convenient, in measuring the ratio of the radii, to be able to vary the number of turns, as is done effectively when the coils are joined in parallel instead of in series. (*c*) By sending the current in opposite directions in the two windings, the full heating effect may be produced without any magnetic effect. This is convenient in testing the balance, and in detecting a change in the radius of the coil with different currents.

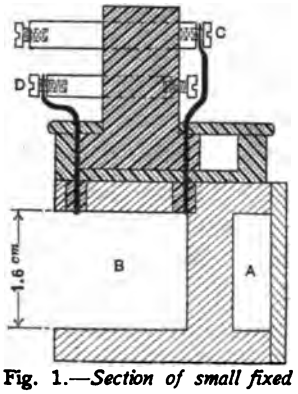

Fig. 1.—*Section of small fixed coil showing first form of terminal block*

A is the water channel. B is the channel for the wire

The enamel covering of the wire has the advantage over a silk covering in being thinner, as well as hard and unyielding, and of very uniform thickness. It can be wound as uniformly as bare wire, although it should be handled carefully to prevent injury to the insulation. When thoroughly dry, the insulation resistance is very high.

For every coil the insulation resistances between the two wires, and between each wire and the form, were frequently tested with a potential difference of 40 volts, and were found to be very good throughout the work, the insulation seldom falling below 100 megohms.

The ends of one wire are brought out through two small, axial holes bushed with ebonite and lying accurately upon the same radius of the form; the ends of the other wire of the bifilar are similarly brought out at the other extremity of the same diameter. After passing through the brass forms, the ends of the wires either are attached by spade terminals and small screws to short brass rods, supported in a radial and axial plane by ebonite posts attached to the form (Fig. 1) and forming the true terminal blocks of the coil, or they pass through an ebonite box attached to the form, as shown in Fig. 2, connections to the leads being made by drops of solder. These forms of terminals were adopted because of the facility with which they allow one terminal of the coil winding to be disconnected from the leads, and the latter short-circuited (for measuring the lead effect) without disturbing the position of the leads.

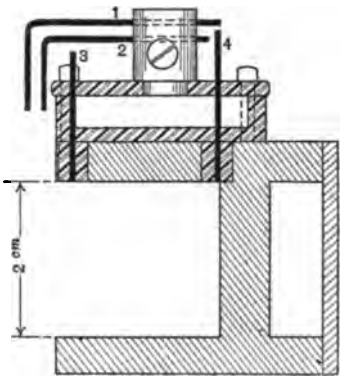

Fig. 2.—*Section of large fixed coil showing second form of terminal block*

Connections are made as desired by drops of solder

Two pair of coils, known as $S1$, $S2$, $L1$, and $L2$, were wound in 1907 upon forms of cast brass. A specimen of the brass, cast as a bar, was tested by means of an induction balance, and the forms were tested by a magnetometer, and were pronounced good. Later, by means of the extremely sensitive instrument described below, these coils were found to be slightly magnetic. In order to determine whether or not this might be an appreciable source of error it was deemed advisable to build a pair of coils having forms that were still less magnetic, and accordingly coils $L3$ and $L4$ were built. The forms for these coils were built up entirely of rolled brass, riveted together, and soft soldered. These forms are somewhat better than $S1$, the best of the old coils, and only about one-eightieth as magnetic as $L1$, the worst of the old coils.

The most satisfactory means found for testing the magnetic properties of conductors, especially of various portions of large masses, is a delicately suspended astatic magnetic system, so arranged that the body under test can be brought very close to

one pole. The instrument used had needles about 6 mm long mounted on opposite ends of a slender glass rod 5 cm long. The whole was suspended by a silk fiber 10 cm long. The suspension was completely inclosed and the lower needle hung near the bottom of a glass tube 1 cm in diameter. By suitably leveling the instrument, one pole of the lower needle can be placed very near the wall of the tube so that the test object can be brought within a few millimeters of the pole. The deflections were read by means of a mirror, telescope, and scale.

With such an instrument, and with a scale distance of 2 meters, deflections of 5 cm were obtained when a tube containing ferrous sulphate was presented to the needle. This sensibility is ample for our purpose, but it could undoubtedly be increased without very great trouble. Though we have tested many kinds of brass, we have never found a specimen that did not produce a slight deflection, nor one that could not be permanently magnetized by subjecting it to a strong magnetic field. We have found rolled brass to be the most uniformly good magnetically, though excellent cast brass can be obtained, as is shown by *S*1.

In order to improve the insulation of the coils, the wire channels of the forms were lined with paper attached to the metal with thin shellac; in the case of *L*3 and *L*4 thin paper, soaked in hot paraffin, was ironed down to the bottoms of the channels. As each layer of wire was wound it was covered with a strip of onion-skin paper (0.05 mm thick). Owing to the fact that the coils were not sealed air tight, the paper absorbed moisture to a degree depending on the average humidity of the atmosphere, and swelled slightly in consequence. This caused very slight but appreciable changes in the mean radii of the coils. Consequently, in the summer of 1910 the paper covering the outer layer of wire was saturated with paraffin melted in with a clean hot soldering copper, the paraffin being well melted to the sides of the form. Then a strip of muslin, well soaked in a hot mixture of beeswax and Venice turpentine, was wrapped around the coil and melted to the underlying paraffin; over the whole was wrapped a strip of binder's cloth soaked in hot paraffin and melted to the muslin and form. This sealed the coils very effectually against the absorption of moisture from the air.

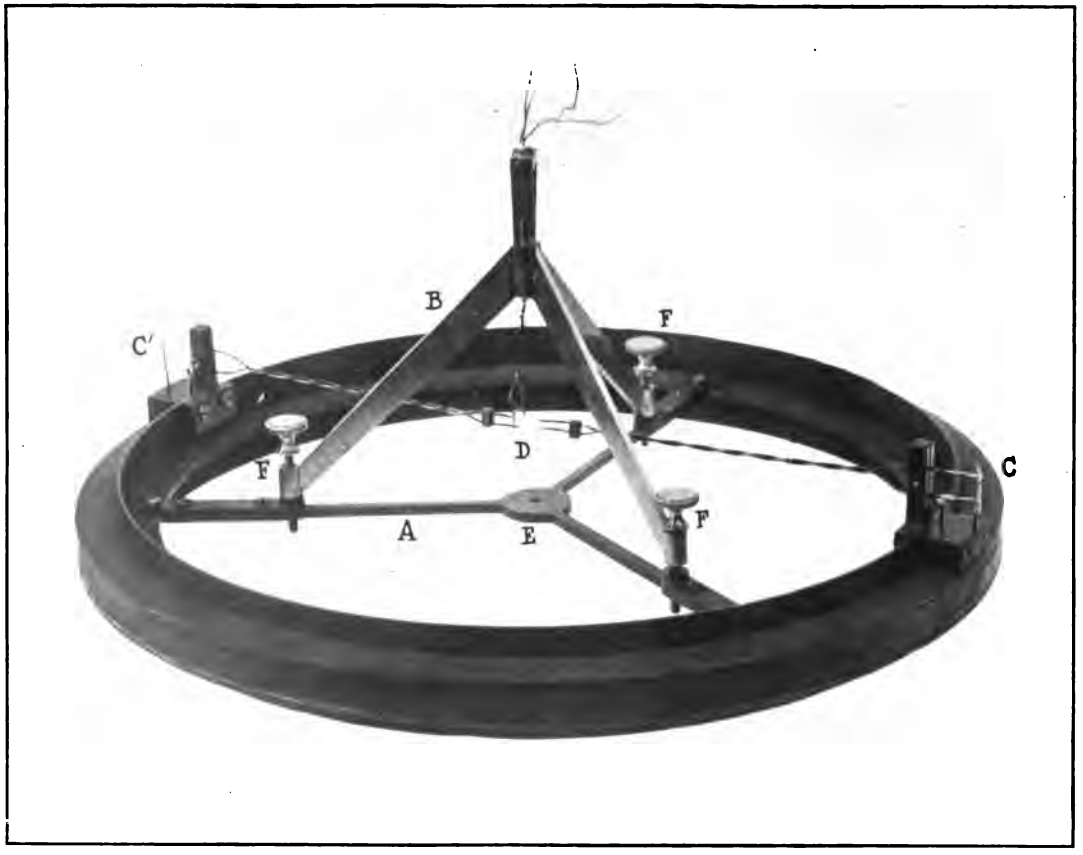

Fig. 3.—*Photograph of a moving coil, showing the leads, the star, and the tripod*

The width of the channels after lining was designed to take an integral number of turns of the bifilar winding. Owing, however, to slight variations in the thickness of the insulated wire, the fit was better in some places than in others. In all cases the wire was transferred from one layer to the next at points previously determined upon.

The wire was approximately 0.5 mm in diameter, and was wound under a tension of about 1 kilogram. The diameters of the forms and of each layer of wire were measured as the coils were wound, so as to obtain a very approximate measure of the mean diameter of each coil, and to keep a check on the uniformity of the winding. These measurements are given in the appendix, page 385, but were not used in the calculation of the constants of the balance because they can not compare in accuracy with the measurements by the electrical method. In winding the earlier coils ($S1$, $S2$, $L1$, $L2$) an integral number of turns of each wire was placed in each layer. This necessitated bringing up the wire from one layer to the next, always at the extremities of the same diameter, and gradually the winding became slightly elliptical with the long axis along this diameter. Consequently, when winding $L3$ and $L4$, the wire was brought up from one layer to the next at a point one thirty-sixth of a revolution short of the point at which it was brought up the layer before. Thus, the coil is kept circular and each winding has one turn less than by the old method.

7. THE MOVING COILS

Four moving coils have been built at various times during the progress of the work. They are all wound bifilar, as were the fixed coils, of enamel insulated wire upon brass forms finished dead black and having a section as shown in Fig. 4. The black finish and the winglike projections facilitate the dissipation of heat. The two windings of the moving coils were generally joined in parallel during the weighings, but were used in series in some cases to obtain a check upon the work. The method of connecting the windings to the leads is shown in Figs. 3 and 4. Here also the leads may be short-circuited by drops of solder, and their effect

in situ determined without any current flowing through the windings.

The three coils, $M1$, $M2$, and $M3$, are wound upon forms of cast brass, the winding, the treatment of the terminals, and the sealing being done in essentially the manner already described for the fixed coils. The fourth coil, $M4$, was wound upon a form cast of rolled brass. Its windings differed from the others in that the windings in each layer were not uniformly distributed, but in adjacent layers were crowded toward opposite sides of the channel; hot paraffin was painted on and into each layer.

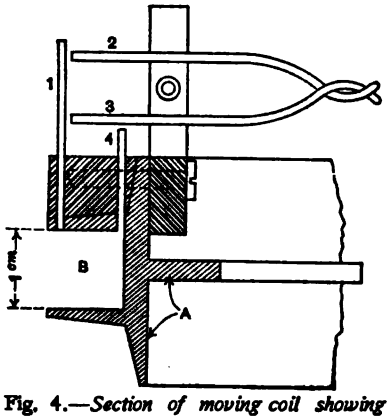

Fig. 4.—*Section of moving coil showing third form of teminal block*

Connections are made as desired by drops of solder. *A* are flanges to secure stiffness. *B* is channel for the wire

Over each layer a strip of onion-skin paper (0.03 mm thick) soaked in paraffin was pressed down closely with a hot copper; over the top layer the coil was sealed like the others. Thus the windings of this coil are embedded in a solid block of paraffin, and most effectively protected from atmospheric action. As shown in Section III, the failure to distribute the wires uniformly across the channel was a source of uncertainty not recognized at the time.

These coils were tested by the magnetometer in the same manner as were the fixed coils, and, excepting $M1$, were found to be very good. $M1$ was much worse than $L1$, and so was not used in the later work.

The insulation was frequently tested as for the fixed coils, and always found to be very high.

The results of the direct measurements of the coils will be found in the appendix, page 386.

8. THE COOLING SYSTEM

As shown in Figs. 1 and 2, each form on which a fixed coil is wound has back of the slot in which the wire is wound a channel

Fig. 5.—*Photograph of the current balance as used in the final measurements*

The moving coil is suspended from the pan of the balance. A is the water jacket surrounding the moving coil, B is the tank containing the thermostat which controls the temperature of the water

through which water can be passed to remove the heat generated by the current and, therefore, to control the temperature of the coils.

During the weighings, the moving coil is surrounded by a cylindrical copper jacket, double-walled on the sides, completely closed at the bottom, and covered by a lid having a hole about a centimeter in diameter in its center, through which passes the tube from which the moving coil is suspended. The space between the two cylindrical walls is filled with circulating water, which carries away the heat generated by the current in the moving coil, and maintains the jacket at a constant temperature so that a temperature equilibrium can be attained without undue delay. Such an equilibrium is exceedingly important, for, unless the convection currents set up by the heating of the moving coil are maintained constant, the weighings will be erratic. By the use of such a jacket we have succeeded in obtaining weighings of such concordance that the mean departure of the individual results from the mean of a group is only about ·1 per cent of the total force exerted by the convection air currents.

Water from a cylindrical tank, 25 cm in diameter by 40 cm deep, is forced, by means of an electrically driven turbine pump, through three pipes to the water channels in the forms of the two fixed coils, and to the space between the walls of the water jacket. After passing through the channels and the jacket, respectively, the water, now warmed a few degrees by the heat generated by the current, is conveyed by three pipes, each provided with a valve, to a small trough at the top of a second tank similar to the first. From this trough it passes through a pipe immersed in iced water contained in the second tank. During its passage through this pipe the water is cooled to a temperature somewhat lower than the water in the first tank into which it then enters. This overcooling of the water is compensated for by electrical heating, thermostatically controlled, thus giving a supply of water at a constant temperature. Since the two coils and the jacket are all supplied in parallel, the two coils may be maintained at very nearly the same temperature.

The water tanks and connections may be seen in Fig. 5.

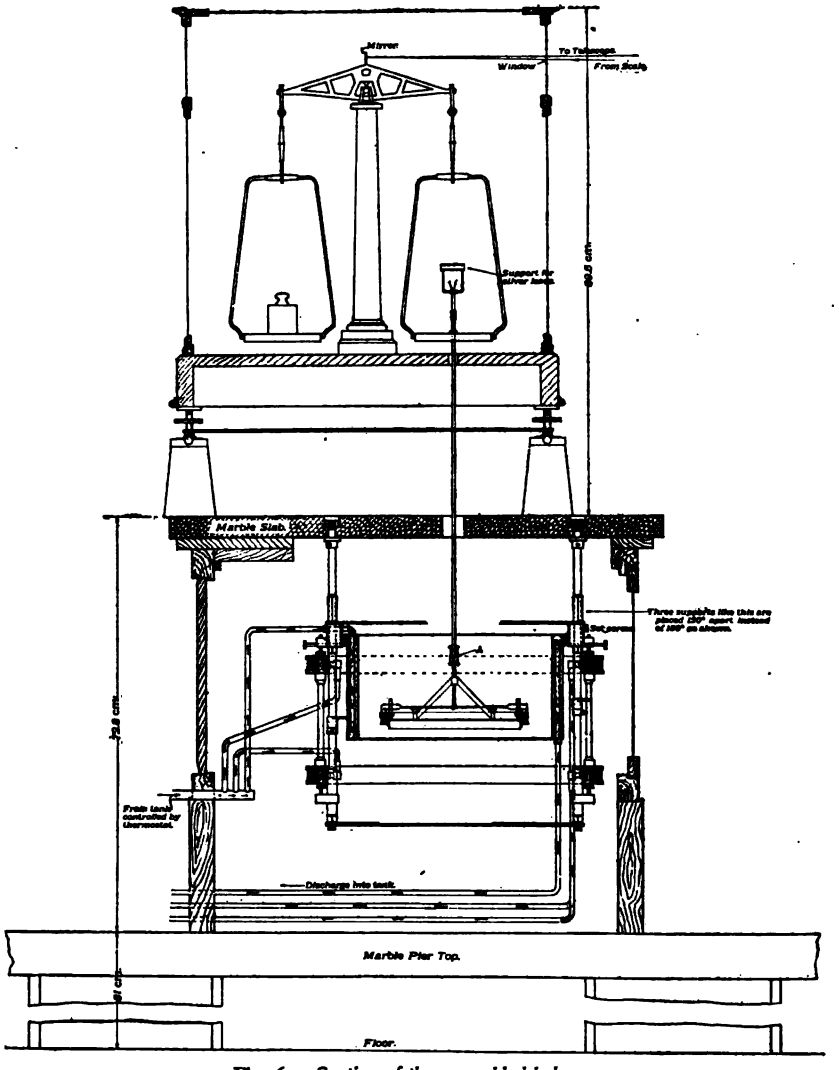

Fig. 6.—*Section of the assembled balance*

The fixed coils are suspended from the marble slab. The water circulation through the jacket surrounding the moving coil and through the channels of the fixed coils is shown

9. THE ASSEMBLED CURRENT BALANCE

The assembled balance is shown in Figs. 5 and 6. As may be seen, three brass rods suspend the fixed coils from the marble top of the case which surrounds them. This case has no magnetic material in its construction, and all the material used in the supports, pipes, etc., was tested with the magnetometer and found to be good. The balance rests on oak pyramids fastened to the marble top of the coil case.

The method of supporting the large fixed coils is shown in Fig. 7. *C* is one of the rods bolted to the marble top of the case; it carries two collars, *M* and *K*, to which are attached arms carrying the leveling screws *F* and *F′*, between which the coils and their distance piece *B* are clamped. A light brass rod, *J*, which fits rather snugly into holes drilled through the coils, passes through the coils and their distance piece and into holes of the proper size in the ends of the leveling screws. The ends of the distance piece *B* are faced accurately parallel to one another. By proper manipulation of the three pairs of leveling screws it is easy to level the coils and to clamp them rigidly together.

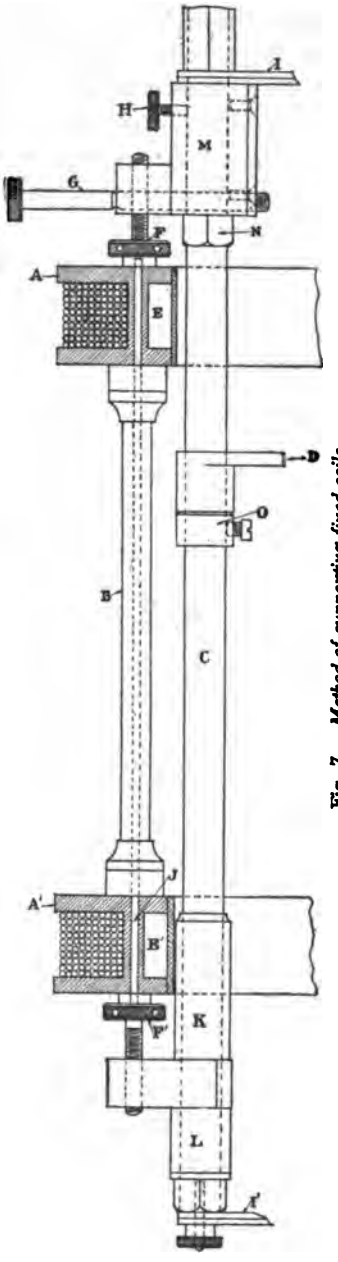

Fig. 7.—*Method of supporting fixed coils*

C is a stout brass rod bolted to the marble top of the coil case, *A*, *A′* are sections of the fixed coils, *B* is a distance piece separating the coils, *F*, *F′* are leveling screws, and *J* is a slender rod threading the coils and distance piece, and ending in cavities in the leveling screws

The holes through the coil forms having been accurately spaced when the forms were made, the coils will be very nearly coaxial when they are thus clamped. An electrical method for testing the accuracy of this adjustment is described in Section V. Here it will suffice to say that in no case was this adjustment found to be sufficiently in error to cause an error of 5 parts in a million in the force. The collar *M* rests upon a shoulder *N*, and when adjusted it can be clamped to the rod *C* by means of the set screw *H*. *D* is a flange attached to a collar fitting *C* and resting upon an adjustable collar *O*. It can be turned out under the upper coil so as to support it while the lower coil is being put in place. By using

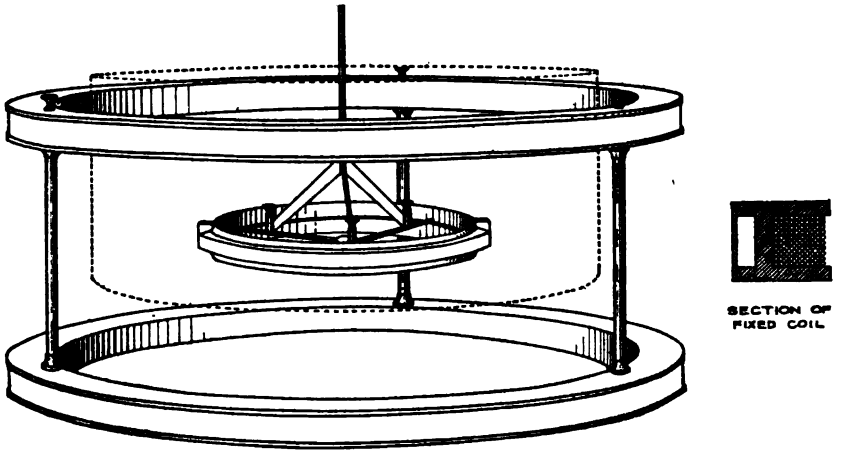

Fig. 7a.—*Perspective drawing of the coils. The dotted line shows the position of the water-jacket*

collars (*L*) of different lengths, distance pieces (*B*) of various lengths may be used. The three rods (*C*) are connected to one another both above and below the coils by means of light triangles of brass, *I*, *I'*, so that the entire mounting is very rigid.

The mounting for the smaller fixed coils differs from that just described merely in having the lower portions of the rods *C* offset with reference to the upper so that they can go inside the coils and still be attached to the marble at the same places.

To the inner flange (Fig. 3) of the moving coil is attached a light three-armed star *A*, having a 3-mm hole at its center, and to this star is attached, by means of three long leveling screws, a

tripod *B* carrying a short section of thin-walled 5-mm brass tubing. By means of a collar and a pair of brass links this piece of tubing may be attached to a collar on a piece of similar tubing, which passes through the center of the right pan of the balance and is supported by a nut which rests on the top of the pan. By means of the three screws attaching the tripod to the star the height of the moving coil can be adjusted and the coil leveled. When in position and leveled, the axis of the coil passes through the tube at the top of the tripod. By undoing the links it is very easy to replace one moving coil by another.

Since the fixed coils are rigidly attached to the coil case, the lateral adjustment of the moving coil can be accomplished only by moving the balance. In order to be able to make changes of known amount in this adjustment, the plates, on which the front leveling screws of the balance rest, are provided with screw motion (Fig. 5), the left plate in a direction parallel to the length of the case and the right plate in a direction perpendicular to the latter. In order to eliminate any trouble from the springing of the leveling screws, they are connected to one another near their lower ends by means of collars and a rigid frame of brass.

The water jacket used with the large fixed coils is supported by the rods carrying the fixed coils; a smaller one used with *S*1 and *S*2 is supported on oak blocks resting on the pier top.

Connections between the brass pipes of the water-circulating system, and the coils and the water jacket, are made by means of glass and rubber tubing.

The coil case rests on a marble slab 152 by 76 by 7.5 cm, supported by two heavy oak piers resting on the concrete floor. In the earlier part of the work piers of white enameled brick set in Portland cement were used; but as the brick, and especially the sand used in the cement, were later found to be slightly magnetic, it was deemed advisable in the subsequent work to replace these piers by wooden ones. With the coils situated as shown in Figs. 5 and 6, this change produced a perceptible but very slight change in the force. The iron in the floor construction has been found to produce no effect upon the measurement of current. (See p. 347.)

10. THE ELECTRICAL CONNECTIONS

The two windings of each fixed coil are connected in parallel by means of a pair of closely twisted enamel-insulated wires, passing halfway around the circumference of the coil; a similar pair of twisted leads runs from the nearer terminals to binding posts set in the left wall of the coil case and connected with the commutator on the outside of the case. These posts and their connections to the commutator are carefully insulated with ebonite. Where the twisted leads expand into loops to connect with the coil terminals, the planes of the loops are carefully adjusted so as to lie as nearly as possible in a radial-axial plane. The outstanding effect of these loops is determined experimentally by measuring the force exerted upon the moving coil when the leads are short-circuited (without changing their positions) and a rather heavy current is passed through them, with no current passing through the fixed coils and with the normal working current flowing in the moving coil. The effect is always very small, but the proper correction has been applied to the observations.

Twisted leads pass from the terminals of the moving coil along a diameter of the coil to the axis, where they are soldered to twisted leads passing along the axis of the coil through the suspending tube into the balance case above the right pan of the balance. Here they are connected to the terminals of two sets of silver leads, one above the other, each consisting of 25 wires 0.02 mm in diameter and 8 cm long. The terminals at the other ends of these leads are connected by means of twisted leads to the commutator on the end of the coil case.

Where the leads pass through the brass tube, they are wrapped with a strip of paraffined paper so as to protect them from abrasion by the tube. They have soldered joints at the point *A*, Fig. 6, so that the coil and the lower section of the leads may be removed without disturbing the upper section. Wherever loops occur, they are placed as nearly as may be in radial-axial planes, and the outstanding effect is determined and allowed for by a method analogous to that described for the fixed coils. In order to be able to measure the resistance of the moving coil when carrying the working current, a pair of fine twisted leads runs

up the tube with the current leads from the axial junction of the latter, and is connected with the outside terminals by means of a single pair of fine silver wires.

The terminals of the two sets of silver leads are insulated from one another by a block of ivory; the outer pair of terminals is carried by a stand which rests upon the floor of the balance case and can be moved so as to adjust the slackness of the wires. These leads affect the sensibility of the balance only slightly. The sensibility given on page 283 is with the leads in position.

The mercury commutator attached to the end of the coil case

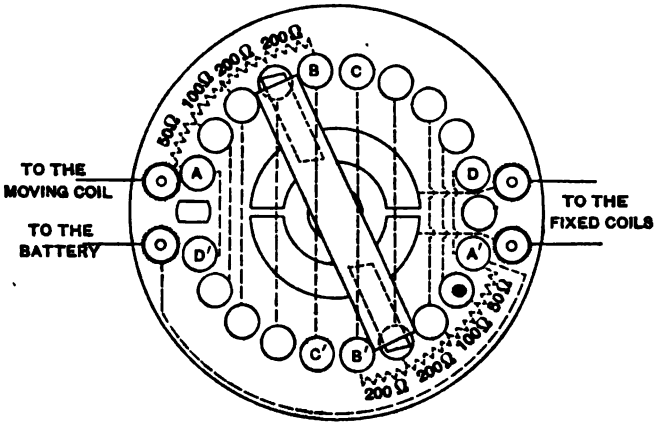

Fig. 8.—*Circular reversing switch*

As the handle is rotated from AA' to DD', the current is gradually reduced, reversed through the fixed coils, and then increased to its original value

enables the current in any coil to be reversed without changing the direction of the current in any other coil. From suitable terminals of this commutator, leads run to the circular reversing switch shown in Fig. 8. By means of this switch, which is placed on the end of the coil case where it can be readily reached by the observer at the balance, the current can be rapidly and smoothly reduced to a very low value, reversed through the fixed coils only, and then increased to its normal value. A reversal of this nature eliminates the danger of damaging the coils by a high induced electromotive force, and greatly facilitates the manipulation of the balance.

From the commutator, twisted leads with heavy rubber insulation run to the floor, and then through a conduit, one going directly to one terminal of a 25-ampere 120-volt storage battery, and the other to the standard resistance, regulating resistances, small fuse block, and switch, to the other terminal of the battery. The complete set-up is shown in Fig. 9. The resistances and potentiometer for measuring the current in terms of the international ohm and the Weston normal cell are on the table placed parallel to the pier, with its nearer end about 1.5 meters west of the axis of the coils. The working standard of electromotive force was given by four Weston normal cells kept in a thermostatically regulated kerosene bath on the small table in front. These cells and the working standard of resistance—an Otto Wolff 1-ohm coil—were frequently compared with the standards of this Bureau, and have been found to remain very constant.

The potentiometer was frequently calibrated and has been found to remain very constant. Its coils have been dipped in hot paraffin to prevent changes in resistance due to changes in atmospheric humidity. During the first portion of the work, the potentiometer was used to measure directly the voltage at the terminals of the 1-ohm coil; during the latter portion it has been customary to measure the small difference between this voltage and that of the standard cell, the two being connected in series and in opposition. This increases the sensibility and minimizes the errors introduced by slight variations in the potentiometer current.

By using as a source of current a storage battery of large capacity, which is entirely disconnected from all other circuits, and by waiting until temperature equilibrium of the coils of the balance and the ballast resistance has been attained, it has been possible to obtain a very constant current. Normally, it is possible to control the current so that its oscillations will amount to not more than one or two millionths of an ampere.

In order to facilitate the balance work, the current has always been adjusted so as to give a force of 1.5 or of 3.0 g, giving a change of force on reversal of the current of 3.0 or 6.0 g, respectively. A single platinum weight, of 3 or of 6 g, can therefore be used in each case in weighing the force.

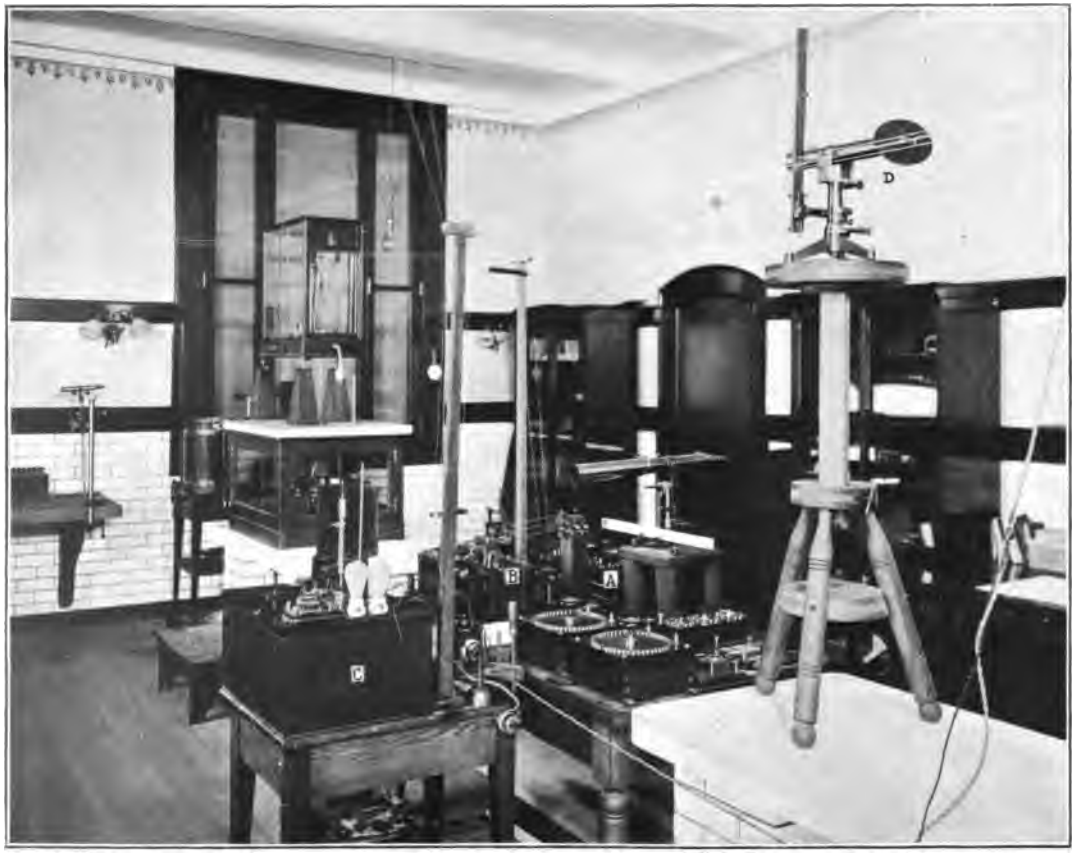

Fig. 9.—*Photograph of the auxiliary apparatus and of the current balance as mounted for the preliminary work*

A is a Wolff potentiometer, at B are standard resistances, C is oil tank containing the standard cells and thermostat, D is telescope and scale used in reading the rest points of the balance

III. THE RATIO OF THE RADII

11. DEVELOPMENT OF THE METHOD

Since the mutual force between given currents in two *coaxial circular circuits* varies in such a way as to become a maximum for a certain value of the distance between the planes of the circuits, the distance for maximum force, as well as the magnitude of this maximum force, must be a function solely of the radii of the coils. (That the force has a maximum is at once evident from the fact that it is zero both when the planes of the coils coincide and when they are at an infinite distance apart, but is finite for intermediate positions.) Furthermore, knowing that the dimensions of the square of a current in electromagnetic units are the same as those of a force, and that the force between two circuits such as we are considering is equal to the product of the two currents into some function of the radii of the circuits and of their distance apart, it is evident that this maximum force, of which we have spoken, is a function solely of the *ratio* of the radii of the two circuits.

Excepting for correction terms depending upon their finite section, the same is true for coaxial circular *coils*. Hence, the determination of the constant of the balance, in which the coils are so spaced as to exert their maximum mutual force, depends solely upon a knowledge of the sectional dimensions of the coils and upon the ratios of the mean radii of the fixed coils to the mean radius of the moving coil.

The value of this ratio can be obtained from the direct measurement of the mean radii of the coils, but such measurements for multiple-layer coils are very difficult to obtain and are at best of a relatively low order of accuracy.

A far better and more accurate method is to measure directly the ratio of the galvanometer constants of the two coils. From this ratio and a knowledge of their sectional dimensions we obtain at once the ratio of the two mean radii, the quantity upon which the constant of the balance depends.

In taking account of the sectional dimensions of the coils in the computation of the constant of the balance, it is, however, simpler—at least, the steps appear more tangible—if we deal with the

radii directly rather than with their ratios. In order to obtain numbers corresponding to the mean radii, the absolute values of these radii affecting only the correction terms necessitated by the finite sectional areas of the coils, it is allowable to assume *any* approximate value for the mean radius of one coil, and from this and the measured ratios to deduce the corresponding radii for the other coils.

In the present work we have assumed that the mean radius, corrected for the temperature, of the coil designated as *S*1 is, and has always remained, exactly that calculated from the direct measurements made upon the various layers of the coil as it was wound.

Methods—In 1854 Bosscha [9] described a method for the determination of the ratio of the galvanometer constants of two coils.

This method, which was employed by Lord Rayleigh to obtain the radius of the moving coil of his balance in 1884, consists in placing the two coils concentric and with their planes in the magnetic meridian; in connecting the coils, with suitable resistance in series with each, in parallel and in such a way that when a current is passed through this compound circuit the magnetic fields at the center of the coils will be opposed; and then in adjusting the resistance in one branch of the compound circuit until the torque exerted upon a small magnetic needle suspended at the center of the coils is zero. When this condition is attained, the ratio of the current in the two circuits, and consequently the inverse ratio of the resistances of the two circuits, will be equal to the inverse ratio of the two galvanometer constants (except for a correction term depending upon the length of the magnetic needle employed). Hence, excepting for this correction term, the ratio of the galvanometer constants is equal to the ratio of the two resistances, which can be measured.

The method as described is ideally simple and easy of application where extreme accuracy is not needed. When, however, an accuracy of 1 in a million, or even of 1 in 100 000 is desired, the heating of the coils by the measuring current introduces almost insuperable difficulties.

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[9] Pogg. Ann., 93, p. 402; 1854.

In order to minimize these difficulties, Lord Rayleigh so arranged the coils and standard resistances that, after the balancing of the magnetic fields, the removal of a single link sufficed to convert the resistances, of which we desire the ratio, into the adjacent arms of a Wheatstone bridge, the opposite arms of which were composed of standard coils. Thus, it was possible to pass rapidly from one measurement to the other, and so to reduce the change of temperature to a relatively small amount. He was thus able to obtain an accuracy that was ample at that period.

In order to increase the accuracy still further, the link used by Rayleigh was omitted (thus keeping the coils in the Wheatstone net during the process of balancing the magnetometer), and a simultaneous balance of both the bridge and the magnetometer was obtained. Then the ratio of the resistances in the arms containing the coils will, at the instant of balance, be exactly that of the ratio of the other two arms of the bridge. These, which we shall call the ratio arms, must be of low resistance, and must have small temperature coefficients, as they carry the full currents passing through the coils. They must further be capable of fine adjustment in order to obtain an exact magnetometer balance.

These conditions can scarcely be fulfilled by coils designed for precision resistance measurements, so arrangement was made for quickly transferring these ratio arms to a second bridge in which they can be measured against precision resistances.

This method, though workable and yielding results of higher accuracy than those previously used, was abandoned early in the work for three reasons: (*a*) It is slow, and therefore variations in the earth's field are a grave source of error; (*b*) it is difficult to avoid slight trouble due to the heating of the ratio arms; and, most important of all, (*c*) the current being alternately off and on, the coils to be compared never attain a stationary temperature condition, and hence an exceedingly accurate interpretation of the results is not possible.

In recent times the facilities for the measurement of current have improved to such an extent that it is now almost as easy to measure two currents by means of potentiometers and to determine

their ratio as it is to measure the ratio of two resistances. This very obvious change in the details of the method involves simultaneous observations by three observers, but is capable of the most extreme accuracy. This may be called the *potentiometer method.*

Another modification of the details of the method renders it even more simple than as used by Bosscha, but is applicable only to coils having very nearly the same galvanometer constant. Two such coils are connected in series, and the coil with the larger galvanometer constant, together with an added resistance, is shunted so as to obtain a zero field at the center of the coils. If the galvanometer constants of the coils differ by a small amount it is easy to adjust the resistance in series with the shunted coil so that the shunt is large enough to give the desired sensibility of balance. Neither the shunt nor the shunted resistance need be known accurately, since a very small part of the total current flows through the shunt.

This method may be called the *shunt method;* it requires but a single observer. By using the shunt as a volt box it is easy to determine under working conditions the resistance of the circuit shunted in terms of a standard coil through which the total current passes. Since the galvanometer constants are assumed to be very nearly equal, very little current will flow through the shunt and, consequently, it will not be appreciably heated.

Under certain conditions a combination of these two methods may be desirable. This may be called the *combination method.* It is described in a later portion of the paper.

In order to obtain the best results it is necessary to make the observations at a time when the earth's magnetic field is as steady as possible. This is usually at night.

12. THE POTENTIOMETER METHOD

The connections for the potentiometer method are shown in Figure 10. M and F are the two coils; they are actually coplanar and concentric, though, for simplicity, shown otherwise. R_1 and R_2 are two standard resistances from which leads run to the potentiometer for measuring the currents. From that resistance which is in series with the fixed coil, auxiliary leads run to a

second potentiometer. By means of this potentiometer and a continuously variable resistance in circuit with the fixed coil the current can be held constant at any desired value. C_2 is a commutator by means of which the two resistances may be interchanged with reference to the coils M and F; r, r_1, and r_2 are adjusting resistances. L is a self-inductance. C_1 is a pair of commutators for reversing the current through the coils. Since it is impossible to so construct C_1 that it will make and break both circuits at identically the same time, it is connected with the switch S so that they can all be thrown at one operation, and

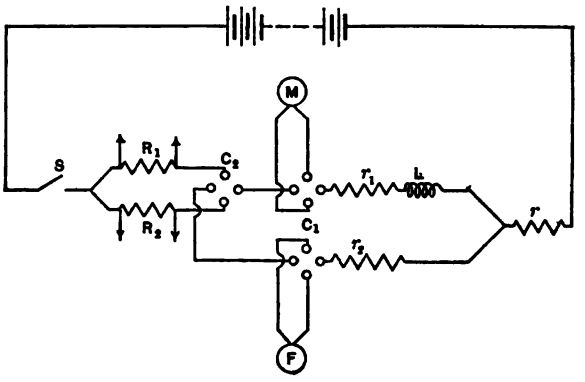

Fig. 10.—*Connections for the potentiometer method of measuring the ratio of the radii*

so that S is opened before C_1 and closed after C_1, thus avoiding severe deflections of the magnetometer.

In the present case the coils M and F are wound on metal forms, and, consequently, owing to eddy currents, the time constant of the field at the center of each, when alone, is quite large. But as adjusted for the ratio of the radii work, the field inside M is very small, being largely neutralized by the field due to F; that inside F, but outside M, is much greater than it would be if M were absent. Hence, as thus adjusted, the eddy currents in the form of M will be small, while those in the form of F will be about as strong as if M were absent. Hence, the time constant of the field of F will be much greater than that of the field of M. This will cause a deflection of the magnetometer whenever the current is stopped or started. Furthermore, when the coils are connected for the two-potentiometer method there is an interval

between the opening of S and that of C_1, during which the circuit of F is closed through M so that the induced electromotive force in the former gives rise to a current through both F and M in such directions that the two fields thus produced at the center are in the same direction. This adds to the effect just mentioned, and causes a very violent deflection of the magnetometer whenever the current is made or broken during the use of this method. To reduce this deflection, and so increase the speed of the work, a suitable *large* inductance L is introduced in the moving-coil circuit. It is placed at a distance from the magnetometer so as to reduce its effect upon the needle, it being desirable to have the zero of the magnetometer on closed circuit the same as on open circuit. However, by the procedure adopted in this work, in no case can the field of the inductance produce any effect upon the observed ratio of the galvanometer constants.

The method of procedure was as follows: The resistances r, r_1, and r_2 were adjusted so as to make the two fields approximately equal and of such a strength as to give the desired magnetometer sensibility (a change in reading of 1 mm on reversal corresponding to a difference in the two fields of about 3 or 4 in a million). The current was then left on continuously, with water of the proper temperature circulating through the fixed coil, for about an hour before observations were begun. Then the resistances of the coils were measured, the thermometers attached to the coils were read, and while one observer held the current through the fixed coil constant and at the desired value, a second observer, with a potentiometer which could be connected to either R_1 or R_2, connected his potentiometer across the standard resistance in series with the fixed coil, and adjusted the potentiometer current until he obtained the nearest possible balance with some even setting of the potentiometer dials; the lack of exact balance was measured by the galvanometer deflection and allowed for in the reduction. He then threw his potentiometer across the standard resistance in the moving-coil circuit, adjusted this current approximately to its correct value, and then allowed it to drift slowly toward his potentiometer balance, while a third observer damped and read the magnetometer. The third observer gave a signal at

the instant of reading the magnetometer, and the second observer noted the galvanometer deflection at that instant. The first observer thoughout this time held the current through the fixed coil at a constant value. The switches C_1 are now thrown so as to reverse the currents through both coils, and the operation is repeated. After taking several such pairs of observations, the second observer again puts his potentiometer across the standard resistance in the fixed-coil circuit, and observes the deflection when the dials are set as at first, and the first observer's indicating apparatus is balanced as at the start. The very slight difference between this and the deflection observed at the start is due to the relative change in the two potentiometer currents, and is allowed for in the reduction. The thermometers are then read, C_2 is reversed so as to interchange the standard resistances with reference to the balance coils, and the operation is repeated. Taking the mean of these two sets of observations eliminates the values of the two standard coils, and hence makes it unnecessary to know their values accurately. Two such sets, each of five pairs of magnetometer readings, takes from 15 to 20 minutes. The resistances of the coils are then measured.

By this method of procedure, it is evident that the only stray field that can affect our results is that due to the leads from C_1 to the coils. The effect of these is always very small, and was always measured by noting the magnetometer deflection produced by reversing, through the leads alone, short-circuited at the coil terminals, a current of 4 or 5 amperes. Of course, one terminal of each winding was entirely disconnected from the leads, so that there was no possibility of any of the current passing through the coil itself.

The relative values of the standard resistances R_1 and R_2 can be eliminated by interchanging them only if they are of the same denomination. In some cases standard resistances of different denominations would have to be used for the measurement of the ratio of the galvanometer constants; in which cases the relative values of R_1 and R_2 would have to be known with high precision. The necessity of using standard resistances of different denominations arises from the fact that the lower setting on the

potentiometer must be great enough for its value to be determined to the required degree of accuracy, and the higher one must, of course, not exceed the range of the instrument.

13. THE SHUNT METHOD

In the present work $M3$ was so constructed that when its windings are in series it has very nearly the same galvanometer constant as $S1$ or $S2$, when their windings are in parallel. Consequently, the ratio of its constant to that of either $S1$ or $S2$ could be determined by the shunt method. In fact, both methods were used, and checked very closely.

The connections used for the shunt method are shown in Fig. 11. As in the previous method C_1 and S were thrown simultaneously

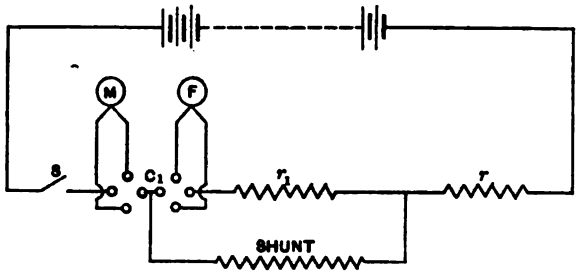

Fig. 11.—*Connections for the shunt method*

and in such a way that S broke circuit before either of the commutators C_1, and closed circuit after C_1.

After the temperature equilibrium had been reached, the shunt was adjusted for approximate balance, and the magnetometer was read; then the currents through the coils were reversed, and the magnetometer was read again, and so on. About nine such pairs of observations can be made in five minutes.

14. THE COMBINATION METHOD

If the constants differ so much that in order to obtain the accuracy required in the value of the ratio it is necessary to know r_1 (Fig. 11) to an inconveniently high degree of accuracy, we can use the same schematic arrangement, but by means of a potentiometer and of standard coils in r_1 and in r we can measure directly the currents in the two coils, as in the potentiometer method; or,

if the constants are quite different we can incorporate the standard coils with r_1, and with the shunt, so that we can measure the currents in these branches. This is an especially desirable arrangement if the constant of F is twice that of M.

In these combination methods, it is desirable that the observer who holds the current constant should have his potentiometer so connected as to control the *total* current. The second observer, with his potentiometer across a standard coil in r_1, will then observe practically no drift, and so can read his deflection at the desired instant with a maximum of accuracy; he then throws his potentiometer across the other standard coil and measures that current. If the constant of F is very nearly equal to twice that of M, then these two standard coils should have the same value and the two potentiometer settings will be nearly the same; everything will be suitable for a well-balanced measurement. The main objection to this method is the purely practical one of facility in manipulation. As every change in r_1 or in the shunt affects the current through M, the balancings of the two potentiometers have to proceed simultaneously. This makes the attainment of a balance distinctly slower than in the potentiometer method, where the two circuits are almost independent of one another.

All of these methods have been used, but as no case existed in which the ordinary potentiometer method could not be satisfactorily employed, we have in the final work used that and the shunt method only.

15. ADJUSTMENTS OF COILS AND NEEDLE

As shown in Fig. 12, the coils were clamped concentric with one another by means of three brass blocks fastened to a marble slab. The slab is supported vertically on a wooden stand having leveling screws. All portions of the mounting were tested by the astatic magnetometer and were found to be satisfactory. The magnetometer needle was of tungsten steel, about 1.9 mm long, 1 mm wide, and not over 0.2 mm thick, fastened to a slender glass rod, to the lower end of which was attached a small plane mirror. By means of a quartz fiber, about 5 cm long, the whole was suspended in a brass tube, 17 mm in diameter, which was attached to the

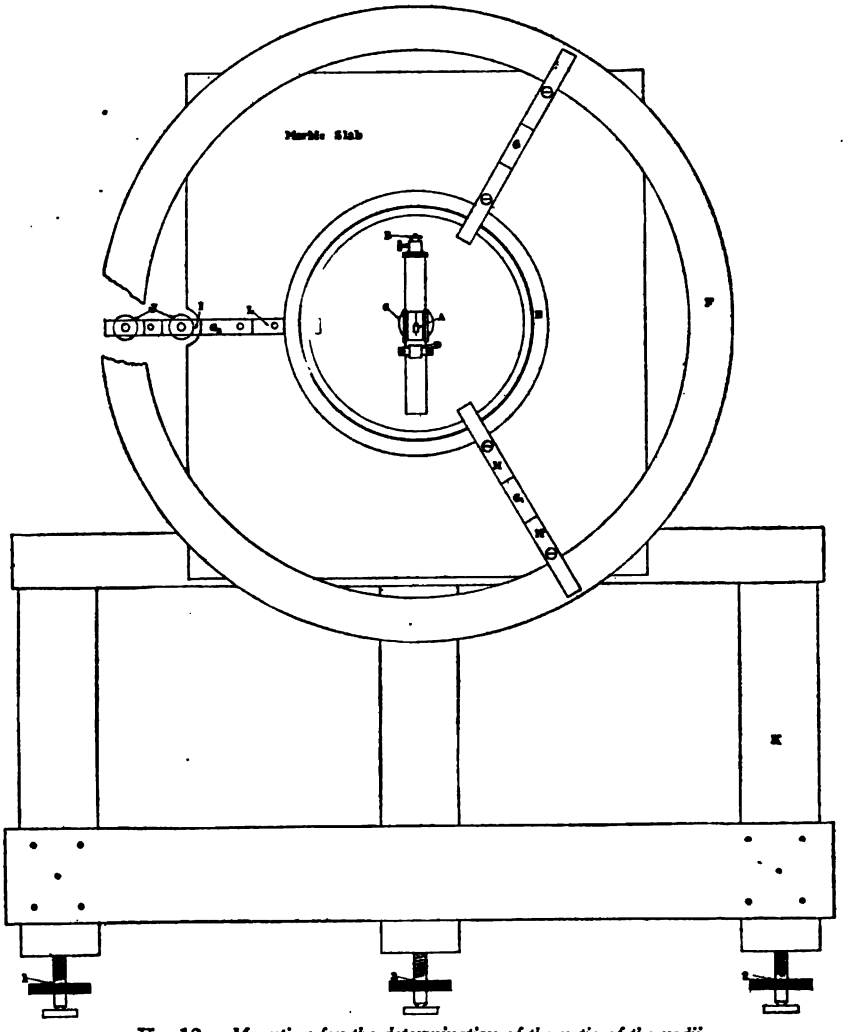

Fig. 12.—*Mounting for the determination of the ratio of the radii*

A is suspended magnet and mirror, *B* is screw for adjusting height of magnet, *C* is hole in marble slab with cross-hairs for preliminary adjustment of magnet, *D* is fixed mirror for determining position of stand, E is 20-cm moving coil, *F* is 50-cm fixed coil, *G*, G_1, G_2 are brass blocks attached to the marble slab, *H*, *H'* are clamps, *I* is slot for 40-cm fixed coil, *J* are micrometers for rotating fixed coils around a vertical axis, *L* is removable block for admitting 25-cm moving coil

marble so that the needle hung near the axis of the coils. By means of a screw *B* (having a graduated head) the vertical position of the needle could be adjusted over a range of several millimeters. The needle could be displaced horizontally parallel to the planes of the coils by means of the leveling screws 1 and 2. By means of a graduated series of brass liners, each coil could be independently adjusted so that its plane passed through the needle. By means of the micrometer screws *J* with threads of 0.5-mm pitch, the large coil could be delicately adjusted by rotation about a vertical axis so as to place its plane parallel to the plane of the small coil; and by rotating the entire stand the planes of the coils could be set parallel to the needle.

Now it is evident that if the mean planes of the two coils exactly coincide, then the ratio of the moments exerted on the needle for given currents in the two coils will be independent of the angular position of the needle with reference to the coils, except for a very small second-order correction term, depending upon the variation in the correction for the length of the needle.

If the needle lies along the bisector of the angle between the planes of the two coils, then the ratio of the moments will be independent of this angle, except for the same small correction term.

But if neither of these conditions is fulfilled, then that coil which is the more nearly parallel to the needle will exert a relatively greater moment, and for two reasons. First, the poles of the needle lie in a field which, relative to the other coil, is stronger; and, second, the direction of this field is more nearly normal to the length of the needle than is that of the other coil. The first of these effects is very small, as it concerns the second-order correction term spoken of above. The second effect results from the fact that the moment which either coil exerts upon the needle is proportional to the cosine of the angle between the magnetic axis of the needle and the plane of the coil, so that the ratio of the two moments is proportional to the ratio of these two cosines. For example, suppose the needle makes an angle of 2° with the plane of one coil and 2° 1′ with the plane of the other—that is, the planes of the coils are inclined at an angle of 1′—then the value of the ratio of the galvanometer constants, calculated from the currents on the assumption

that the adjustments are perfect, will be in error by a factor equal to the ratio of cos 2° to cos 2° 1′—that is, it will be in error by 1 in 100 000.

With a needle but 2 mm long, and a fiber which must be stout in order to withstand the rather rough treatment to which it is necessarily subjected, it is evident that the adjustment of the needle to within 2° by any process of inspection is out of the question. The same is true with respect to the adjustment of the coils to within a single minute of arc. But, by making use of this very error due to maladjustment, it is a comparatively simple process to set *both* coils and needle in parallel planes. We have adopted the following procedure:

The coils are adjusted so that their planes are vertical and approximately coincide in the magnetic meridian. The planes of the coils are made vertical with sufficient precision by means of a plumb line. The vertical height of the needle is adjusted so that the latter lies on a level with the centers of the coils, and the fiber is free from torsion. The large coil is rotated about a vertical axis through an angle of a few minutes by means of the adjusting screw. Then, by means of an auxiliary coil, suitably placed, and a constant battery, the needle is deflected from its normal position by about 24°, and the apparent ratio of the galvanometer constants is measured with the needle in this deflected position. The current through the auxiliary coil is now reversed so as to deflect the needle by about the same amount in the other direction, and the ratio is again measured. Suppose the angle between the coils is 3′; then, when the needle was deflected (with reference to the small coil) in the same direction as the large coil, the moment exerted by the latter relative to that exerted by the small coil was approximately 39 parts in 100 000 too great; when deflected in the other direction, it was too small by the same amount, provided that the undeflected position of the needle bisected the 3′ angle between the coils and the deflections were equal in both directions. Otherwise one of these quantities will be increased, and the other decreased, by slightly different amounts. If we plot these observed ratios of the galvanometer constants as ordinates against angular positions of the needle as abscissæ, the straight line connecting

them will pass very nearly through the point representing the correct ratio at that abscissa which corresponds to the position of the needle when it bisects the angle between the two coils.

Now rotate the fixed coil in the opposite direction, and repeat the measurements, and (if the coil has been turned beyond the position where they are coplanar) we obtain a line sloping in the other direction. The abscissa of the intersection of these two lines is very nearly that setting of the needle which makes it parallel to the moving coil. It is exactly that setting if the lines intersect at their middle points. Then the stand carrying the coils and magnetometer is rotated in the proper direction through an angle, as indicated by the reflection of the scale from a mirror fixed to the magnetometer tube, that is equal to the ascertained angle between the coils and the needle. The observations are then repeated in order to test the accuracy of the setting. Thus, the needle is placed in the plane of the coils to within the accuracy with which we can adjust the stand. In practice, 1 cm on the scale was considered sufficiently accurate; at a scale distance of 3 m this corresponds to an angle of about 6′. With an error of this magnitude in the position of the needle, an angle of 2′ between the planes of the coils will give an error of 1 in a million in the result. Hence, it is necessary to assure ourselves that the coils are parallel to within this limit.

From what has been said above, it is evident that it is impossible for the apparent ratio with the needle deflected $+25°$ to be the same as that with the needle deflected $-25°$, unless the coils are parallel to one another. Hence, all that is necessary in order to set the coils parallel to one another is to rotate the large coil until this condition is fulfilled. The amount and sense of this rotation are indicated by the slopes of the two lines on the plot. Using a 25° deflection, and 1 in 100 000 as an accuracy rather easily obtained without refined reductions, we see that we can detect an angle of 2.4×10^{-5} radians, or about 5″ between the planes of the coils. This, for the mounting of the small fixed coils, corresponds to an advance of the adjusting screw of only 7 microns. Occasionally an accuracy of this amount was accidentally attained, but usually the accuracy of setting was about one-fifth of

this, or 25 seconds. With an error of 6′ in the setting of the needle, and 25″ between the coils, the observed ratio of the galvanometer constants will be in error by 2 in ten million. This is about the error that this maladjustment may introduce into our latest determinations of the ratio of the radii.

In the earlier work the importance of this adjustment was not recognized, and as a consequence those measurements may, from this cause, be in error by over 1 in 100 000.

Besides the orientation of the coils, four other adjustments must be made, viz, the vertical adjustment of the needle, the adjustment of each coil so that its mean plane passes through the needle, and the lateral adjustment of the needle in the plane of the coils. In each case measurements of the ratio were made for various adjustments of the type under investigation and the results were plotted. The theoretical curve, representing the variation of the ratio with variations in this type of adjustment and calculated from the known dimensions of the apparatus, was fitted as well as possible to the plotted results, and the particular adjustment was made to correspond to the vertex of the curve. Any slight failure in realizing this exact adjustment, at the time the observations were taken, was allowed for in the final precise reductions.

The adjustment of the angle between the planes of the coils was always repeated after all other adjustments had been made.

16. CORRECTION TERMS

In addition to the corrections already considered, there are three others of prime importance.

(*a*) *Needle correction.*—This depends upon the distance between the effective poles of the needle. For large magnets this has been found to be about five-sixths the total length of the magnet. This value was used in the earlier portion of the work. Later it was noticed that the amount of the correction can be determined directly by a comparison of the apparent ratio of the constants when the needle is deflected with that found when the needle is not deflected. A series of observations has been made with this object in view, and the corrections to the observed ratios

as determined experimentally are given in Table I. The effect of the length of the needle is such as to increase the measured ratio of the radii of the larger to the smaller coil, and the correction, therefore, has the negative sign.

TABLE I

Correction for the Length of the Needle (Needle Length = 2.0 mm)

Radii of coils		Parts per million
20 cm	10 cm	−56.3
25	10	−63.0
25	12.5	−35.9

These corrections correspond to a polar distance of 2.00 mm; five-sixths of the total length of the needle is 1.93 mm, but it is evident that the vertical width of the needle must increase its effective length; hence the discrepancy between the observed polar distance and that calculated from data obtained with much larger magnets is but 3 per cent. We are not acquainted with any previous attempt to measure the polar length of such small magnets.

(*b*) *Correction for the temperatures of the coils.*—Since it was found that the temperature indicated by a thermometer attached to the form of a fixed coil varied appreciably as the position of the thermometer was changed, and as it was obviously impossible to read a thermometer attached to the moving coil when the latter was surrounded by the water jacket, it was deemed best in the latest work to derive all temperatures from the resistances of the coils. This involves a knowledge of the resistances of the coils at some known temperature and of the temperature coefficient of the resistance of the coil. The first is easily obtained, and the latter can be found by a preliminary measurement. But, as it is necessary to determine experimentally the coefficient of expansion of the coils, it was considered simpler and better to assume a value of the resistance temperature coefficient that is approximately right, and to determine the coefficients of expansion in terms of the temperatures calculated on the basis of this coefficient. Then

any error in the assumed resistance coefficient will be exactly compensated for by the observed coefficient of expansion. We have taken as our temperature of reference 22° C, and as the assumed resistance temperature coefficient per 1° C the value 3.9×10^{-3} of the resistance at 22°.0 C. We have called the temperatures thus determined the "electrical temperatures" of the coils.

The values of the coefficients of expansion of the coils have been determined from measurements of the ratio of the galvanometer constants for various temperatures of one coil, the other changing in temperature but little. The temperatures of the fixed coils were regulated by the temperature of the water circulating through them; those of the moving coils by the temperature of the room. It was found experimentally that the temperature coefficients as thus determined are independent of the actual temperature and of the rapidity of the flow of the water within the limits of accuracy with which we are concerned. The resistance of a coil at any time was determined by potentiometer measurements of the current through it and of the drop of potential across its terminals. The values of the coefficients of expansion of the coils are given in Table VII, page 322.

(*c*) *Correction for the load.*—It is evident that the wire when carrying a current will be at a higher temperature than the form on which it is wound, and this excess will be proportional to the load—that is, to the number of watts expended upon the coil. But the radius of the coil is determined by neither of these temperatures, but by some temperature between them. Hence the radius of a coil will not be determined solely by its electrical temperature, but after the latter correction is applied there will be still another one needed to take account of the load. The amount of the latter correction will depend upon the heat insulation of the windings from the form and may be expected to vary greatly from coil to coil.

To determine this correction, it is sufficient to measure the ratio of the galvanometer constants with various loads on one coil and a constant load on the other. To realize such a condition, we have made use of the property of conjugate conductors in a

Wheatstone bridge. The leads from the reversing switch, C_1, of Fig. 10 (p. 301), instead of going directly to the terminals of F, with its two windings in parallel, run, as shown in Fig. 13, to the opposite corners of a Wheatstone net. Two arms of this net are composed of the two windings A and B of the coil F, the others of resistances r_a and r_b. R_a and R_b are two standard 1-ohm coils

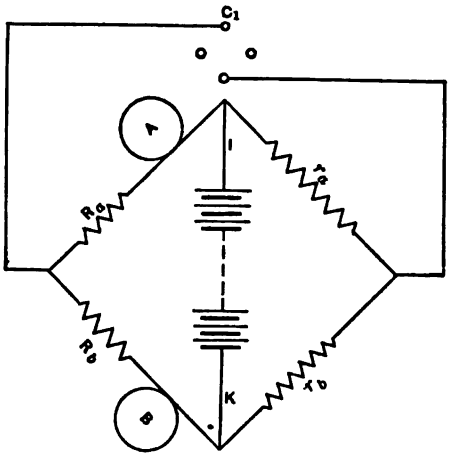

Fig. 13.—*Bridge used for electrically loading a coil in the determination of the correction for the load*

by means of which the current through each winding can be measured. The windings are so connected that the current from C_1 passes through them in the same direction—that is, their magnetic fields add together. By adjusting r_a and r_b, the bridge is balanced, and then any electromotive force desired may be placed across IK without affecting the current from C_1, which measures the galvanometer constant. Furthermore, the constants of the two windings of any coil are practically identical, and, as would be expected, the current from the IK battery flowing through them in opposite directions has been found to produce no effect on the magnetometer. Also, since measurements of the constants are made by reversing C_1, IK remaining unchanged, the only possible effect the current from IK could produce upon the magnetometer would be a permanent displacement of the zero. Furthermore, any slight lack of balance of

the bridge (called, in general, the heating bridge) will produce no error, for the resulting current through the bridge will add itself with proper sign to the measuring current and after passing C_1 will be measured with the latter. Hence the only point requiring especial care is that the *IK* battery shall be well insulated from the battery furnishing the current for the measurement of the ratio. Thus we are enabled to study the effect of the load upon each coil. It has been found that the apparent radius of a coil after being corrected to a constant electrical temperature decreases linearly as the load increases. Hence the radius of the coil is given by the expression:

$$A = A_0 \{1 + \tau (t - 22) - \lambda w\}$$

where A is the radius at the electrical temperature t, and with the load of w watts, and A_0 is the radius at 22°00 C and with no load. The values of the coefficients τ and λ are given in Table VII, page 322.

17. SOURCES OF ERROR

The sources of error to be considered are (1) maladjustment, (2) effect of leads, (3) changes in the size of the coil, (4) error in the measurement of the sectional dimensions, (5) magnetization of the forms.

The first two have already been considered. The third may arise from either of two causes; either from the strains incident to winding the coil, or from changes in the humidity. The latter may be obviated by sealing the coils, and the former by heating the coil (by means of a current) for several hours to a temperature appreciably higher than any at which it will afterwards be used.

The magnitude of the fourth source may be estimated from a consideration of the equation expressing the relation between the galvanometer constant of a coil and its sectional dimensions, viz:

$$A_0 = \frac{2\pi n}{G_0}\left\{1 - \tfrac{1}{2}\left(\frac{\alpha_0}{A_0}\right)^2 + \tfrac{1}{3}\left(\frac{\rho_0}{A_0}\right)^2 + \tfrac{3}{8}\left(\frac{\alpha_0}{A_0}\right)^4 + \tfrac{1}{5}\left(\frac{\rho_0}{A_0}\right)^4 - \frac{\alpha_0^2\rho_0^2}{A_0^4} + \cdots\right\}$$

where A_0 is the mean radius of the coil, n is the number of turns of wire, G_0 is the galvanometer constant, $2\alpha_0$ is the axial breadth of the windings, $2\rho_0$ is the radial depth of the windings.

If, owing to errors in measurement, we assume the values $\alpha = \alpha_0 + \delta\alpha$, $\rho = \rho_0 + \delta\rho$ instead of α_0 and ρ_0 we shall find from the observed galvanometer constant not A_0 but A. Expanding the above expression by Maclaurin's theorem we find

$$\frac{A - A_0}{A_0} = -\left(\frac{\alpha_0}{A_0}\right)^2 \frac{\delta\alpha}{\alpha_0} + \frac{2}{3}\left(\frac{\rho_0}{A_0}\right)^2 \frac{\delta\rho}{\rho_0}$$

In the case of the coils used in this work $\frac{\alpha_0}{A_0}$ and $\frac{\rho_0}{A_0}$ were approximately either 0.04 or 0.05, so that the per cent error in A is about 0.002 times the per cent error in either dimension of the section. Hence, an error of 0.01 mm in the width or in the depth of the windings of a coil with a section 1 cm square will produce an error of about 2 in a million in the radius. Excepting the moving coils, all coils have a larger section than this, and so the effect of an error of this magnitude in the measurements will produce for them a smaller error in A.

As pointed out by one of us,[10] and as shown at length in the appendix to this paper, page 375, the sectional dimensions of the windings of any coil are to be understood as determined by the expression ns where n is the number of wires in the direction of the dimension considered and s is the distance between the axes of consecutive wires.

Hence, in the above expression, $2\rho_0 = \frac{n}{n-1}(r_0 - r_i)$, where n is the total number of layers, r_0 is the radius to the axis of the wire in the outer layer, and r_i is the radius to the axis of the wire in the bottom layer. These quantities can be readily measured with considerable accuracy, probably to within 0.01 mm. For details, see page 390 in the appendix.

As already stated, the coils were wound with enamel-covered copper wire, which winds more smoothly and uniformly than silk-covered wire, the finished coils presenting a very beautiful appearance. However, adjacent spires are not in actual contact throughout, so that the distance between the axes of adjacent wires slightly exceeds the diameter of the wire

[10] This Bulletin, 2, pp. 77, 413; 1906, and 3, p. 235; 1907.

(about 0.01 mm to 0.03 mm). This will produce no error, provided the spacing is uniform and half as wide a space occurs on the average between the end turns and the sides of the channel; $2\alpha_0$ is then equal to the breadth of the channel.

After calculating the magnitude of the maximum error that could occur due to this cause, we think it improbable that an error greater than 3 parts in a million exists in any coil, except moving coil $M4$.

Moving coil $M4$ was wound in such a way that in each layer the wires were crowded toward one side of the channel, adjacent layers being displaced in opposite directions. The wires were maintained in the position which they occupied on the completion of the layer, by the paraffin which was thoroughly melted into the layer. Owing to very slight, almost microscopic, kinks in the wire, and to its tendency to slide into the spaces between the wires in the underlying layer, it is practically impossible to pack the wires so closely that the actual width of the space in which the n wires lie is equal to nd, where d is the diameter of the wire. Hence, the value $2\alpha_0$ for this coil lies between the values nd and the width of the channel. Now it is easy to show that the galvanometer constant of a coil having layers of width nd, and so wound that each layer is displaced in its plane by an amount h, with reference to the adjacent layers, in the manner in which $M4$ is wound, is approximately equal to that of a coil of breath $\sqrt{n^2d^2+3h^2}$. For $M4$ the breadth of the channel is 10.877 mm, $nd=10.50$ mm, $h=0.377$ mm. Hence, $\sqrt{n^2d^2+3h^2}=10.520$ mm. This differs from the breadth of the channel by 0.357 mm, which, by what precedes, corresponds to a difference of about 63 in a million in the radius as calculated from the observed galvanometer constant.

This considerable range, within which the radius may lie, makes the results obtained by this coil of no value so far as the absolute measurement of current is concerned, but as it is satisfactory in every other respect, and is especially well protected from the effects of atmospheric humidity, observations with it will serve as a test of the constancy of the other coils.

In the computation of the constants for this coil, we have assumed, as for the others, that the effective axial breadth is equal

to the breadth of the channel. As shown above, this is too great in the present case and the effect will be to yield too high a value for the standard cell.

As regards the fifth source of error, that due to the magnetization of the forms, it is difficult to obtain an estimate of its magnitude. It is evidently very small, for otherwise the variations in the results obtained in the measurements of current should have some relation to the magnetic properties of the coils. No such relation is evident, and we conclude that even the most magnetic of the fixed coils causes no appreciable error on this account.

18. RESULTS

In order to exhibit the method of reduction adopted, and to give an idea of the reproducibility of the observations, a few of the reductions and results of the most recent determinations are given in Tables II and III.

In Table II are given the determinations of the ratio of the galvanometer constants of M_3 and S_1 for various temperatures, and loads on the latter. These particular observations were obtained by the shunt method, and the values of the ratios of the current in M_3 to that in S_1, as given in column 6, have been corrected for the lack of balance of the magnetometer. For the potentiometer method, the numbers in this column would be the mean of two ratios, each corrected for the errors of the potentiometer; otherwise, the table would be the same.

TABLE II

Ratio of Galvanometer Constants, S1 to M3 (Determined by the Shunt Method) [11]

Date	Temperature		Load in watts		Observed ratio	Temperature correction		Load correction		Corrected ratio	Δ
	M	F	M	F		M	F	M	F		
Nov. 28	20°95	20°17	0.285	0.928	1.004604_1	+20.7	−31.1	+2.3	− 0.4	1.004596	3
	21.61	23.46	0.285	0.928	4559_5	+ 7.7	+24.8	+2.3	− 0.4	94	1
	22.99	20.79	1.200	3.860	4627_0	−19.5	−20.6	+9.8	− 1.8	95	2
	22.98	20.76	1.194	3.872	4626_8	−19.3	−21.1	+9.8	− 1.8	94	1
Nov. 29	21.06	23.78	0.294	21.25	4551_6	+18.5	+30.2	+2.4	−10.0	93	0
	21.06	23.78	0.290	21.15	4548_1	+18.5	+30.2	+2.4	− 9.9	89	4
	21.15	22.76	0.288	15.20	4568_7	+16.8	+12.9	+2.4	− 7.1	94	1
	.....	.....	.....	.....	4567_7	.....	.....	.....	.....	93	0
	21.12	22.75	0.286	15.19	4566_8	+17.3	+12.7	+2.3	− 7.1	92	1
	21.05	20.94	0.286	4.56	4594_4	+18.7	−18.0	+2.3	− 2.1	95	2
	.....	.....	.....	.....	4596_6	+17.5	−18.0	+2.3	− 2.1	96	3
	.....	.....	.....	.....	4593_5	+16.0	−18.1	+2.3	− 2.1	92	1
	.....	.....	.....	.....	4595_3	+14.5	−18.1	+2.3	− 2.1	92	1
	.....	.....	.....	.....	4595_7	+13.7	−18.2	+2.3	− 2.1	91	2
	21.35	20.93	0.286	4.57	4596_6	+12.8	−18.2	+2.3	− 2.1	91	2
	21.39	20.32	0.286	0.929	4609_2	+12.0	−28.6	+2.3	− 0.4	94	1
Dec. 2	21.65	20.34	0.282	0.918	4612_8	+ 6.9	−28.2	+2.3	− 0.4	93	0
	21.65	20.34	0.282	0.918	4610_9	+ 6.9	−28.2	+2.3	− 0.4	92	1
Mean	.....	.....	.....	.....	.....	.....	.....	.....	.....	1.004593	1.4

[11] In the last column are given the differences from the mean value in parts in a million.

From these observations the temperature coefficient and the load correction for *S*1 are determined; these quantities for *M*3 are similarly determined from values not given in this table. By the use of these coefficients we deduce from the observed values the values that would have been found had both coils been at 22°00 C and without load—were such a measurement possible. These corrected values are given in the column headed "Corrected ratio," and their agreement is an indication of the accuracy of the measurement; their deviations from the mean average only 1.4 in 1 000 000. The mean of the values in this column after being corrected for the length of the needle, for maladjustment as determined by preliminary observations, and for the effect of the leads as determined just before removing the coils, gives us the most probable value of the ratio of the galvanometer constants of these two coils.

In Table III are given these ratios for the various combinations studied.

TABLE III

Corrected Observed Ratios of Galvanometer Constants

Fixed	M2	M3	M4
S1	.....	1.004652	.....
S2	.....	1.005101	.....
L1	2.247584	1.325326	2.273009
L2	2.247672	1.325371	2.273088
L3	2.247122	1.325054	2.272534
L4	2.246676	1.324790	2.272087

In order to obtain a better insight into the reliability of this work, and to average out the accidental errors as much as possible, it appeared desirable to deduce from these values as many nominally identical ratios as possible, and to use the means of these nominally identical ratios in the computation of the radii. For example, by dividing the numbers in column 2 by those in column 3, we obtain four independent values of the ratio of the galvanometer constant of $M3$ to that of $M2$; similarly, we can get four values of the ratio of the constants of $M3$ to $M4$. By a similar treatment of the rows, we obtain sets of three nominally equal ratios. Thus, we find the values given in Tables IV and V.

TABLE IV

Ratios of the Galvanometer Constants of M3 to those of M2 and of M4 as obtained from observations using coils L1, L2, L3, L4

	L1	L2	L3	L4	Mean
$\frac{M3}{M2}$	1.695872_6	1.695881_3	1.695872_0	1.695873_3	1.695874_8
$\frac{M3}{M4}$	1.715056_6	1.715058_0	1.715050_2	1.715054_6	1.715054_6

TABLE V

Ratios of the Galvanometer Constants of L1, L2, and L3 to L4, obtained from observations with M2, M3, and M4

	M2	M3	M4	Mean
$\frac{L1}{L4}$	1.000404_1	1.000404_6	1.000405_8	1.000404_8
$\frac{L2}{L4}$	1.000443_8	1.000438_6	1.000440_6	1.000440_9
$\frac{L3}{L4}$	1.000198_4	1.000199_2	1.000196_7	1.000198_2

From these tables it appears unlikely that the means are in error by more than about 2 in 1 000 000. This accuracy could not have been obtained had the work not been done at night, and only during steady magnetic periods. Frequently the work was suspended during times of magnetic disturbance.

In order to obtain the radii from these values it is necessary to know the radius of one coil and to know the value of

$$\frac{AG}{2\pi}=n\left\{1-\frac{1}{2}\left(\frac{\alpha}{A}\right)^2+\frac{1}{3}\left(\frac{\rho}{A}\right)^2+\frac{3}{8}\left(\frac{\alpha}{A}\right)^4+\frac{1}{5}\left(\frac{\rho}{A}\right)^4-\frac{\alpha^2\rho^2}{A^4}+\ldots\right\}$$

for every coil.

As already remarked, no knowledge of the absolute value of the radius is required except in the calculation of the corrections for the sectional dimensions of the coils, and, consequently, it is allowable to assume any approximate value for the radius of the coil of reference. We have accordingly assumed that the mean radius of S1 at 22°00 C, and with no load, is exactly equal to 19.97510 cm, the value calculated from the direct measurement of the coil.

The data for the computation of AG is given in the following table.

TABLE VI

Correction to Galvanometer Constant

Coil		Turns	Axial breadth 2α	Radial depth 2ρ	Radius A	$\frac{AG}{2\pi n}$
			cm	cm	cm	
Moving	M1	2×72	0.985_9	0.908_9	9.961	0.9994682
	M2	2×72	0.956_4	0.954_8	12.499	0.9997535
	M3	2×98	0.996_7	1.029_4	10.030	0.9996406
	M4	2×71	1.087_7	1.073_7	12.460	0.9996648
Fixed	S1	2×392	1.580	1.528	19.97	0.9997045
	S2	2×392	1.579	1.522	19.96	0.9997013
	L1	2×648	2.027	2.035	25.03	0.9997299
	L2	2×648	2.027	2.054	25.03	0.9997402
	L3	2×647	1.969	1.943	25.00	0.9997270
	L4	2×647	1.965	1.925	25.00	0.9997209

The correction factor given in the last column would be unity for a coil of zero section.

From these values and the assumed radius of $S1$ we can at once obtain from the first two ratios given in Table III the value of the radii of $M3$ and of $S2$. From this value of $M3$ and the mean ratios of Table IV, we get the radii of $M2$ and of $M4$. From the radii of the three moving coils and the last line of Table III we obtain three values for the radius of $L4$; and from the mean of the latter and the mean ratios of Table V we get the radii of $L1$, $L2$, and $L3$. The three values found for $L4$ were 25.002 47, 25.002 47, and 25.002 48 cm.

These are the most probable values that can be obtained from our observations. These radii, as well as the coefficients of expansion and the load corrections, are given in Table VII.

The radius in cm of a coil at the electrical temperature $t°$ and with a load of w watts is related to the quantities given in this table, as shown (p. 314) by equation

$$A = A_0\{1 + \tau(t - 22.0) - \lambda w\}$$

In the earlier work the importance of the exact parallelism of the coils and the needle was not realized, and the coils were not

sealed until the summer of 1910. Hence, the earlier measurements of the radii are subject to uncertainties. However, the

TABLE VII

Radii and Coefficients

Coil	Radius A_0	Temperature coefficient τ	Lead coefficient λ
	cm		
M2	12.50248	20.3×10^{-6}	2.9×10^{-6}
M3	10.03337	19.7×10^{-6}	8.2×10^{-6}
M4	12.46716	19.7×10^{-6}	10.7×10^{-6}
S1	19.97510	17.0×10^{-6}	0.47×10^{-6}
S2	19.96611	17.6×10^{-6}	0.93×10^{-6}
L1	25.03121	17.8×10^{-6}	0.61×10^{-6}
L2	25.03056	.....	0.75×10^{-6}
L3	24.99767	18.9×10^{-6}	0.81×10^{-6}
L4	25.00247	18.5×10^{-6}	0.84×10^{-6}

measurements were all made in the autumn or winter, when the humidity in the laboratory is low and not subject to great variations, so that the discrepancies between the various measurements should be a minimum. Merely in order to show the variations observed under such conditions, these older values are given in the following table:

TABLE VIII

Comparison of Radii Determinations

Coils	Mechanical [12]	Electrical			
		Oct., 1908	Feb., 1909	Jan., 1910	Nov., 1910, to Jan., 1911
	cm	cm	cm	cm	
M1	9.9684	9.96389	9.96374	9.96362	.....
M2	12.4989	.....	12.50290	12.50250	12.50248
M3	10.0308	10.03342	10.03347	10.03330	10.03337
M4	12.4713	.....	.....	.....	12.46716
S1	19.9751	19.97510	19.97510	19.97510	19.97510
S2	19.9658	19.96636	19.96612	19.96601	19.96611
L1	25.0216	.....	25.03109	25.03019	25.03121
L2	25.0282	.....	25.02971	25.02976	25.03056
L3	25.0030	.....	.....	[13] 24.99851	24.99767
L4	25.0021	.....	.....	[13] 25.00362	25.00247

[12] These measurements were not used in the computation of the constants.
[13] These measurements were made Mar. 29, 1910.

The radius of $S1$ is assumed to remain constant, and the table shows that in spite of the unfavorable conditions just mentioned the greatest range relative to $S1$ is only about 5 in 100 000.

IV. THEORY OF THE BALANCE AND COMPUTATION OF CONSTANTS

19. THEORY OF THE RAYLEIGH BALANCE

The mutual energy of two coils being equal to the mutual inductance of the coils multiplied by the product of the two currents, it is evident that the force, in any direction, which one coil exerts upon the other is equal to the product of the currents multiplied by the rate of change of the mutual inductance, as the second coil moves in the direction specified.

Hence the expression for the mutual force per unit current in each may be obtained by differentiating any of the expressions which may be derived for the mutual inductance.

The derivation of all of these expressions includes two distinct steps, namely, the formation of an expression for the mutual inductance of linear circuits, and the modification of this expression, necessitated by the finite section of the coil.

One of the simplest methods that has been suggested to correct for the sectional dimensions of the coil is the method, described by Lyle [14] in 1902, which makes use of an "equivalent radius." Lyle showed that so far as its external field is concerned, a circular coil of square section is equivalent to a linear circular circuit of radius a_e, where $a_e = a\left(\frac{1+4\alpha^2}{24a^2}\right) = a + \frac{\alpha^2}{6a}$ and a is the mean radius of the coil having a section $2\alpha \times 2\alpha$. This approximation neglects quantities of the order of the fourth and higher powers of the ratios of 2α to the radius of the coil, and of 2α to the distance of the point considered from the plane of the coil. The quantity we have denoted by a_e is called the "equivalent radius" of the square coil.

Similarly it is shown that if the axial breadth of the coil is 2α, its radial depth is 2ρ, and its mean radius a, then if 2α is

[14] Phil. Mag., 3, p. 310; 1902; this Bulletin, 2, pp. 374-378; 1906.

greater than 2ρ the coil is equivalent to two linear circular circuits, each of radius $a_e = a\left(1 + \frac{4\rho^2}{24a^2}\right) = a + \frac{\rho^2}{6a}$, the planes of these circles being a distance 2β apart and symmetrically placed with reference to the mean plane of the coil. The value of β is given by the expression

$$\beta^2 = \frac{4\alpha^2 - 4\rho^2}{12} = \tfrac{1}{3}(\alpha^2 - \rho^2)$$

If 2α is less than 2ρ, the coil is equivalent to two coplanar circular circuits lying in the mean plane of the coil and having radii

$$a'_e = a + \frac{\alpha^2}{6a} + \delta$$

$$a''_e = a + \frac{\alpha^2}{6a} - \delta$$

where

$$\delta^2 = \frac{4\rho^2 - 4\alpha^2}{12} = \tfrac{1}{3}(\rho^2 - \alpha^2)$$

Should α and ρ be so great that this order of approximation is not sufficient, then the coil can be subdivided until the sections of the subdivisions are sufficiently small to give the desired accuracy, and each section be replaced by its equivalent circles. The total current turns in the actual coil must, of course, be equally divided among the various circles into which the coil is thus resolved.

In this manner the computation is reduced to the case of linear circular circuits.

Probably the simplest expression for the mutual inductance of two coaxial circular filaments is its spherical harmonic expansion given by Maxwell.[15] When this is expanded in terms of the linear quantities involved, and differentiated with reference to the distance between the coils, it yields the following expression

[15] Elec. and Mag., §§ 696–699.

for the mutual force of attraction between the circles per unit current in each.[16]

$$F=\frac{\pi^2A^2a^2}{C^4}\left\{1.2.3\frac{B}{C}+2.3.4\frac{(B^2-\frac{1}{4}A^2)b}{C^3}+3.4.5\frac{B(B^2-\frac{3}{4}A^2)(b^2-\frac{1}{4}a^2)}{C^5}\right.$$
$$+4.5.6\frac{(B^4-\frac{3}{8}B^2A^2+\frac{1}{8}A^4)(b^2-\frac{3}{4}a^2)b}{C^7}$$
$$\left.+5.6.7\frac{B(B^4-\frac{4}{3}B^2A^2+\frac{5}{8}A^4)(b^4-{}^3b^2a^2+\frac{1}{8}a^4)}{C^9}+\ldots\right\}$$

where A and a are the radii of the circles, A is assumed to be the larger, B and b are the distances of the centers of these respective circles from some point on their common axis, $C^2=A^2+B^2$.

Owing to the slow convergence of this series, it is impracticable to use it in the computation of the force when high accuracy is desired. However, this series is of great value in the study of the relation of the force to the various linear quantities involved, and in this connection has been carefully studied by Lord Rayleigh in the paper referred to, in which will be found some of the conclusions given below.

From this expansion it is at once evident that F is of zero dimensions in length; that is, it depends solely upon the ratios of the linear quantities involved. Hence, increasing all the dimensions in the same proportion will leave F unchanged.

Again for very small values of b (in the limit for $b=0$), the derivative of F with respect to b becomes

$$\left(\frac{dF}{db}\right)_{b=0}=\frac{\pi^2A^2a^2}{C^4}\left\{2.3.4\frac{B^2-\frac{1}{4}A^2}{C^3}-3.5.6\frac{a^2(B^4-\frac{3}{8}B^2A^2+\frac{1}{8}A^4)}{C^7}+\cdots\right\}$$

This will equal zero, and hence F will be a maximum, for a value of B nearly equal to $0.5A$ (that it is really a maximum can be seen from a consideration of the third term in the expansion for F). For a second approximation give B this value where it occurs in the second term in the above expression, equate to zero, and we find for the new value

$$B^2=\tfrac{1}{4}A^2\left\{1-\tfrac{2}{5}\cdot\frac{a^2}{A^2}\right\}$$
$$B=\tfrac{1}{2}A\left\{1-\tfrac{2}{10}\cdot\frac{a^2}{A^2}\right\}$$

[16] Maxwell, Elec. and Mag., §699. Rayleigh, Phil. Trans., 175, pp. 411–460; 1884; Scientific Papers, 2 pp. 278–332.

That is, if the distance between the planes of the circles be equal to

$$\tfrac{1}{2}A\left\{1-\tfrac{9}{10}\cdot\frac{a^2}{A^2}\right\}$$

the force between them will be very nearly a maximum for variations in B, and consequently will vary but slowly with changes in the value of the latter; thus, the exact measurement of B becomes of relatively slight importance. This is greatly to be desired in practice, as B is the distance between the mean planes of the coils and, consequently, can not be measured directly with high precision.

It is also evident that when B is such that F is a true maximum, then b will enter into the expression only as the square, cube, and higher powers. Of these the square term will be much the most important.

If there are two coaxial circles of radius A, and the origin is taken midway between them, then, from symmetry, the odd powers of b will be absent from the expression which represents the sum of the forces exerted upon the small circle by the two large ones (the currents being such that these forces are in the same direction). This force will be a maximum when the distance from the small circle to each of the large ones is given by the expression found above, and variations of the force from this maximum will be an even function of the displacement (b) of the moving coil from this position. For small displacements, the change in the force is proportional to the square of the displacement.

If the currents in the two large circles are so directed that the forces which they exert on the small circle are opposed, we obtain the resultant force by subtracting the expressions of F for the two circles, and thus find that the difference in the forces is an *odd* function of the displacement of the small circle. If the spacing is that which gives the maximum force, the first term in the series for the difference in the forces involves the cube of the displacement of the small circle. Hence, for small values of this displacement, the difference is nearly independent of the displacement. If the spacing is different from that assumed, the difference in the forces is nearly a linear function of the displacement.

It will also be noticed that when the two large circles are so spaced that

$$B^2 = \tfrac{3}{4}A^2$$

the square term in the expression for the sum of the forces vanishes, and therefore the expression for the total force is reduced to a constant term, and those involving the fourth and higher powers of the displacement of the small circle. Hence, for this adjustment small displacements of the smaller circle produce a minimum effect upon the force.

For this reason this adjustment has been used in some cases, but in our opinion the advantage gained in this respect is, in practice, more than outweighed by the fact that, when not working at the position of maximum force, the distance between the mean planes of the fixed coils must be measured with considerable accuracy. Consequently the coils have been spaced in our work so as to obtain the maximum force.

Any of the other series formulas for the mutual inductance of coaxial circular circuits may be used with more or less facility in the computation of the force, but the only exact expression for the force that has been given is the one in terms of elliptic integrals which was given by Maxwell,[17] namely,

$$F = \frac{\pi B \sin \gamma}{\sqrt{Aa}} \left\{ 2F_\gamma - (1 + \sec^2\gamma) E_\gamma \right\}$$

where A and a are the radii of the two coaxial circles, B is the distance between their planes, F_γ and E_γ are the two complete elliptic integrals of argument γ, and

$$\sin \gamma = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + B^2}}$$

20. METHOD EMPLOYED IN THE CALCULATION OF THE CONSTANTS

Each coil has been replaced by the eight equivalent circles obtained by quartering the coil and replacing each quarter by its two equivalent circles (as defined by Lyle) in the manner already described. The force between the various circles was calculated by means of the elliptic integral expression just given.

[17] Electricity and Magnetism, 2, § 701.

This computation is greatly facilitated by the tabulation of log $[\sin \gamma \{2F_r - (1 + \sec^2\gamma)E_r\}]$ for values of γ progressing by equal steps. Such a table has been constructed by Lord Rayleigh[18] using seven-place logarithms. Owing, however, to the accumulation of small errors, many of the values in that table are in error by several units in the seventh place of the logarithms. In a few cases it amounts to 10 units, giving an error of about 3 in a million in the result. While this is a small amount, it appeared desirable, especially in the study of the effect of the subdivision of the coils and in the calculations for the determination of the distance for maximum force, to attain a higher accuracy in the computation; consequently, the table has been recalculated, using Legendre's tables of elliptic integrals and Vega's ten-place table of logarithms. It is given in the appendix to this paper. We believe that the table contains no errors as great as 1 unit in the last place. In order to attain an accuracy of 1 in 1 000 000 in the force, it is necessary to know γ to about 0°.000 008, but a unit in the seventh place of the logarithm of the sine of γ corresponds to about three times this change in angle; hence, in all final computations we have calculated the sine of γ either by means of Vega's ten-place table or by a calculating machine. In both cases ten-place tables were used in passing from sin γ to γ.

21. DISTANCE FOR MAXIMUM FORCE

Before proceeding to the computation of the constants of the balance it is necessary to know the distance between the coils which will give the maximum force, the latter being the quantity directly measured. As seen above, an approximate value of this distance is given by the relation

$$B'_m = \tfrac{1}{2}A\left\{1 - \tfrac{9}{16}\cdot\frac{a^2}{A^2}\right\}$$

In order to obtain a more accurate value for B_m, and at the same time to obtain an accurate expression for the variation of the force with the distance for values differing from B_m by several

[18] Phil. Trans., 175, pp. 411–460; 1884; Scientific Papers, 2, p. 327.

millimeters, calculations of the force were made for various values of B lying on each side of the approximate value B'_m given by the above expression. From these values the distance B_m for maximum force was determined, and the coefficients in the equations representing (1) the variation of the force with the distance $(x-B_m)$, (2) the variations in the magnitude of the maximum force with variations in A, the radius of the moving coil remaining unchanged, and (3) the variations in the distance for maximum force with variations in A, a being constant as before.

In this computation two assumptions have been made for the sake of simplicity. First, that the coils are equivalent in their action to coils of square cross section, and of the same mean radii and same sectional area as the actual coils; second, that any such square coil produces the same effect as its "equivalent" circular current, as defined by Lyle. These assumptions are equivalent to the single assumption that each coil may be regarded as a linear circular current lying in the mean axial plane of the coil, and having a radius (A_e) defined by the equation

$$A_e = A + \frac{\alpha\rho}{6A}$$

where 2α and 2ρ are the sectional dimensions of the coil, and A is its mean radius.

These computations have been performed for each moving coil, and the largest and the smallest of the large fixed coils, and for $M3$ and each of the small fixed coils. The agreement of the coefficients in the variation formula furnishes a good check on the work. The values of B_m for the intermediate values of the large fixed coils were determined by interpolation.

TABLE IX

Equivalent Radii and Distances for Maximum Force

Coil	A_e	Coil	A_e	Coils	B_m	Coils	B_m	Coils	B_m
M2	12.50552	L1	25.03808	M2, L1	9.6092	M3, S2	7.6344	M4, L1	9.6256
M3	10.03763	L2	25.03749	M2, L2	9.6089	M3, L1	10.6673	M4, L2	9.6253
M4	12.47106	L3	25.00405	M2, L3	9.5882	M3, L2	10.6669	M4, L3	9.6047
S1	19.98014	L4	25.00877	M2, L4	9.5912	M3, L3	10.6476	M4, L4	9.6077
S2	19.97113			M3, S1	7.6400	M3, L4	10.6502		

The equivalent radii A_e used, and the values found for the distances for maximum force B_m, are given in Table IX.

The values of the coefficients in the following variation formulæ are given in Table X:

$$F = F_m\{1 - \gamma(x - B_m)^2 + \delta(x - B_m)^3\}$$

$$\frac{\Delta F}{F} = -\epsilon\frac{\Delta A}{A}, \qquad a \text{ constant}$$

$$\frac{\Delta B_m}{B_m} = \eta\frac{\Delta A}{A}, \qquad a \text{ constant}$$

TABLE X

Variation Coefficients

Radii	1000γ	1000δ	ϵ	η
50 cm, 25 cm	7.523 cm^{-2}	0.566 cm^{-3}	2.555	1.62
50 cm, 20 cm	6.433 cm^{-2}	0.407 cm^{-3}	2.317	1.35
40 cm, 20 cm	11.874 cm^{-2}	1.101 cm^{-3}	2.555	1.62

The value of B_m can also be obtained from the fact that it must be such a distance that each coil will exert the same force upon each of the two flat annular current sheets, which form the upper and the lower boundaries, respectively, of the other coil. The necessary computation may be readily made by replacing one coil by its equivalent turn, and each of the two current sheets by their two equivalent turns, as defined by Lyle. This has been done in one case, as a check upon the other work, and yielded a result differing but 2 microns from the value as found by the other method.

22. COMPUTATION OF FORCE

Having found the value for the distance for maximum force, and having found by a preliminary computation that ample accuracy is obtained by assuming that the fixed coil is equivalent to the eight turns obtained by quartering it and replacing each quarter by its two equivalent turns,[19] and that the moving coil is likewise equivalent to the eight turns obtained by replacing each quarter by two equivalent turns,[20] the computation was made in the following two ways:

[19] Although the sections of the coils were nearly square, they departed too much from a square to permit replacing each quarter by a single turn.

[20] It would probably have been sufficiently accurate in most cases to replace the moving coil by two turns without quartering.

(1) Instead of making 64 computations (for the force of each of the 8 equivalent turns of the fixed coil upon the 8 turns of the moving coil) for each pair of coils used, it is simpler to replace each coil by a single equivalent turn on the assumption that it has a square section and calculate the force F_1 for each combination of coils used. Then with any moving coil replaced by a single turn as before, we may calculate for each fixed coil used with the moving coil, the force F_f obtained by replacing the fixed coil by 8 turns. The percentage difference between F_1 and F_f for any coil—that is, the correction for its sectional dimensions—will be the same for every moving coil having approximately the same radius as that used in the computation of F_f. Hence, this correction need not be redetermined for other moving coils of the same size.

Having thus determined the sectional correction for each fixed coil for each group of moving coils with which it is to be used, we in a similar manner determine the sectional corrections for each moving coil. Calling F_m the force obtained when the fixed coil is replaced by a single turn and the moving coil is quartered, we have for the corrected force, both coils subdivided, the expression

$$F = F_1(1 + \delta_f + \delta_m)$$

where $\delta_f = \frac{F_f - F_1}{F_1}$, and is determined by calculation with the special fixed coil under discussion but with any moving coil having approximately the same radius as that concerned in F; and $\delta_m = \frac{F_m - F_1}{F_1}$ and is determined for the special moving coil concerned in F, but for any fixed coil of approximately the same radius as that used in F.

(2) Or, after calculating the value of F_1 as defined above, we may proceed by the method just described to the determination of the effect of the area of section for a coil of approximately the same radius and of strictly square cross section of approximately the same sectional area as the actual coils. This effect will be very small, and over a considerable range will be independent of the radius of the coil. Then, by means of the variation formula given below (and derived in the appendix, p. 376, where this method

is considered more in detail), the correction for the departure of the actual section from an exact square may be determined and applied for each coil. This method greatly lightens the labor of computation, but was not used in the present work except as a check, because of a failure to realize earlier the range of application and the accuracy of the method. A thorough test of the method has been made, and in the future it can be used with all confidence.

Using the values of the mean radii as given in Table VII, and the sectional dimensions as given in Table VI, and the distances for maximum force as just determined, we find the values (F_1) for the forces in dynes per C. G. S. current turn, as given in Table XI. In the column headed "I for 1.5 g" in this table are given the currents in absolute amperes which, when passed in series through the given pairs of coils, will give in each case a force corresponding to 1.5 g at a place where the acceleration of gravity is 980.091 cm per second per second, the windings of each coil being in parallel and the radii the same as if the coils were at 22°.00 C and carried no load

TABLE XI

Constants of the Balance

Coils	F_1	$\delta_f \times 10^{+6}$	$\delta_m \times 10^{+6}$	F	I (in absolute amperes) for 1.5 g
M2, L3	5.355834	− 48.1	− 3.4	5.355558	0.767641_0
M2, L4	5.353251	− 80.9	− 3.4	5.352799	0.767838_7
M3, S1	5.417298	−136.8	+100.2	5.417100	0.840503_1
M3, S2	5.423552	−149.9	+100.2	5.423282	0.840023_9
M3, L1	3.137781	+ 16.6	+ 69.6	3.138051	0.858911_1
M3, L2	3.137953	+ 50.7	+ 69.6	3.138331	0.858872_6
M3, L3	3.147665	− 44.5	+ 69.6	3.147744	0.858250_4
M3, L4	3.146292	− 70.9	+ 69.6	3.146282	0.858448_9
M4, L3	5.318310	− 48.1	− 28.0	5.317906	0.775759_7
M4, L4	5.315754	− 80.9	− 28.0	5.315175	0.775959_2

23. EFFECT OF ERRORS IN SECTIONAL DIMENSIONS

In the section dealing with the determination of the ratio of the radii, we have considered how an error in the sectional dimensions affects the radius as computed from the observed ratio of the galvanometer constants. This per cent error in the radius multi-

plied by the factor ϵ (Table X) gives the per cent error, thus introduced into the force.

But having assumed values for the radii, the sectional dimensions again enter in the computation of the force, and so an error in them produces a second error in the force. The most direct estimation of this error is obtained by a consideration of the spherical harmonic expansion of the force between two coils of finite section. This expression may be obtained from that for two linear circular circuits either by direct integration, or by the use of Taylor's theorem as was done by Maxwell (Elec. and Mag., Vol. II, §700), and is of the form

$$F_1 = F_0 + \lambda'_1 \frac{\rho_1^2}{A^2} + \lambda'_2 \frac{\alpha_1^2 + \alpha_2^2}{a^2} + \lambda'_3 \frac{\rho_2^2}{a^2}$$

Whence, by Maclaurin's theorem, the value F of the force given by $\rho = \rho_1 + \delta\rho_1$, etc., is given by the expression

$$\frac{\Delta F}{F} = \frac{F - F_1}{F_1} = \lambda_1 \frac{\rho_1^2}{A^2} \cdot \frac{\delta\rho_1}{\rho_1} + \lambda_2 \frac{\alpha_1^2}{a^2} \cdot \frac{\delta\alpha_1}{\alpha_1} + \lambda_3 \frac{\rho_2^2}{a^2} \cdot \frac{\delta\rho_2}{\rho_2} + \lambda_2 \frac{\alpha_2^2}{a^2} \cdot \frac{\delta\alpha_2}{\alpha_2}$$

While it is impracticable to calculate the coefficients with sufficient accuracy to enable us to determine F_1 from F_0 and a knowledge of the sectional dimensions of the coils, they can be readily determined with ample accuracy for use in the variational equation just given. In the sectional dimensions, the subscript *1* refers to the fixed coil and the subscript *2* to the movable one. The expressions for the coefficients in terms of the radii of the coils and of their distance apart are given in the appendix, page 378, and when evaluated yield the numbers given in Table XII.

TABLE XII

Variation of Force with Sectional Dimensions (Radii Constant)

$\frac{A}{a}$	λ_1	λ_2	λ_3
2	+2.523	−0.8478	+1.704
2.5	+1.954	−0.4361	+1.201

Denoting the errors thus introduced by the letter C, and those introduced by the error in the determination of the radius from the galvanometer constant by the letter G, we find that an error of 0.01 mm in the axial breadth 2α or in the radial depth 2ρ produces the errors in the force given in Table XIII.

TABLE XIII

Effect of Errors in the Sectional Dimensions

Radii		Dimensions of square section		δF in parts per million for a variation of 0.001 cm in the radial depth or axial breadth											
				Fixed coil						Moving coil					
Fixed	Moving	Fixed	Moving	For variation in depth			For variation in breadth			For variation in depth			For variation in breadth		
				G	C	Sum	G	C	Sum	G	C	Sum	G	C	Sum
cm	cm	cm	cm												
20	10	1.58	1.0	−1.68	+2.50	+0.82	+2.53	−3.35	−0.82	+4.26	+4.26	+8.52	−6.39	−2.12	−8.51
25	12.5	2.00	1.0	−1.36	+2.02	+0.66	+2.04	−2.71	−0.67	+2.72	+2.73	+5.45	−4.09	−1.36	−5.45
25	10	2.00	1.0	−1.23	+1.56	+0.33	+1.85	−2.18	−0.33	+3.86	+3.02	+6.88	−5.79	−1.09	−6.88

From Table XIII it appears that while the two errors nearly balance one another in the case of the fixed coils, they add together in the case of the moving coils, and even for the small assumed error of 0.01 mm in 2α or 2ρ may amount to over 8 in a million in the force, or 4 in a million in the current. This is about the magnitude of the uncertainty in the present work, due to errors in measuring the cross sections of the coils. If the cross sections were larger, this error would be increased, and hence the cross section should be kept as small as practicable.

V. THE MEASUREMENT OF CURRENT

24. GENERAL DESCRIPTION

As already stated, the current which passed through the coils of the current balance also passed through a standard resistance, and, by means of a calibrated potentiometer, the drop of poten-

tial between the terminals of this resistance was measured in terms of a Weston normal cell.

Knowing the electromotive force of this cell in international volts, as defined by that value for the mean Weston normal cell (1.0183, at 20°.00), which was adopted by the international committee to take effect after January 1, 1911, and knowing the resistance in international ohms, the value of the current in international amperes (1911) is known. The term "international ampere (1911)" is used in this paper to denote that current which, when passed through a resistance of 1 international ohm, will produce between the extremities of the latter a potential difference of 1 international volt, as defined by the value adopted for the mean Weston normal cell.

This current, measured in *international amperes*, as just described, is measured in *absolute amperes* by means of the current balance. To find the ratio of these two numerical values of the same current is the object of this investigation.

The same result may be stated in a different form by saying that the quotient of this ratio into the electromotive force of the mean Weston normal cell at 20°.00 C (by definition 1.0183 international volts) is equal to the electromotive force of this mean cell in terms of the absolute ampere and the international ohm. The unit of electromotive force as thus expressed depends both upon the absolute and the international system, and consequently is called, in this paper, a "semiabsolute volt." In this investigation we have found that the electromotive force of the mean Weston normal cell at 20°.00 C is 1.018 22 semiabsolute volts.

If the silver voltameter were as convenient an instrument as the standard cell and were defined by complete official specifications, a better method would be to state the result in terms of the mass of silver deposited in a silver voltameter per second by the absolute ampere, as Rayleigh did thirty years ago. But the measurement of current with high precision by the silver voltameter is still an investigation rather than an operation, and the complete specifications are as yet undefined; consequently, for precise measurements in practice, the standard cell and resistance are always employed. We have, however, in section 38 given our final result in terms of the electrochemical equivalent of silver as

found by a large number of observations with two types of voltameters at the Bureau of Standards.

In the actual work the potentiometer was connected to measure directly the drop of potential across the standard resistance, or to measure the small difference between this drop in potential and the electromotive force of the cell. In the first case a sensibility of about 8 in 1 000 000 for 1 mm, and in the second of about 3 in 1 000 000 for 1 mm deflection, was secured. The scale distance was 270 cm. By means of a continuously variable resistance, the deflection could be kept, as a rule, within a quarter of a millimeter during the time necessary for the determination of the rest point of the balance, and frequently it could be kept apparently zero for this length of time. Hence, there is no appreciable error in the work due to variation of the current during measurement.

25. MANIPULATION OF THE BALANCE

In the manipulation of the balance three points, in addition to the control of the current, have to be considered.

First.—Slight changes in the zero of the balance may be produced by arresting it. Consequently, it is better to change the weight and to reverse the current without arresting the balance. By means of a suitable arrangement for manipulating the weight without opening the balance case, the circular reversing switch already described, and an air jet (produced by squeezing a rubber bulb) for deflecting the balance beam, it is possible with practice to make the reversal of the current and the change in weight simultaneous, so that the balance receives no bump or jar in the process. This procedure has been followed throughout this work.

Second.—Though the change in the weight does not jar the balance, it does usually start the pan and the coil to swinging slightly. This vibration is stopped by gently touching with a camel's-hair brush the tube by which the coil is suspended from the pan. This is done in the space between the balance case and the coil case.

Third.—Owing to slight changes in temperature or other causes, the zero of the balance is usually continually drifting. Also, the earth's field exerts a slight force upon the current in the moving coil. Both of these effects may be eliminated at the same time if a series of weighings are made with the weight alternately on

and off, the current in the moving coil being always in the same direction and that in the fixed coils being alternately direct and reverse. As stated in an earlier portion of this paper, the current is always adjusted to such a value that these simultaneous changes in the weight and the direction of current produce but a slight change in the rest point of the balance.

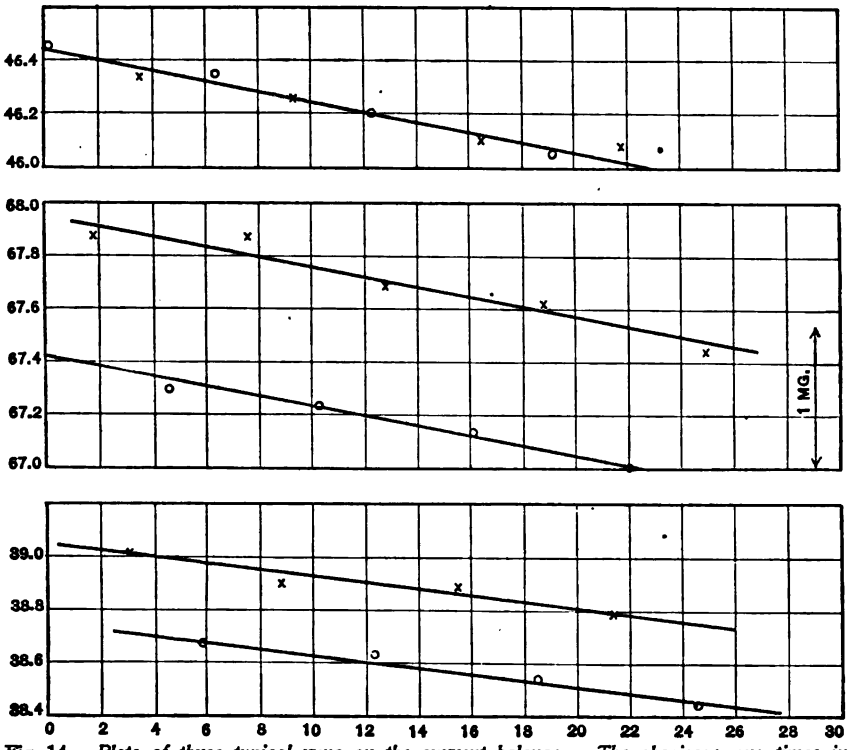

Fig. 14.—*Plots of three typical runs on the current balance. The abscissae are times in minutes; the ordinates are doubled scale readings in centimeters*

By this procedure the effect of the earth's field is reduced to a very slight permanent displacement of the zero of the balance. The difference between the change in the weight and twice the force exerted by the fixed coils on the moving coil is directly given by the difference in the rest points of the balance.

To eliminate the drift, the time at which each rest point is taken is noted, and the rest points are plotted as functions of the time. Under favorable conditions they will lie upon two parallel

straight lines, generally slightly inclined to the time axis. The distance between these lines measured perpendicularly to the time axis is proportional to the amount by which twice the force due to the current differs from the gravitational force exerted upon the weight used. A set of four pairs of observations can be obtained in about 30 minutes. Three such "runs" are shown in Fig. 14.

For two reasons this graphical treatment of the observations is considered superior to the mere numerical averaging, as is usual in the determination of the rest point of a balance. First, the observations can not be taken at strictly equal intervals of time; and, second, irregular behavior due to imperfect current control, swinging of the coil, variations in the air currents, change in the rate of drift, etc., is detected much more readily when the results are plotted; and single erratic points can be readily given reduced weight.

As already stated, the deflection of the balance beam is read by a mirror and telescope and scale, at a scale distance of 250 cm. Each resting point is determined from at least five turning points. The amplitude of oscillation is usually such that the displacement of the moving coil from its position of rest does not exceed about 0.2 mm.

The weights used were of platinum, and the correction for the buoyancy of the air was taken as 56 parts per million.

26. ADJUSTMENT OF THE COILS

Having carefully leveled and clamped the fixed coils in position, as already described, the moving coil is adjusted to approximately the proper height by means of the nut screwing on the suspending tube and resting upon the pan of the balance. Then, by means of the long screws by which the coil is connected to the suspending tripod, it is leveled and adjusted (by direct measurement), as near as may be, midway between the fixed coils. These adjustments are tested by means of a surface gauge resting on the surface of the lower coil, and moved from point to point. While testing this adjustment, the balance pan should be hanging freely on its knife-edge, and the beam should be just locked.

Then a star, which just fits inside the forms of the fixed coils and which has a hole in its center through which passes a conically pointed brass rod, is placed in the lower coil. The star is held in place by flanges resting on the surface of the coil, and when so placed the rod in its center coincides with the axis of the coil. While the balance pans are freely suspended from their knife-edges, this rod is pushed up until its tip passes into the 3-mm hole in the center of the star by which the moving coil is suspended. If the tip of the rod is not central to the hole, the balance is properly shifted until the observer is no longer able to decide which side of the hole is first touched by the conical tip as the rod is raised. When this condition is attained, the center of the moving coil must lie very nearly upon the axis of the lower coil.

Since, when properly adjusted, the force for a given current is a minimum for lateral and a maximum for vertical displacements of the moving coil, it is very easy to determine the proper adjustment from a study of the variations in the force as the coil is slightly displaced in different directions. This was always done.

For each of the three mutually perpendicular adjustments weighings were made with the moving coil in at least three positions, of which at least two were approximately equidistant from and on opposite sides of the position corresponding to the maximum (or minimum) force. These forces reduced, if necessary, to the same current, were plotted, on a suitable scale, in terms of the position of the coil; the theoretical curve, connecting the force with the displacement of the moving coil, was plotted on the same scale and fitted to the observations, as well as possible. The position, thus determined, of the vertex of the curve is the proper setting for the moving coil. Four of these curves are shown in Figs. 15 and 16. The two curves in Fig. 15 are drawn to the same scale; A applies to the small moving and the small fixed coils, B to the same moving coil and the large fixed coils. In Fig. 16, C is drawn to the same scale as A and B, and applies to large moving coil and the large fixed coils. For D the scale of the abscissas is reduced by the factor $\sqrt{2}$; otherwise D is the same as C, but

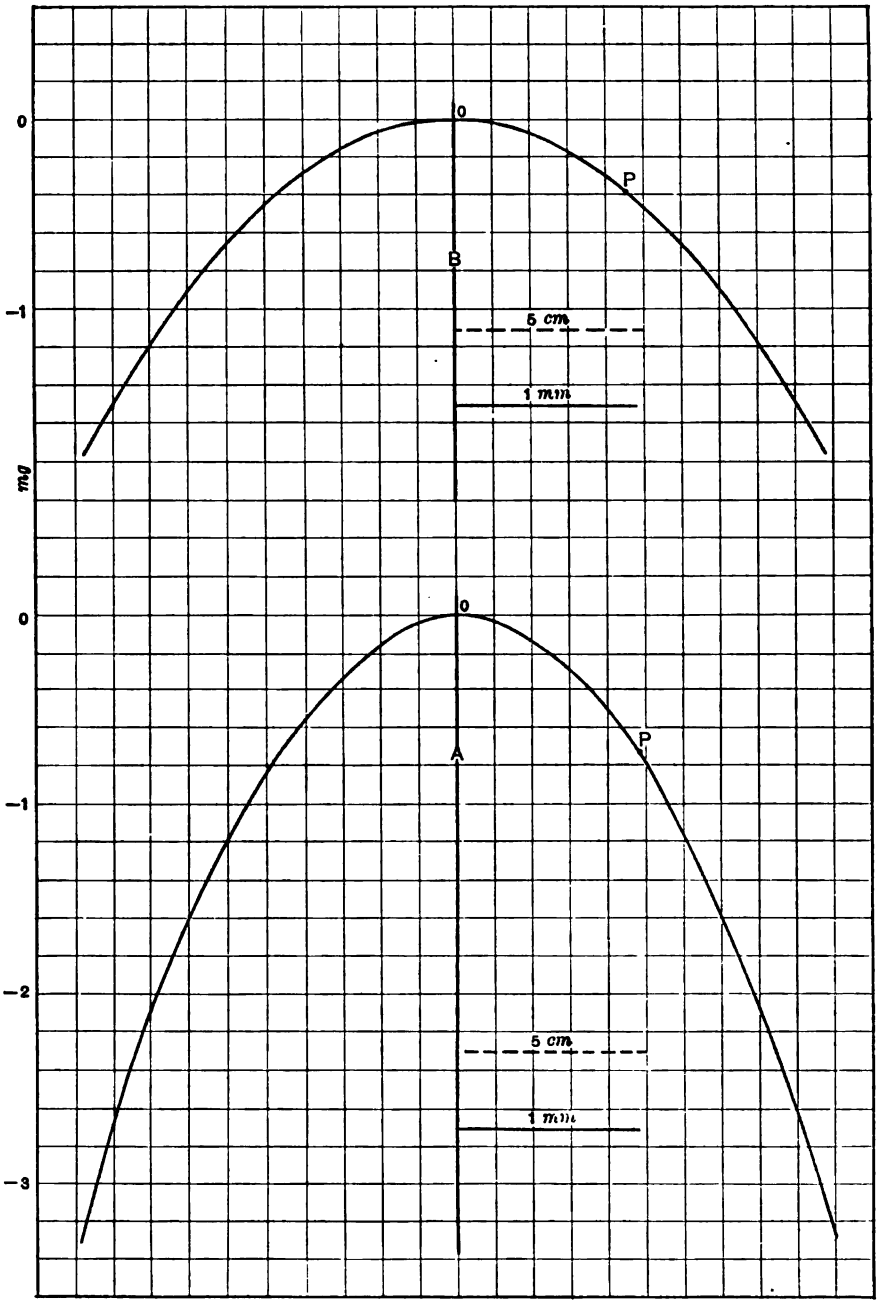

Fig 15.—*Vertical displacement curves*

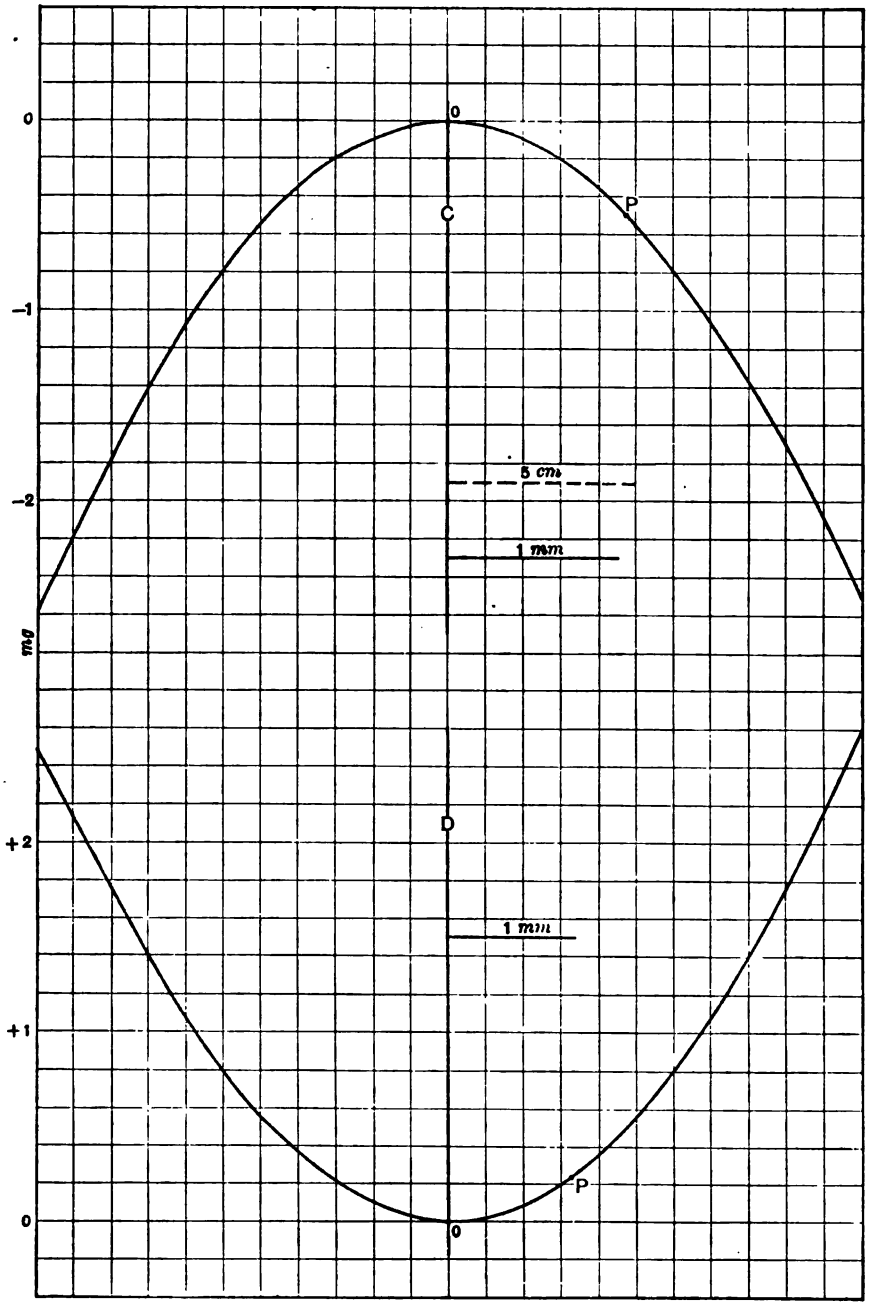

Fig. 16.—*Vertical and horizontal displacement curves*

inverted.[21] This change of scale is made so that the same theoretical curve can be used for the observations on lateral as on vertical displacements. The point P gives the force when the moving coil is displaced 1 mm from the maximum. This corresponds to a change of about 5 cm on the scale, which is used to read the rest points of the balance.

As there appeared to be no reasonable chance of the balance being accidentally displaced horizontally after it was once adjusted and the screws locked, and as numerous observations during the preliminary work had failed to disclose any such shift, this horizontal adjustment was in the final observations usually made but once for each combination of coils. On the other hand, a displacement of the reading telescope might occur, and this would change the reading for the correct vertical position of the coil. Consequently, the correct vertical position of the coil was determined daily; seldom, however, was any accidental change observed.

27. INSULATION OF THE COILS AND ELECTROSTATIC EFFECT

Each time the coils were adjusted, their insulation resistances were measured just before proceeding to the electrical adjustment of the moving coil. To do this each coil, taken one at a time, had one terminal of each winding disconnected from the leads. Then the insulation resistance between the two windings was tested by a direct deflection method, using a well insulated portable battery of 40 volts. The sensibility was such that 1 mm deflection corresponded to 25 000 megohms.

When the circuit connecting the balance with the apparatus on the table was opened by disconnecting one coil from the commu-

[21] From considerations of symmetry and the fact that the force satisfies Laplace's equation, we have the expression

$$F = F_0 + k\,\delta^2 P_2(\theta) + \textit{terms in fourth and higher powers of } \delta$$

connecting the force F_0 acting on the moving coil when symmetrically placed with reference to the two fixed coils and the force F when the coil is displaced from this position by the amount δ, along a line inclined at an angle θ with respect to the common axis of the fixed coils. $P_2(\theta)$ is the second zonal harmonic. Hence, denoting the change in F for an axial displacement by dF_a and the change for a radial displacement by dF_r, we have (since $P_2(\theta) = 1$ when $\theta = 0°$ and is equal to $-\frac{1}{2}$ for $\theta = 90°$)

$$dF_a = k\delta^2_a$$

$$dF_r = -\tfrac{1}{2}k\delta^2_r = -k\left(\frac{\delta_r}{\sqrt{2}}\right)^2$$

which shows that, in order to use the same curve (only inverted) for both the axial and the radial displacements, the scale of abscissas must be changed by the factor $\sqrt{2}$.

tator, its resistance was never low enough to give a deflection exceeding 3 mm for 40 volts.

The large coils showed considerable absorption, the deflection on closing the circuit (after the initial charge due to the electrostatic capacity) being large, this deflection rapidly decreased, and after a minute seldom exceeded 10 cm. In the later measurements it but once amounted to so much as 25 cm; that is, the insulation resistance was but once so low as 100 megohms. The capacity and the leakage between the windings and the form on which they were wound were somewhat greater than those between the windings, but seldom did the insulation resistance fall appreciably below 100 megohms. The capacity effect for the moving coils was always much smaller than for the fixed coils, and the insulation resistance was considerably higher. It thus appears that the insulation of the coils and connections throughout was so high that there was no error due to leakage.

To obviate any electrostatic force upon the moving coil, its windings were at all times directly connected to the water jacket by means of a wire running from the commutator. Thus the water jacket and all of the surrounding framework were brought to the potential of the moving coil. A number of tests showed that no appreciable current flowed through this wire.

28. TEMPERATURES

In the earlier work, the measurements of the temperatures of the coils were somewhat uncertain; the temperatures of the fixed coils were determined by thermometers attached to the forms, and that of the moving coil was inferred from the temperature of the surrounding water jacket, a preliminary experiment being made to determine the difference between them under working conditions.

In the later work—that is, after the summer of 1910—all temperatures are based upon the electrical resistances of the coils as described in the section on the ratio of the radii. The resistances at a known temperature near 22°.0 C, and with no load, were measured when the temperature conditions were steady, usually in the morning. It was done by a potentiometer method, using a low current (0.1 ampere) and keeping it on for as short a time as

possible. In the case of the fixed coils, the measurements were made at the proper terminals of the mercury commutator attached to the end of the coil case. For the moving coil, a pair of potential leads extends down the suspending tube and connects with the vertical current leads, where the latter join the horizontal leads extending across the coil.

These measurements, together with the temperatures of the coils as given by the thermometers attached to the fixed coils, and inserted in the water jacket surrounding the moving coil, enable us to calculate the electrical temperature of the coils (as defined in the section on the ratio of the radii) from any future measurement of the resistance, provided the lead connections have not been changed in the interval. The resistance with no load was measured on at least two days for each combination of coils used; these observations were always quite concordant.

No weighings were taken until the working current had been on the circuit, and water at the desired temperature had been circulating through the system for an hour or an hour and a half. By this time temperature equilibrium has been about established, the drift of the balance has become slight, and the resistances of the coils have become nearly constant.

During the weighings, the resistances of the coils were measured several times a day, and the thermometers in the oil baths containing the standard resistance and the cells, those attached to the fixed coils and those in the water jacket, and the coil case, were read after nearly every run. The latter observations were for the purpose of checking the constancy of the conditions, and were not used in the actual reduction of the results.

The temperature of the oil in the cell bath, and of the water circulating through the coils and water jacket, were thermostatically controlled. Both were well stirred by propellers driven by electric motors.

The oil in the bath containing the standard resistance was stirred by a propeller pump driven by a small electric motor. The excess in temperature of the wire of the resistance above that of the oil was determined experimentally to be $0^{\circ}.56$ C per watt under working conditions, and the proper correction has been applied.

29. SUM AND DIFFERENCE OF FORCES

By means of the mercury commutator attached to the end of the coil case, we can readily connect the fixed coils either so that the forces which they exert upon the moving coil are in the same direction, and thus can measure their sum; or by reversing either of the fixed coils we can oppose the forces and so measure their difference. By measuring both the sum and the difference we can readily determine the force exerted by each of the two coils.

The measurement of the difference in the forces, combined with the effect of interchanging the upper and lower coils, gives a check on the accuracy in the computation of the two constants, and a splendid check on the presence of any unsymmetrical disturbance due to fixed magnetic masses. Such a disturbance will cause the force exerted by the lower coil to be too small or too great always by the same amount.

As we shall see in a later section, measurements of the difference in the forces of the two coils also furnish valuable information as to the magnitude and effect of certain maladjustments.

30. MAGNETIC TESTS BY BALANCE

As regards their effect upon the measurement of an electric current by the balance, magnetic bodies other than the coils themselves may be divided into two classes.

The first class includes those attached to the swinging portions of the balance, beam, pointer, pans, etc. These produce their effect solely in virtue of the forces which the fixed coils exert upon them.

The second class includes all other bodies; what we may call the fixed masses. Their effect arises from the forces which they exert upon the moving coil, and upon bodies of the first class. When observations are taken in the manner we have described, the effect of these bodies upon the final result depends solely upon that portion of their magnetization, which is reversed when the current through the fixed coils is reversed.

The effect of bodies of the first class, in so far as it is independent of the current in the moving coil, can be determined by observing the deflection of the balance when a current passes through the fixed coils only. It is, in fact, determined along

with the leads of the moving coil, and is separated from the latter by making two sets of observations, differing only in the direction of the current through the leads of the moving coil. This was always done and the proper correction, though seldom amounting to 0.01 mg, has always been applied. It seems scarcely probable that the value of this correction can be appreciably changed by the presence of a current in the moving coil.

As we have seen, measurements upon the difference in the forces exerted by the two fixed coils will serve to detect the presence of bodies of the second class, except in so far as their effect is symmetrical with reference to the two fixed coils. Indeed, the presence of unsymmetrically distributed fixed magnetic masses (especially above and below the coils) affects the difference in the forces to a much greater extent than it does the sum; for, in the latter case, the axial fields of the fixed coils are opposed, and so the bodies are but slightly magnetized, while in the former the fields are added and, consequently, the bodies are much more strongly magnetized. This magnification of the disturbance increases the value of this method for detecting the presence of unsuspected magnetic material and of testing the effect of known magnetic bodies.

The chance of magnetic material being symmetrically distributed is very small, and such a possibility can be tested by taking observations with the coils in two different vertical positions, everything else being the same.

All these tests have been applied with satisfactory results.

In order to form an idea of the magnitude of the possible effect of the reenforcing in the concrete construction supporting the floor, weighings were made with a number of iron rods aggregating about 75 kg placed under the coils and at various distances below them. Both the sum and the difference in the forces were measured. The results are given in Table XIV.

These observations were made with moving coil $M3$ and the large fixed coils $L3$ and $L4$, $L3$ being the upper coil; the current used was such that the doubled force was 6 g; the distance between the mean planes of the fixed coils was about 21.3 cm.

From these observations it is evident, as would be anticipated, that the effect of iron below the coils is to decrease the force exerted by the upper coil and to increase that exerted by the lower

coil. Being nearer the lower coil, the increase in the latter's force exceeds the decrease in the former's, the difference being the resultant effect upon the sum of the forces. As the distance of the iron from the coils increases the latter becomes very small. In the present case the effect of this large mass of iron upon the sum of the forces is imperceptible at 116 cm. On the other hand, at this distance it affects the difference in the forces by 0.10 mg; in this position the effect on the difference is about 10 times that on the sum.

TABLE XIV

The Increase which 75 kg of Iron at Different Distances below the Coils produces in the Double Force

Distance of iron below moving coil	Increase produced by iron			
	Sum L3+L4	Difference L3−L4	Force by L3	Force by L4
cm	mg	mg	mg	mg
116	+0.01	− 0.10	−0.05	+ 0.05
109	.....	− 0.14	.....	.....
100	+0.12	− 0.22	−0.05	+ 0.17
86	+0.22	− 0.74	−0.26	+ 0.48
70	+1.00	− 2.79	−0.90	+ 1.90
50	+8.20	−17.76	−4.78	+12.98

The iron reenforcing in the concrete floor is about 140 cm below the moving coil. There is about 13 kg of iron per meter of each girder and 2 kg per square meter between girders. The girders are about 1.1 m apart. Containing so much less iron than we used in the test and lying at a greater distance from the coils, it would appear that the reenforcing can produce no perceptible effect upon the sum of the forces exerted by the two coils, though it may produce a slight effect upon their difference.

Weighings made with pieces of iron and steel, that were small, but distinctly larger than the knife-edges of the balance, and that were placed in various positions in the balance case, and between the latter and the coil case, proved conclusively that the magnetic susceptibility of the knife-edges was without effect upon the observed force.

In order to obtain an idea of the effect that might be produced by the slight magnetic susceptibility of the forms on which the

fixed coils were wound, weighings were made with the fixed coil forms covered with strips of cloth to which iron filings had been stuck by means of paraffin. Though the cloth was thickly powdered, the effect on the force was very small. The best criterion, however, seems to be the agreement between the results obtained with the various combinations of coils. Although the magnetic susceptibility of all the coils is extremely slight, $L1$ has nearly 100 times the susceptibility of $L3$ and $L4$, and yet the results obtained agree very closely. This would indicate that there can be no error due to this cause.

In order to see if the wire used in winding the coils is sufficiently nonmagnetic, measurements of the difference in the forces of the two fixed coils were made both with and without a marble form wound with about 7 kg of the same wire introduced into the lower coil. There was no observable difference in the two cases.

31. CORRECTIONS TO THE OBSERVED FORCE

(*a*) *For the leads.*—The portion of the observed force which was contributed by the leads was determined experimentally in the manner already described and the proper correction was always applied.

(*b*) *For spacing and lack of coaxiality of the fixed coils.*—The constant of the balance has been computed for the moving coil in the position of maximum force. But, unless the distance between the fixed coils is exactly right, this condition can not be simultaneously satisfied for both of the fixed coils. Hence, it becomes necessary to consider the relation that exists between the maximum observed force and that which would have been observed had the distance between the fixed coils been exactly that assumed in the computation. In practice the distance between the two fixed coils was adjusted so as to make the two maxima coincide as accurately as possible. The distance apart of these two maximum points was generally not more than a few tenths of a millimeter. The correction for this displacement was determined in every case. A mathematical discussion of this subject is given in the appendix to this paper, page 380. The experimental method of determining this correction is as follows:

The two fixed coils used together were always very nearly identical. In this case, the distance for maximum force, the value

of the maximum force, and the variation in the force for small displacements of the moving coil from the position of maximum are very nearly the same for both fixed coils.

Consequently, the maximum observed force is obtained when the moving coil is at C, Fig. 17, midway between the maxima A and B of the two coils respectively, for a slight displacement from this position will cause the force exerted by one fixed coil to increase by the same amount as that exerted by the other

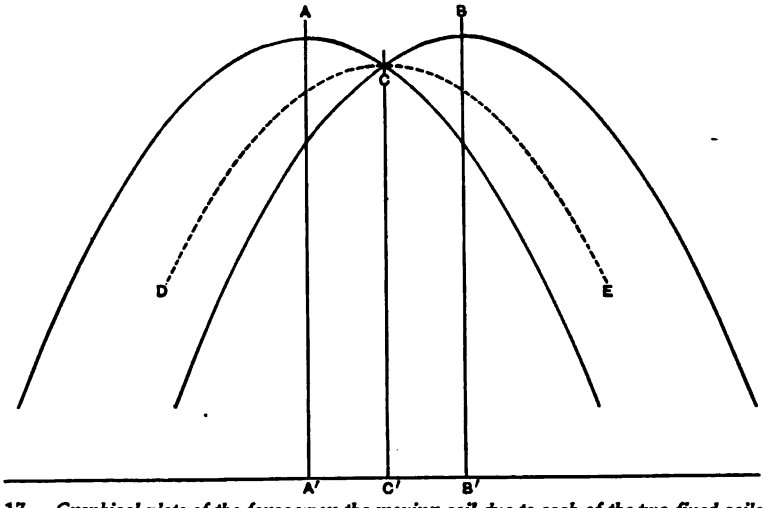

Fig. 17.—*Graphical plots of the force upon the moving coil due to each of the two fixed coils, and of one-half the sum of their forces, the distance between the fixed coils being such that the force maxima do not coincide*

will decrease. Also, for this position of the moving coil the force CC' exerted by each fixed coil will be less than the corresponding maximum forces AA' and BB' by the same amount; therefore the difference between the two forces which is observed when the moving coil is in this position is equal to the true difference between the two maximum possible forces. Also for small displacements from this position the observed difference between the two forces will vary linearly with the displacement, the rate of variation being proportional to the difference between the actual distance between the fixed coils and that corresponding to the absolute maximum of force for the given current. Hence, the slope of the line representing the difference in the forces as a function

of the position of the moving coil is a criterion of the correctness of the spacing of the fixed coils.

Also, when the moving coil is in the position A, corresponding to the true maximum force of one of the fixed coils, a slight displacement will not affect the force due to this coil, and, consequently, the observed rate of variation of *both* the sum of and the difference in the two forces will be equal to the variation in the force due to the other coil, and so will be equal to one another. Also, if the displacements throughout this range are small enough to

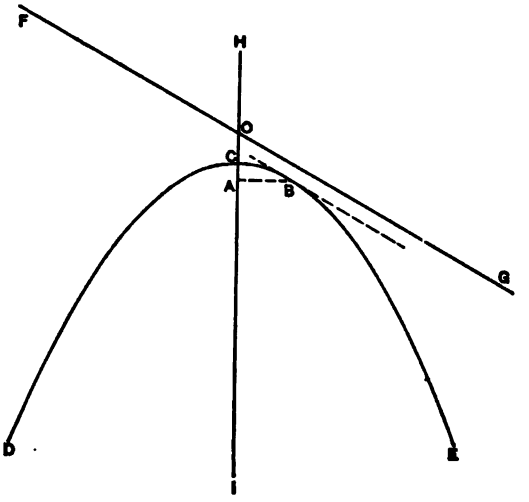

Fig. 18.—*The variation with the position of the moving coil of the observed sum and difference of the forces exerted by the two fixed coils when the distance between them is not that which gives coincidence of the maxima*

enable us with sufficient accuracy to express the variation of the force from the maximum as proportional to the square of the displacement, then it is easy to see that the sum of the true maximum forces is as much greater than the observed maximum of the sum as the latter exceeds the observed sum of the forces when the moving coil is in the position of maximum for either coil; that is, when the moving coil is in such a position that the rate of variation of the observed sum of the forces equals the rate of variation of the observed difference. Consequently, we have the following graphical method for determining the sum and the

difference of the true maximum forces from the observed values. In Fig. 18 let *DCE* represent the relation between the observed sum of the forces, as ordinates, and the position of the moving coil as abscissas; *DCE* will be a parabola, *HI* its axis. Similarly and on the *same scale* let *FOG* represent the observed difference between the forces (the origin from which the difference of the forces is measured will naturally not coincide with that used for the sum of the forces). Let *B* be the point at which the tangent to the *DCE* curve is parallel to *FOG*. Draw *BA* parallel to the axis of abscissas. Then the value of the difference in the force observed at the point *O* will be the difference between the true maximum forces, the sum of the two maximum forces will exceed the force at *C* by the amount *AC*, and the distance between the fixed coils differs from that corresponding to the true maximum by twice the distance represented by *AB*.

The next approximation in which the force of each coil is represented by a cubic is considered in the appendix, page 383, where it is shown that in all cases encountered in the present work the construction just given is sufficiently exact.

A similar discussion applies to the case when the fixed coils are not coaxial. This may be investigated experimentally by displacing the moving coil in a horizontal plane. If the moving coil is coaxial with *one* of the fixed coils, the force due to the latter is a minimum. If the moving coil is slightly displaced in a horizontal plane, the force increases proportionally to the square of the displacement. If the *two fixed coils* are not coaxial, the sum of their forces will be too great even for that position of the moving coil in which the sum of the forces is a minimum. This is a parallel case to the one considered above, where the maxima do not coincide, excepting that the error is of the opposite sign. The error due to the lack of coaxiality may likewise be determined experimentally by the curve of the difference of the forces exerted by the two fixed coils. If these coils are coaxial, their difference will be constant for small displacements of the moving coils in a horizontal plane. Otherwise the difference will vary linearly with the displacement of the moving coil, the slope of the line representing the difference will be a maximum for displacements of the

moving coil in the same direction in which the axes of the fixed coils are displaced relative to one another. Two experimental curves for the difference taken in a horizontal plane and in directions at right angle to each other, allow the lack of coaxiality to be measured, and the corresponding correction to be applied to the sum of the forces determined. The mechanical adjustments of the fixed coils were so good that this correction, as already stated, always amounted to less than five parts in a million in the force.

(*c*) *For load and temperature of coils.*—We have calculated the constants of the balances for 22°00 C and no load. As they were used at different temperatures and with different loads, it is necessary to apply temperature and load corrections.

Either of two methods may be adopted in calculating these corrections. We may correct the calculated constant for the load and the temperatures, and thus obtain the actual constant of the balance under working conditions; or we may correct the observed force for these quantities and thus obtain the force that would have been observed had the coils been of the radii assumed in the calculation of the constant. Since the temperatures of the coils differ slightly from time to time, and it is frequently desirable to study the variations in or the constancy of the force for a given current and a given balance constant, we have adopted the second method and have corrected the observed force for the temperatures and load. Knowing the coefficient of expansion of the coils, the load correction and the rate of variation of the force with the radius (Tables VII and X), this correction is readily determined.

32. THE OBSERVED DIFFERENCE IN THE FORCES

Since the study of the difference in the forces exerted upon the moving coil by the single fixed coils is of the nature of an adjustment or check measurement, it appears desirable to consider briefly at this point the results which were obtained by this study during the final set of observations. These are set forth in Table XV. Proper corrections for leads, temperatures, etc., have been applied.

TABLE XV

The Difference in the Forces (Milligrams)

Date	Coils	2×sum of forces	Order of subtracting	Difference in forces		Change on interchanging	Mean $\frac{A+B}{2}$	Calculated difference	M-C
		g		Upper coil is 2 or 3 A	Upper coil is 1 or 4 B	A-B	M	C	
Nov., 1910	S1, S2, M3	6	2-1	+3.45	+3.43	+0.02	.....	.....	.....
Mar., 1911	S1, S2, M3	6	2-1	+3.45	.....	.....	.....	.....	.....
Mean	.....	.....	.....	+3.45	.....	.....	+3.44	+3.43	+0.01
Nov., 1910	L1, L2, M3	6	2-1	−0.04	+0.12	−0.16	.....	.....	.....
Mar., 1911	L1, L2, M3	6	2-1	+0.11	.....	.....	.....	.....	.....
Mean	.....	.....	.....	+0.04	.....	.....	+0.08	+0.27	−0.19
Nov., 1910	L3, L4, M3	6	3-4	+1.26	+1.49	−0.23	.....	.....	.....
Feb., 1911	L3, L4, M3	6	3-4	.....	+1.46	.....	.....	.....	.....
Mar., 1911	L3, L4, M3	6	3-4	.....	+1.61	.....	.....	.....	.....
Mean	.....	.....	.....	.....	+1.52	.....	+1.39	+1.39	±0.00
Sept., 1910	L3, L4, M2	6	3-4	+1.58	+1.60	−0.02	.....	.....	.....
Feb., 1911	L3, L4, M2	6	3-4	.....	+1.57	.....	.....	.....	.....
Mean	.....	.....	.....	.....	+1.58	.....	+1.58	+1.54	+0.04
Oct., 1910	L3, L4, M2	3	3-4	+0.78	+0.79	−0.01	+0.78	+0.77	+0.01
Oct., 1910	L3, L4, M4	6	3-4	+1.46	+1.60	−0.14	.....	.....	.....
Feb., 1911	L3, L4, M4	6	3-4	.....	+1.69	.....	.....	.....	.....
Mean	.....	.....	.....	.....	+1.64	.....	+1.55	+1.54	+0.01
Oct., 1910	L3, L4, M4	3	3-4	.....	+0.79	.....	.....	+0.77	.....

It will be noticed that the excess of the mean difference (M) observed over the difference (C) calculated from the dimensions of the coils exceeds the experimental error in the weighing in but a single case. This is a good check on the accuracy of the radii determinations, and on the computations. Even in the case of the single exception the discrepancy is so small that were it entirely due to an error in the computed constant of the balance it could affect the measurement of the current by but 15 parts in a million.

In three cases the effect of interchanging the fixed coils lies well within the experimental error, showing the complete absence of any asymmetrical distribution of fixed magnetic masses that can affect the measurement of the current.

In the other three cases this effect, though quite measurable, is still so small that even if it affected the sum of the forces by its full amount it would in the worst case amount to but 2 in a hundred thousand in the measurement of the current. While the source of the trouble is not entirely clear, we believe, in view of the enhanced sensibility of the method of difference, that whatever its source it can affect the measurement of the current by but a small fraction of this amount. A further study of this point is to be made shortly.

33. RESULTS

The observations naturally fall into three distinct groups. The first includes those taken before November 30, 1908. These were of a strictly preliminary character, and served the purpose of discovering defects in the design and in the manipulation of the balance.

Then followed a long series of nearly 500 measurements of current made between November 30, 1908, and May 7, 1910. During this period many changes in the details of the work and of the mounting of the coils were made, and the method of procedure adopted in the final work was developed and thoroughly tested, the effects of small maladjustments were studied, and the question of the possible effect of slight magnetic impurities in the apparatus and of a permeability of the surroundings slightly greater than unity was investigated.

Unfortunately, the possible variations in the radii of the coils with the humidity (p. 286) had not yet been recognized, and the coils were not sealed. Consequently, these results are of uncertain value, and, as stated in the section dealing with the ratio of the radii, the earlier ones are also somewhat uncertain owing to the importance in that work of the parallelism of the coils (p. 307) being unrecognized at the time.

A summary of the values obtained in this interval is, however, given in Table XVI. To each value is assigned a weight equal to the total number of days on which observations were taken. While the values have a much greater range than in the later work, the mean of all is remarkably close to that obtained in the final measurements.

TABLE XVI

Results of Observations Previous to the Summer of 1910

Date	Cells	Number of weighings	Weight of observation	I_i Current in international amperes (1911)	Doubled force mg	I_A Current in absolute amperes	$\frac{I_i - I_A}{I_A} \times 10^6$	Deviation
Nov. 30–Dec. 12, 1908	L2, M1	60	4	1.010370	3000.11	1.010211	16	8
Dec. 21, 1908–Jan. 5, 1909	S1, S2, M1	29	4	0.989318	6000.01	0.989215	10	2
Mar. 24–Apr. 12, 1909	L1, L2, M1	61	13	1.010370	6000.23	1.010322	5	3
Apr. 13–July 16, 1909	L1, L2, M2	102	24	0.768384	6000.29	0.768308	10	2
July 17–Aug. 3, 1909	L1, L2, M2	39	8	0.768384	6000.66	0.768331	7	1
Aug. 7, 1909	L1, L2, M3	5	1	0.858897	5999.81	0.858848	6	2
Oct. 1–5, 1909	L1, L2, M3	15	3	0.858897	5999.28	0.858811	10	2
Oct. 11–14, 1909	L1, L2, M1	23	4	1.010370	6000.69	1.010280	9	1
Oct. 19–25, 1909	L1, L2, M2	19	6	0.768384	6000.40	0.768330	7	1
Oct. 28–29, 1909	L1, L2, M3	10	2	0.858897	5999.51	0.858827	8	0
Nov. 6–11, 1909	S1, S2, M3	17	4	0.840273	5999.37	0.840224	6	2
Nov. 17–23, 1909	S1, S2, M1	16	4	0.989318	6000.79	0.989399	−8	.....
Apr. 4–13, 1910	L3, L4, M3	36	7	0.858448	5999.83	0.858386	7	1
Apr. 15–16, 1910	L1, L3, M3	10	2	0.858638	5999.96	0.858581	7	1
Apr. 18–May 7, 1910	L3, L4, M2	40	10	0.767855	6000.44	0.767806	6	2
Weighted mean for the total of 482 weighings	.....	.....	.....	.....	.....	.....	7.4	.....
Weighted mean, omitting Nov. 17–23	.....	.....	.....	.....	.....	.....	8.0	2

The emf of the Weston normal cell at 20°.00 derived from the above results is 1.018 30 − 0.000 08 = **1.018 22** semiabsolute volts.

The last three values are especially interesting, as they indicate that the result given by $L1$, the most magnetic of the coils, does not differ appreciably from that given by $L4$, the best coil.

In reducing these results we have used the radii observations of October, 1908, for the observations between November 30, 1908, and January 5, 1909; those of February, 1909, for the observations between March 24 and August 7, 1909; and those of January, 1910, for the remainder.

The average difference of the results obtained in the 14 sets of measurements shown in Table XVI (excluding the one set of November 17–23, 1909) from the mean of the whole series is 2 parts in 100 000. If all the errors had been purely accidental, we could safely assume that in so large a number of weighings (482) these errors would be very largely eliminated, and the final result, 1.018 22 volts at 22° C for the Weston normal cell, would be considered as determined with very high accuracy.

The probable error of this weighted mean is less than 1 part in a million, and it would in that case be useless to make any further current weighings. But there was evidence in this work of certain small systematic errors which we believed could be eliminated so that a more satisfactory series of results could be obtained. Hence, after the peculiar effects, finally traced to changes in atmospheric humidity, had been overcome, it was decided to make certain changes for removing the possibility of these slight errors, and to undertake such a final series.

In May, 1910, these peculiar effects became very pronounced, and the rest of the spring and summer was devoted to an investigation of their cause (see appendix, p. 368) and to the sealing of the coils, the winding of moving coil $M4$, the replacing of the brick piers by wooden ones, and general preparation for exact measurements in the autumn and winter.

The third and final group of measurements was made between September 24, 1910, and March 25, 1911, and is composed of two parts. The first extends to November 17, 1910. Then the balance and the current connections were entirely taken apart, and the ratio of the radii of the coils, the needle correction, the tempera-

ture coefficients, and the load corrections were carefully measured. This work, with its reduction, lasted until February. The constants as thus determined were used in the reduction of all the weighings composing this group and have been given in a former section of this paper (Sec. III).

The balance was then reassembled and the remaining observations were made.

All of the results are given in considerable detail in Table XVII. As already stated, at least two additional weighings were made each day with the moving coil displaced considerably from the position of maximum force. As these observations were taken merely for the purpose of locating the position of the maximum, they are of the nature of observations for adjustment, and so are not given in the table nor considered in the determination of the value of the cell.

TABLE XVII

Third Series of Measurements of the Electromotive Force of the Weston Normal Cell in Semiabsolute Volts, Sept. 24, 1910–Mar. 25, 1911

Date	Cells			Temperatures		Double force		Current (absolute amperes)	Current amp. Int. (1911)	Int.—abs.	Parts per 10^6	Excess above mean	Emf of cell (semiabsolute volts)	Excess above mean in millionths
	Upper	Lower	Moving	Fixed	Moving	Observed	Corrected							
1910						mg	mg							
Sept. 24	L3	L4	M2	28°10	26°38	6000.08	6000.12	0.767748	0.767802	54	70	+ 6		
				28°10	26°38	.09	.13	49	02	53	69	+ 5		
Sept. 26				28°16	26°62	.10	.07	45	798	53	69	+ 5		
				28°17	26°59	.18	.17	51	95	44	57	− 7		
				28°16	26°56	.16	.15	50	97	47	61	− 3		
				28°14	26°53	.09	.08	46	97	51	66	+ 2		
Sept. 27	L4	L3	M2	28°14	26°56	.14	.07	45	99	54	70	+ 6		
				28°10	26°46	.32	.23	55	800	45	59	− 5		
				28°12	26°47	.28	.22	54	799	45	59	− 5		
				28°12	26°47	.23	.17	51	99	48	63	− 1		
Sept. 30				28°19	26°47	5999.43	5999.46	06	58	52	68	+ 4		
				28°18	26°48	6000.12	6000.14	49	96	47	61	− 3		
Oct. 1				28°16	26°37	.21	.27	58	802	44	57	− 7		
				28°16	26°37	.22	.27	58	02	44	57	− 7		
				28°16	26°37	.11	.17	51	02	51	66	+ 2		
				28°16	26°37	.09	.15	50	02	52	68	+ 4		
Mean											64	4_5	1.01823_6	+10

TABLE XVII—Continued

Date	Coils			Temperatures		Double force		Current (absolute amperes)	Current int. amp. (1911)	Int.—abs.	Parts per 10^6	Excess above mean	Emf of coil (semiabsolute volts)	Excess above mean in millionths
	Upper	Lower	Moving	Fixed	Moving	Observed	Corrected							
1910						mg	mg							
Oct.[22] 6	L4	L3	M2	25°82	26°21	3000.41	3000.26	0.383887	0.383914	27	70	− 5		
				25°81	26°24	.42	.27	87	14	27	70	− 5		
				25°82	26°24	.47	.32	91	14	23	60	−15		
				25°80	26°21	.41	.26	87	14	27	70	− 5		
Oct.[22] 10	L3	L4	M2	23°02	23°42	.39	.28	88	23	35	91	+16		
				23°03	23°44	.47	.35	92	22	30	78	+ 3		
				23°03	23°47	.41	.29	89	22	33	86	+11		
Mean											75	9	1.01822_4	−1
Oct. 14	L3	L4	M4	26°26	23°94	5999.93	6000.35	0.775881	0.775914	33	43	− 3		
Oct. 15				26°24	23°98	.73	.13	67	12	45	58	+12		
				26°24	23°98	.76	.16	69	12	43	55	+ 9		
Oct. 17				23°48	21°65	.91	.22	73	09	36	46	0		
				23°49	21°68	.88	.18	70	05	35	45	− 1		
				23°49	21°68	.92	.22	73	04	31	40	− 6		
Oct. 20	L4	L3	M4	23°45	21°56	.31	5999.65	36	868	32	41	− 5		
				23°44	21°56	.98	6000.31	78	907	29	37	− 9		
Mean											46	6	$[1.01825_3]$	
Oct. 18	L3	L4	M4	21°84	21°03	2999.92	2999.99	0.548614	0.548631	17	31	−15		
				21°80	20°97	.54	.64	582	10	28	55	+ 5		
				21°80	20°97	3000.28	3000.38	649	69	20	36	−10		
				21°86	21°07	.24	.31	43	67	24	44	− 3		
Oct. 24	L4	L3	M4	21°73	20°68	.29	.42	53	76	23	42	− 4		
				21°74	20°70	.24	.36	48	75	27	49	+ 3		
Oct. 25				21°70	20°56	.23	.37	49	80	31	56	+10		
				21°72	20°63	.21	.33	45	76	31	56	+10		
				21°72	20°65	.27	.38	49	76	27	49	+ 3		
Mean											46	7	$[1.01825_3]$	
Oct. 28	L4	L3	M3	24°24	23°68	6000.51	6000.27	0.858366	0.858434	68	79	− 9		
Oct. 31				24°22	23°44	.33	.30	69	44	75	87	− 1		
				24°24	23°55	.33	.25	65	38	73	85	− 3		
				24°25	23°60	.33	.24	64	35	71	83	− 5		
Nov. 2	L3	L4	M3	24°23	23°65	.24	.13	56	37	81	94	+ 6		
				24°22	23°67	.28	.16	59	36	77	90	+ 2		
				24°22	23°67	.30	.18	60	32	72	84	− 4		
Nov. 3				28°65	27°70	.10	.02	49	32	83	97	+ 9		
				28°66	27°72	.10	.01	48	31	83	97	+ 9		
Mean											88	5	1.01821_6	−15

[22] These observations were made with the windings of the moving coil in series.

TABLE XVII—Continued

Date	Coils			Temperatures		Double force		Current (absolute amperes)	Current int. amp. (1911)	Int.—abs.	Parts per 10^6	Excess above mean	Emf of cell (semi-absolute volts)	Excess above mean in millionths
	Upper	Lower	Moving	Fixed	Moving	Observed	Corrected							
1910						mg	mg							
Nov. 4	L2	L1	M3	24°84	23°95	5999.71	5999.79	0.858878	0.858935	57	66	+ 1		
				24°86	23°96	6000.13	6000.13	902	54	52	61	− 4		
Nov. 5				24°86	23°94	5999.35	5999.36	847	01	54	63	− 2		
				24°90	24°04	.07	.07	26	880	54	63	− 2		
Nov. 7	L1	L2	M3	25°03	23°86	.13	.19	35	90	55	64	− 1		
				25°06	23°94	.97	6000.01	93	957	64	74	+ 9		
Mean											65	3	1.01823_4	+ 9
Nov. 11	S1	S2	M3	22°46	23°27	5998.86	5998.61	0.840166	0.840218	52	62	+ 6		
Nov. 12				22°46	23°25	.89	.70	72	20	48	57	+ 1		
				22°46	23°24	.92	.74	75	19	44	52	− 4		
Nov. 14				22°36	23°18	5999.57	5999.37	219	72	53	63	+ 7		
				22°36	23°21	6000.00	.80	49	302	53	63	+ 7		
				22°38	23°23	.05	.85	53	297	44	52	− 4		
				22°38	23°23	5999.98	.78	48	97	49	58	+ 2		
Nov. 16	S2	S1	M3	22°34	23°23	.01	5998.80	179	18	39	46	−10		
Nov. 17				22°33	23°22	6000.06	5999.85	253	303	50	60	+ 4		
				22°30	23°22	.10	.88	55	299	44	52	− 4		
				22°30	23°22	.06	.84	52	99	47	56	0		
Mean											56	4	1.01824_0	+18
1911														
Feb. 3	L4	L3	M4	23°28	21°22	5999.64	5999.99	0.775858	0.775905	47	61	+ 7		
				23°28	21°30	.08	.39	19	862	43	55	+ 1		
				23°29	21°29	.16	.48	25	59	34	44	−10		
Feb. 4				23°28	21°14	.04	.40	20	67	47	61	+ 7		
				23°30	21°30	.60	.92	53	902	49	63	+ 9		
				23°31	21°30	.60	.92	53	898	45	58	+ 4		
				23°32	21°30	.62	.95	55	98	43	55	+ 1		
Feb. 6				23°20	20°98	.76	6000.16	74	914	40	52	− 2		
				23°25	21°09	.66	.03	66	06	40	52	− 2		
				23°28	21°16	.64	.00	64	02	38	49	− 5		
				23°30	21°22	.63	5999.98	62	01	39	50	− 4		
Feb. 7				23°28	21°15	.68	6000.04	61	03	42	54	0		
				23°29	21°22	.66	.00	58	899	41	53	− 1		
				23°30	21°29	.66	5999.98	57	95	38	49	− 5		
Feb. 8				23°28	21°20	.70	6000.04	61	902	41	53	− 1		
				23°30	21°23	.70	.04	61	01	40	52	− 2		
Mean											54	4	[1.01824_0]	

TABLE XVII—Continued

Date	Coils			Temperatures		Double force		Current (absolute amperes)	Current int. amp. (1911)	Int.—abs.	Parts per 10^6	Excess above mean	Emf of coil (semiabsolute volts)	Excess above mean in millionths
	Upper	Lower	Moving	Fixed.	Moving	Observed	Corrected							
1911						*mg*	*mg*							
Feb. 14	L4	L3	M2	23°24	21°73	5999.98	6000.10	0.767747	0.767801	54	70	− 3		
				23°24	21°76	.94	.05	44	00	56	73	0		
Feb. 15				23°20	21°65	.49	5999.62	16	766	50	65	− 8		
				23°20	21°67	.97	6000.11	47	804	57	74	+ 1		
Feb. 16				23°20	21°59	.94	.10	47	08	61	79	+ 6		
				23°21	21°65	.92	.06	44	04	60	78	+ 5		
				23°23	21°68	.96	.10	47	01	54	70	− 3		
Mean.											73	4	1.01822_6	+ 1
Feb. 18	L4	L3	M3	24°10	23°47	5999.99	5999.83	0.858335	0.858426	91	106	+ 5		
				24°78	24°07	.27	.10	283	372	89	104	+ 3		
Feb. 20				24°69	23°82	.46	.37	302	87	85	99	− 2		
				24°72	23°89	6000.03	.93	42	428	86	100	− 1		
				24°75	23°98	5999.93	.82	34	23	89	104	+ 3		
Feb. 21				24°64	23°85	6000.13	6000.02	49	31	82	96	− 5		
				24°64	23°87	.14	.02	49	27	78	91	−10		
Feb. 25				24°70	23°99	.19	.05	51	38	87	101	0		
				24°71	24°00	5999.92	5999.78	31	20	89	104	+ 3		
				24°72	24°01	.93	.79	32	20	88	103	+ 2		
Feb. 28				24°72	24°06	.90	.75	29	19	90	105	+ 4		
				24°72	24°07	6000.21	6000.07	52	36	84	98	− 3		
Mar. 1				24°70	24°04	.05	5999.90	40	28	88	103	+ 2		
				24°72	24°05	.05	.90	40	22	82	96	− 5		
				24°72	24°06	.03	.88	39	22	83	97	− 4		
Mean.											101	4	1.01819_7	−28
Mar. 3	L2	L1	M3	25°17	24°06	5998.92	5998.91	0.858815	0.858871	56	65	− 7		
Mar. 4				25°15	24°10	5999.49	5999.52	58	918	60	70	− 2		
Mar. 6				25°06	23°85	.53	.60	64	29	65	76	+ 4		
				25°08	23°90	.51	.57	62	25	63	73	+ 1		
				25°10	23°95	.57	.62	65	22	57	66	− 6		
Mar. 7				25°08	23°88	.55	.62	65	28	63	73	+ 1		
				25°06	23°91	.53	.58	63	24	61	71	− 1		
				25°06	23°98	.58	.61	65	21	56	65	− 7		
				25°07	23°99	.48	.51	58	20	62	72	+ 0		
Mar. 8				25°02	23°94	.97	6000.00	93	61	68	79	+ 7		
				25°07	24°00	.94	5999.97	90	50	60	70	− 2		
				25°07	24°01	.86	.89	85	50	65	76	+ 4		
				25°08	24°02	.86	.89	85	49	64	75	+ 3		
Mean.											72	4	1.01822_7	+ 2

TABLE XVII—Continued

Date	Cells			Temperatures		Double force		Current (absolute amperes)	Current int. amp. (1911)	Int.—abs.	Parts per 10^6	Excess above mean	Emf of cell (semiabsolute volts)	Excess above mean in millionths
	Upper	Lower	Moving	Fixed	Moving	Observed	Corrected							
1911						*mg*	*mg*							
Mar. 13	S1	S2	M3	23°23	24°05	5999.73	5999.45	0.840232	0.840290	58	69	− 3		
				23°27	24°08	5998.68	5998.37	149	07	58	69	− 3		
Mar. 14				23°23	24°07	5999.77	5999.49	228	86	58	69	− 3		
				23°24	24°08	6000.05	.74	45	.304	59	70	− 2		
Mar. 15				23°22	24°04	.37	6000.06	67	32	65	77	+ 5		
				23°24	24°07	36	.05	67	29	62	74	+ 2		
				23°26	24°10	.31	5999.99	63	24	61	73	+ 1		
Mar. 16				23°22	24°07	.42	6000.11	81	32	51	61	− 9		
				23°23	24°05	.29	.02	65	28	63	75	+ 3		
				23°23	24°05	.29	.00	63	26	63	75	+ 3		
Mar. 17				23°22	24°07	.35	.04	66	29	63	75	+ 3		
				23°24	24°07	.34	.03	65	26	61	73	+ 1		
Mean											72	3	1.01822_7	+ 2
Mar. 20	L4	L3	M4	23°96	21°82	5999.76	6000.08	0.775864	0.775902	38	49	0		
				23°94	21°79	.66	5999.98	57	897	40	52	+ 3		
Mar. 21				23°97	21°84	.14	.45	23	61	38	49	0		
Mar. 22				24°00	21°85	.05	.38	18	59	41	53	+ 4		
				24°01	21°89	.77	6000.14	68	902	34	44	− 5		
				24°02	21°95	.79	.13	67	01	34	44	− 5		
Mar. 23				23°98	21°81	.90	.28	77	18	41	53	+ 4		
				23°97	21°82	6000.01	.39	84	17	33	43	− 6		
				23°97	21°83	5999.98	.35	81	14	33	43	− 6		
				23°98	21°84	.92	.29	77	14	37	48	− 1		
Mar. 24				23°96	21°76	.93	.32	79	20	41	53	+ 4		
				23°96	21°77	.89	.28	77	18	41	53	+ 4		
				24°00	21°80	.91	.30	78	14	36	46	− 3		
Mar. 25				23°98	21°87	.92	.29	77	18	41	53	+ 4		
				23°96	21°78	.89	.28	77	16	39	50	+ 1		
				23°98	21°82	.85	.23	73	14	41	53	+ 4		
Mean											49	3	[1.01825_5]	
Mean, omitting those obtained with M4													1.01822_4	10

34. DISCUSSION OF RESULTS

(*a*) *The final mean value.*—In taking the mean we have omitted the values found by the use of moving coil $M4$, owing to the fact, already stated (p. 316), that the effective sectional dimensions of

this coil are not accurately known. The mean of the others is $1.018\ 22_8$. The mean of the 1910 observations is 4 in a million higher and that of the observations of 1911 are 6 in a million lower than the grand mean. The mean variation of the groups from the mean of all is 1 part in 100 000. The mean variation of the individual weighings of any group from the mean of that group is but 4 in 1 000 000. The mean of the four groups of observations obtained with moving coil $M4$ is $1.018\ 25_0$, and the mean variation from this mean is but 3 in 1 000 000, although for three of the groups the coils were independently adjusted. Observations with the other moving coils would probably be as good, but for none of the others have so many distinct groups of observations been taken during this final series. The high value obtained by means of $M4$ is due to the fact that the assumed axial breadth of the coil is greater than the true. From the method of winding we know that this is the case, although it was not foreseen when the coil was wound. There is, however, no way to determine how much too great it is. These values must, therefore, be disregarded in taking the mean. We believe that values found at different periods, but with the same combination of coils, will be relatively correct to within about 5 in 1 000 000, provided the radii are redetermined for each period. The radii should, of course, remain constant, but an occasional redetermination would be necessary to make sure that there has been no change.

The final conclusion is that the electromotive force of the Bureau of Standards' concrete realization of the mean Weston normal cell at 20°.00 C, January 1, 1911, was

1.01822 Semiabsolute[23] Volts

The probable error of the mean $1.018\ 22_8$ as given in the table is 3 parts in 1 000 000; and we believe it is a conservative estimate to assign to the value given above a possible uncertainty, due to all causes, of 2 in 100 000, a quantity equal to six times the computed probable error.

This differs from the result obtained in 1907 at the National Physical Laboratory (1.018 18 volts at 20°) by 4 parts in 100 000.

[23] A "semiabsolute volt" is that potential difference which exists between the terminals of a resistance of one international ohm when the latter carries a current of one absolute ampere.

This difference may, of course, be due to a great many sources combined, but doubtless an appreciable part may be due to a real difference in the cells measured.

(*b*) *Comparison of values obtained with different combinations of coils.*—In Table XVII there are given values obtained with five combinations of coils. Following are the mean values in each of four combinations, omitting the one found with moving coil number 4:

		Millionths
1. M_2, L_3, L_4 gives	$1.018\,22_8$	$\Delta = 4$
2. M_3, L_3, L_4 gives	$1.018\,20_4$	26
3. M_3, L_1, L_2 gives	$1.018\,23_0$	6
4. M_3, S_1, S_2 gives	$1.018\,23_8$	11
Mean.....	$1.018\,22_4$	10

In these four cases there have been employed three pairs of fixed coils, two large and one small, and two moving coils of different radii. The excellent agreement of the results, in which the separate values differ from the mean by only ten parts in a million, shows that the outstanding errors in the determination of the constants of the coils and in the adjustment of the coils in the balance is very small. The weight to be attached to the final mean is far greater than if the work had been done with a single pair of fixed coils and a single moving coil.

35. ACCELERATION OF GRAVITY

The acceleration of gravity in a basement room of the physical laboratory of this Bureau was determined in terms of its value at the gravity pier at the United States Coast and Geodetic Survey by means of relative pendulum observations taken by Mr. Wm. H. Burger, of the Survey, in August, 1910. The value of the acceleration of gravity at the Survey pier is known in terms of that at Potsdam from the relative pendulum observations made in 1900 by Mr. G. R. Putnam, of the Survey. Hence the value at this Bureau can be referred at once to that at Potsdam. As the result of a long series of determinations, the absolute value of the acceleration of gravity at the latter place is believed to be 981.274 cm sec^{-2} with a mean error of but 3 in a million.[24] On this basis

[24] Jahresbericht Preus. Geod. Inst., 1906–7, p. 6.

the value in the basement of this laboratory is 980.094. The balance is 9 m above the point where these observations were taken, and hence the value at the balance is 980.091 cm per second. This is the value that has been used in the reduction of the results. It is probably correct absolutely to within 5 or 6 in a million. The value of the acceleration of gravity used in the reduction of the results obtained at the National Physical Laboratory, and already mentioned several times, is referred to this same basis.

36. THE STANDARD CELLS

As stated in a former section, the cell actually used in these measurements was some one of a group of four Weston normal cells, which were kept in a thermostatically controlled oil bath in the same room as the balance. These, in situ, were compared almost daily with a group of seven Weston normal cells main-

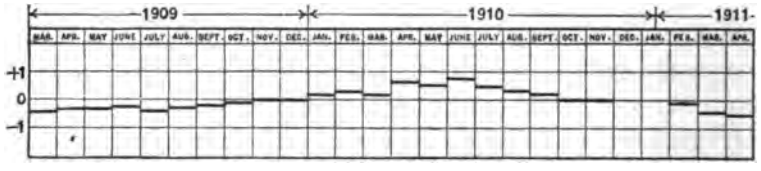

Fig. 19.—*Variation of standard cell W 56 relative to E_7, the reference set of this Bureau*
One division of the ordinates represents ten microvolts

tained continuously at 28°.00 C, and forming the working standard of the Bureau.

These seven cells are known as *W* 7, 10, 13, 20, 21, 23, and 25, and are selected cells. They were chosen in December, 1907, from a large number of cells, and at that time their mean electromotive force was 14 parts in a million lower than the mean of the 12 cells used in the work in this laboratory by F. A. Wolff and C. E. Waters on the Clark and Weston standard cells.[35] The mean of the 12 cells E_{12} has been referred to as the "old basis" of reference.

With reference to the above-named seven cells, those used in the actual measurements have changed very slightly. In Fig. 19 is shown the relative variation during a period of two years of

[35] This Bulletin, 4, pp. 1–81; 1907.

$W56$, the cell which has, except upon rare occasions, been used. It will be seen that throughout this period it has first increased 11 parts in a million and then decreased 12 parts in a million with reference to the seven reference cells.

In the spring of 1910 delegates from the National Physical Laboratory, the Physikalisch-Technische Reichsanstalt, the Laboratoire Central d'Électricité, and of the Bureau of Standards met at this laboratory for cooperative study of the silver voltameter and of the standard cell. Each delegate brought a number of cells from his laboratory, and it was agreed that the mean of the means of each of the four groups of cells (one group from each laboratory) should be regarded as the value of the mean Weston normal cell. The electromotive force of the cell at 20°00 C thus determined has been defined by the International Committee as 1.0183 international volts.

The difference between this value and the mean electromotive force of the seven cells named above was found to be 40 microvolts, the seven being the lower. Hence, denoting the mean electromotive force of the seven by E_7 and that of the mean Weston normal cell by E_N, we have—

$$E_N = E_7 + 40 \text{ microvolts}$$
$$E_N = E_{12} + 26 \text{ microvolts}$$

on June 1, 1910.

From numerous intercomparisons of various groups of cells at this laboratory and from intercomparisons with other laboratories it appears probable that the mean electromotive force of those cells (E_{36}) which were used in the determination of the mean cell and which have since been kept at this laboratory has decreased by about 12 parts in a million in one year; or, assuming a uniform drift, we may say that they have decreased at the rate of one in a million per month.

The difference between E_{36} and E_7 has been determined from time to time by direct comparison; hence, assuming the uniform drift for E_{36}, we can find the value of E_7 in terms of the mean Weston normal cell (E_N). This is done in Table XVIII.

TABLE XVIII

Values of E_7 in terms of E_N

	Oct., 1910	Feb., 1911
E_7	$=E_{36}-32^m$	$=E_{36}-22^m$
E_{36}	$=E_N-4^m$	$=E_N-8^m$
E_7	$=E_N-36^m$	$=E_N-30^m$

From these observations and inferences we must conclude that E_7 has gone up by 6 in a million from October to February, while the balance observations seemed to indicate that it has gone down by about 10 in a million—a discrepancy of 16 in a million.

This discrepancy is very small and is scarcely more than the limits of accuracy with which it is at present possible to perpetuate a given electromotive force by means of standard cells. For this reason we have assumed that $E_7 = E_N - 36$ microvolts throughout the period covered by this work.

In reducing the values to 20°.00 C. we have used the coefficients adopted by the London Conference, viz:

$$E_t = E_{20} - 0.000\,0406(t-20^\circ) + 0.000\,000\,95(t-20^\circ)^2 + 0.000\,000\,001(t-20^\circ)^3$$

37. THE RESISTANCES

The standard of resistance at this Bureau is furnished by a set of sealed wire coils which have been frequently intercompared among themselves, and, by means of traveling standards, compared with the standards of the European laboratories. They have been found to remain very constant. At the above-mentioned meeting in 1910 at Washington a comparison was made of the values of the resistances used at the different national laboratories as representative of the international ohm, and the departures of these various representations from what appeared to be the most probable value of the mean international ohm was determined. The unit as formerly used at this laboratory was found to have a resistance 7 parts in 1 000 000 greater than the adopted value of the mean international ohm. Correction for this has been made in the present work, the results therefore being expressed in terms of the mean international ohm as thus fixed.

38. FINAL RESULT IN TERMS OF THE ELECTROCHEMICAL EQUIVALENT OF SILVER

The above values of the results obtained from the current balance are in terms of the electromotive force of the Weston normal cell and the international ohm. We can also express these results in terms of the electrochemical equivalent of silver, and thereby eliminate the international ohm.

In the course of an extended investigation of the silver voltameter in the laboratories of the Bureau of Standards by Rosa, Vinal, and McDaniel a large number of experiments has been made, comparing different forms of voltameters and different methods of procedure. In three series of deposits during the past 12 months, using very pure electrolyte and following what they consider the best procedure, the following results were obtained for the emf of the Weston normal cell:

		Volts
126 deposits in the porous-pot form of voltameter	mean..	1.01826_2
33 deposits in the nonseptum form of voltameter	mean..	1.01826_5
Mean of all		1.01826_3

This value, of course, is based on the value 1.11800 for the electrochemical equivalent of silver and the international ohm. The above value differs but 4 in 100 000 from the value found from the current balance and international ohm. It follows from the above that the value of the electrochemical equivalent of silver, as derived from our current balance measurements and two types of the silver voltameter as used at the Bureau of Standards, is

1.11804 mg per coulomb

instead of 1.118 00, as adopted by the London Conference. In other words, the international ampere as defined by means of the silver voltameter as used in this Bureau differs from the absolute ampere as realized here by means of our current balance by only 4 parts in 100 000, the international ampere being slightly smaller than the absolute ampere.

APPENDIX

1. EFFECT OF HUMIDITY ON THE RADII OF THE COILS

As stated in the body of this paper, certain abnormalities in the results, which at first seemed inexplicable, were finally found to be due to the moisture absorbed by the thin paper placed between the layers of the coil for the purpose of improving the insulation and of facilitating a uniform winding. Since this effect is quite interesting and of considerable importance, it has seemed well to give the observations with some detail.

In the latter part of April, 1910, using coils $L3$, $L4$, and $M2$, and a current always of the same value, the forces, corrected for coil temperatures, etc., set forth in Table XIX were observed.

TABLE XIX

Corrected Forces, Coils L3, L4, and M2

Date	Force
	mg
April 27, 1910	6000.45
28, 1910	.48
29, 1910	.43
May 9, 1910	.66
10, 1910	.65

Here we see that the force changed by about 0.20 mg between April 29 and May 9, which is 3.3 in 100 000 of the total of 6 g. This is, of course, a very small change, but nevertheless too great to be considered accidental. Previous to April 29 everything had been going nicely and the lower value persisted for several interchanges of the coils; similarly, the higher value of the force persisted after May 10. The insulation was tested (though it was difficult to see how bad insulation of the balance could *increase* the force) and was found to be good, but not quite so good as usual. Then $M2$ was replaced by $M3$, and it also gave a force that was greater than usual. Then the resistance standard was replaced by another, with no change in the result. Other resistance standards were introduced at different points in the circuit so that the current could be measured before going to the balance, between

the fixed coils and the moving coil, and after leaving the balance. All three gave the same result. The potentiometer was changed without effect. As usual, the cells were compared daily and were found to be normal. These observations seemed to prove conclusively that the trouble must be in the balance itself.

With the same current the force that had formerly been 5999.77 mg was now found to be 5999.94 mg; it increased as the damp weather continued and then decreased from May 27 to June 1, when the atmosphere became drier. This suggested that the effect was due to changes in humidity.

Later the humidity again became very high and remained so from June 13 to June 24. Dishes of calcium chloride were then placed in the coil case, and the latter was sealed. The force immediately decreased.

TABLE XX

Force for Constant Current, Coils L_3, L_4, M_3 (1910)

Date	Relative humidity	Weight	Force (normal value 5999.77 mg)
	Per cent	mg	mg
May 18	44		5999.94
19	39		6000.03
20	61		6000.24
	Nearly saturated		
26	40		6000.34
27	40		6000.35
31	41		6000.01
June 1	41		5999.98
13	64		6000.53
14	60		6000.60
15	67		6000.58
16	62		6000.72
17	59		6000.66
18	60		6000.68
20	64		6000.75
21	55		6000.78
22	51	299	6000.72
23	49	286	6000.70
24	40	290	6000.78
25	35	119	6000.47
27	42	22	6000.38

These observations are recorded in Table XX. In the third column are also given the fractional weights in the counterpoise for the moving coil. The decrease in these numbers is exactly equal to the decrease in the *weight* of the coil, but does not, of course, affect the measured force.

It will be noticed that while on May 18th the force was 0.17 mg greater than normal, it increased in one week by 0.4 mg; then it went back, but later increased again until on June 24th it was 1.01 mg greater than normal; that is, it was *17 parts in 100 000* greater than earlier in the season when the atmosphere was dry. Drying the case decreased it 0.4 mg in two days.

An increase of 17 in 100 000 in the force corresponds in this case to an increase in the radius of the moving coil of 7 in 100 000, assuming that the fixed coils remained unchanged; that is, to an increase of 0.007 mm in the radius of the moving coil. A layer of water of this thickness spread over the bottom of the wire channel would have a volume of about 44 cubic millimeters. As seen from the few observations on the changing weight of the coil, this is much smaller than the change in the weight of the coil, but, of course, the moisture absorbed would be chiefly in the outer layers, and hence would produce far less effect than if it all penetrated to the bottom of the coil. From these observations we concluded that the observed changes in the force were due to a very slight swelling of the thin paper between the layers of the moving coil when more moisture was absorbed, and a shrinkage in the volume of the paper as moisture was given off, the latter occurring when the weather became drier. This conclusion was confirmed by the fact that after the coils were sealed air-tight the phenomenon disappeared, no such changes in the force being observed as the humidity of the atmosphere varied.

A similar change in the fixed coils would, of course, act in the opposite direction. Hence, the actual change in the moving coils must have been greater than the above figures indicate. The fixed coils had about three times as many layers of wire as the moving coil and hence would be less affected.

2. THE SECTIONAL DIMENSIONS OF A COIL

The value of the galvanometer constant of a coil of axial breadth 2α, of radial depth 2ρ, of mean radius A, and of n turns is usually deduced from the expression [26]

$$G=\frac{2\pi n}{A}\left\{1-\frac{1}{2}\left(\frac{\alpha}{A}\right)^2+\frac{1}{3}\left(\frac{\rho}{A}\right)^2+\frac{3}{8}\left(\frac{\alpha}{A}\right)^4+\frac{1}{5}\left(\frac{\rho}{A}\right)^4-\frac{\alpha^2\rho^2}{A^4}+\ldots\right\}$$

For the mutual inductance of two such coils Maxwell has given the expression [27]

$$M=G_1g_1P_1(\theta)+G_2g_2P_2(\theta)+\ldots\ldots$$

where G_n g_n are given functions of the sectional dimensions of the coils and of their distances from the origin and $P_n(\theta)$ is the zonal harmonic of the angle θ between the axes of the coils.

Lord Rayleigh and Mrs. Sidgwick in the computation of the constant of the current balance used by them made use of the approximation method (due to Purkiss) expressed by the equality [28]

$$\begin{aligned}6F=&f(A+\rho_1,\,a,\,B)+f(A-\rho_1,\,a,\,B)+f(A,\,a+\rho_2,\,B)\\&+f(A,\,a-\rho_2,\,B)+f(A,\,a,\,B+\alpha_1)+f(A,\,a,\,B-\alpha_1)\\&+f(A,\,a,\,B+\alpha_2)+f(A,\,a,\,B-\alpha_2)-2f(A,\,a,\,B)\end{aligned}$$

Lyle's method of approximation by means of the use of equivalent radii has already been mentioned in Section IV.

These and other methods of approximation are based upon an integration over the section of the coil and thus assume that to the order of approximation desired the actual coil is so constructed that the distribution of current may be regarded as uniform over the entire section covered by the integration.

Hence in the application of these formulae of approximation two questions of prime importance arise, viz: (1) Is the construction of the coil such that the effect of the actual distribution of the current is the same as that of a strictly uniform distribution of the same number of ampere turns? (2) What are the boundaries of the area over which the equivalent uniform distribution is spread?

[26] Rayleigh: Scientific Papers, vol. II, p. 291.
[27] Maxwell: Electricity and Magnetism, § 700.
[28] Maxwell: Electricity and Magnetism, Vol. II, App. II, Ch. XIV, also this Bulletin, 2, p. 370; 1906.

If the wires are uniformly spaced and the distance between adjacent wires is small as compared with the degree of non-homogeneity (in the neighborhood of the wires) of that magnetic field with which we are concerned, then the answer to the first question is *yes*, otherwise *no*. In the case of the galvanometer constant and of the force between two coils not very close together (that is, in the cases with which we are at present concerned), it is evident that the condition specified above is amply fulfilled by coils wound as closely as those used in this work. On the other hand, in the case of the self-inductance of a coil or of the mutual inductance of coils that are very close together, this condition is not fulfilled and the question of the corrections that are thus necessitated has been considered by several writers from Maxwell down to the present time.[29]

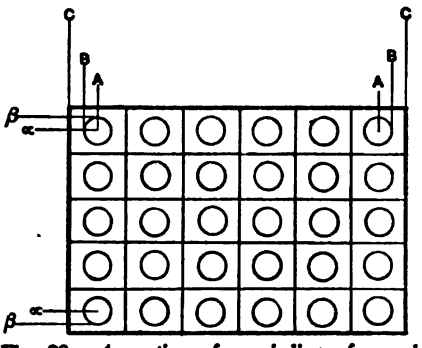

Fig. 20.—*A section of a winding of round wires; the circles represent sections of the conductor and the spaces between are insulation*

When we come to the second question we find considerable difference of opinion among those who have had occasion to use the various approximation formulas. This question has already been considered by one of us,[30] but it appears that another discussion of the subject will not be superfluous.

Limiting ourselves for the present to the case of a uniformly wound coil with the wires arranged in a rectangular array, the distance between the wires in adjacent layers being in general different from that between adjacent wires in the same layer, we may represent a section of the coil as in Fig. 20. The actual boundaries of the channel holding the wire is not shown, as it is immaterial.

Some authors have considered AA and $\alpha\alpha$, the distance measured to the axes of the limiting wires, as the true dimensions of the ideal coil; some have used distances slightly greater than these, but still

[29] Maxwell: Electricity and Magnetism, §§ 691, 692, 693. This Bulletin, 2, p. 161, 1906; 3, p. 1, 1907; 4, pp. 149, 369, 1907–8.

[30] This Bulletin, 2, pp. 77, 413, 1906; 3, p. 235, 1907.

measured to points inside the conductor of the bounding wires; others have considered that the correct dimensions are BB and $\beta\beta$ measured to the outer boundaries of the conducting portions of the terminal wires; some have measured to the outside of the insulation of the terminal wires; in at least one case one dimension was measured from axis to axis and the other from outside of insulation to outside of insulation of the bounding wires. Though apparently a compromise, this in reality introduces a greater error than any of the other methods mentioned above. The correct dimensions, namely, those of the heavy rectangular boundary of the figure, have seldom been used.

That this is the proper boundary for the ideal coil assumed in the integration is readily seen from a consideration of the fact that the only uniform distribution that can possibly be equivalent to the wires in the interior of the section is that having such a density that the amount of current which flows through each of the small rectangles is equal to the current carried by the wire. But in order that such a uniform distribution shall carry the same total ampere turns as the actual coil, it must be extended over the area bounded by the heavy rectangle (Fig. 20). Hence the latter is the correct boundary of the equivalent uniform distribution.

In order to exhibit the nature of the errors introduced by an erroneous choice of section, we shall proceed to a calculation of a few cases by means of the method given by Lyle.[21] He has shown that the magnetic field on the axis of an ideal circular coil of mean radius A and of square section of side b, and over which the current is uniformly distributed, is the same as that due to a linear circular circuit of radius equal to $A\left(1+\frac{b^2}{24A^2}\right)$ to within quantities of the magnitude of $\left(\frac{b}{A}\right)^4$. Hence for $A=10$ and $b=0.3$ the accuracy of replacement is of the order of 1 in a million. This radius of the equivalent linear circular circuit he calls the "equivalent radius" of the coil, and we shall denote it by A_e.

If $b=0.1$ and $A=10$, $A_e=10\ (1+4.16\times10^{-6})$, as b becomes smaller A_e approaches A. But it is evident that the effect of a

[21] Philosophical Mag., 3, p. 310; 1902.

circular wire must lie between those of the square wires having sections of such size that they may be circumscribed about or inscribed in the section of the circular wire. Hence, at least to an accuracy of 4.16 in a million in the radius, a circular filament of 10-cm radius, or any round wire bent so that its axis forms a circle of 10-cm radius and having a sectional radius not exceeding a half a millimeter, may be replaced by a square conductor of 1 mm on the side and bent so that its axis forms a circle of 10-cm radius. Hence for simplicity we shall consider the case of square wires.

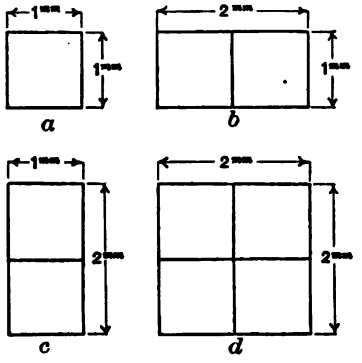

Fig. 21

We shall limit our discussion to a consideration of the galvanometer constant; it can readily be extended to other cases. In Fig. 21 we show sections of but one side of the coils, and shall always assume that the mean radius is 10 cm and that the axes of the coils are horizontal.

(*a*)—For the case represented by (*a*), Fig. 21, we have seen $A_e = 10 \times (1 + 4.16 \times 10^{-6})$.

(*b*)—For each square of (*b*), $A'_e = 10 \times (1 + 4.16 \times 10^{-6})$, but each is displaced 0.05 cm from the mean plane. Hence, in this case, the galvanometer constant for each square is

$$G' = \frac{2\pi}{A'_e}\left\{1 - \frac{3}{2}\left(\frac{0.05}{10}\right)^2 + \ldots\right\} = \frac{2\pi}{A'_e}\left\{1 - 37.5 \times 10^{-6} + \ldots\right\}$$

Hence, for the complete coil of two turns the equivalent radius is $A_e = 10\ (1 + 41.66 \times 10^{-6})$.

(*c*)—In the case represented by (*c*), the two equivalent radii are $A_e' = (10 + 0.05)(1 + 4.16 \times 10^{-6})$ and $A_e'' = (10 - 0.05) \times (1 + 4.16 \times 10^{-6})$, and the galvanometer constant is

$$G = \frac{2\pi}{2}\left\{\frac{1}{A'_e} + \frac{1}{A_e''}\right\} = 2\pi \frac{10(1 + 4.16 \times 10^{-6})}{(100 - 0.0025)(1 + 4.16 \times 10^{-6})^2}$$

$$= \frac{2\pi}{10\ (1 - 25 \times 10^{-6})(1 + 4.16 \times 10^{-6})}$$

Hence, for this coil of two turns the equivalent radius is

$$A_e = 10(1 - 20.84 \times 10^{-6})$$

(*d*)—The square coil shown is equivalent to two coils of the kind just considered, each displaced a distance 0.05 from the mean plane. Hence, for the galvanometer constant, the equivalent radius of this coil is $A_e = 10\ (1 - 20.84 \times 10^{-6})(1 + 37.5 \times 10^{-6}) = 10\ (1 + 16.66 \times 10^{-6})$.

In this case there is no doubt that the sectional dimensions of the coil are the dimensions of the large bounding square, and applying Lyle's method to this large square we find $A_e = 10\ (1 + 16.64 \times 10^{-6})$, sensibly the same as we found by considering the separate squares.

But we have seen that the effect of any square differs by not over about 4 in a million from the effect of a small round wire at its axis. Hence, this large square is what is to be understood when we speak of the section of a coil composed of 4 small wires, each coinciding with the center of one of these squares. This is exactly the conclusion at which we arrived in the early portion of this discussion.

Were we to take the section as determined by the axes of the bounding wires, this last case (*d*) would be equivalent to the first (*a*), and the equivalent radius found would be *too small* by 12.5 in a million.

Were we to regard the axial breadth of the coil as given by the breadth of the large square, and its depth as determined by the axes of the wires forming the upper and the lower layer, it would be equivalent to the second case (*b*) considered, and the equivalent radius found would be *too great* by 25 in a million.

Were we to take the radial depth as determined by the large square and the axial breadth as measured to the axes of the bounding wires, it would be equivalent to the third case (*c*), and the equivalent radius found would be *too small* by 37.5 in a million.

Hence, it is most important to choose the correct dimensions. They are given by the very simple expression $b = ns$, where b is the length of the side desired, n is the number of wires along this side, and s is the distance along this side between the axes of adjacent wires.

3. THE EFFECT OF ERRORS IN THE SECTIONAL DIMENSIONS OF THE COILS

In addition to introducing an error in the radius of a coil as deduced from its galvanometer constant, which effect has been fully considered on page 314, an error in the sectional dimensions of a coil will also introduce an error in the computed value of the force which this coil will exert upon a second coil, though the correct radii are used in the computation. It is this second correction which we desire to consider in this place.

Taking the series given by Maxwell [32] for the coefficient of mutual induction of two coaxial circles, and differentiating with respect to b, we find for the force between two coaxial circular currents the following expression, which was given by Lord Rayleigh.[33]

$$F=\frac{\pi^2A^2a^2}{C^4}\left\{1.2.3\frac{B}{C}+2.3.4\frac{\left(B^2-\frac{1}{4}A^2\right)b}{C^3}+3.4.5\frac{B\left(B^2-\frac{3}{4}A^2\right)\left(b^2-\frac{1}{4}a^2\right)}{C^5}\right.$$
$$+4.5.6\frac{\left(B^4-\frac{3}{2}B^2A^2+\frac{1}{8}A^4\right)\left(b^2-\frac{3}{4}a^2\right)b}{C^7}$$
$$\left.+5.6.7\frac{B\left(B^4-\frac{5}{2}B^2A^2+\frac{5}{8}A^4\right)\left(b^4-\frac{3}{2}b^2a^2+\frac{1}{8}a^4\right)}{C^9}+\ldots\ldots\right\}$$

where A and a are the radii of the two circles, B and b are the distances of their planes from some fixed point upon their common axis, and $C^2=A^2+B^2$.

If instead of linear circular circuits we are concerned with circular coils of axial breadths 2α and $2\alpha'$ and of radial depth 2ρ and $2\rho'$ (the accented letters referring to the smaller coil, assumed to be the one of radius a, and to which b refers), then the total force between the coils per unit current turn in each is given by the expression

$$F_t=\frac{1}{16\alpha\alpha'\rho\rho'}\int_{A-\rho}^{A+\rho}dA\int_{a-\rho'}^{a+\rho'}da\int_{B-\alpha}^{B+\alpha}dB\int_{b-\alpha'}^{b+\alpha'}Fdb$$

[32] Electricity and Magnetism, § 699.

[33] Phil. Trans., 175, pp. 411-460; 1884. Rayleigh, Scientific Paper, II, p. 282.

Performing this integration we shall obtain for the force an expression of the form

$$F_t = F + \lambda_1'\left(\frac{\rho}{A}\right)^2 + \lambda_2'\left(\frac{\alpha}{a}\right)^2 + \lambda_3'\left(\frac{\rho'}{a}\right)^2 + \lambda_4'\left(\frac{\alpha'}{a}\right)^2 + \ldots\ldots$$

whence by Maclaurin's theorem we find

$$\frac{\Delta F}{F} = \frac{F_t - F_{t0}}{F_{t0}} = \frac{1}{F_{t0}}\left\{2\lambda_1'\left(\frac{\rho_0}{A}\right)^2\frac{\delta\rho}{\rho_0} + 2\lambda_2'\left(\frac{\alpha_0}{a}\right)^2\frac{\delta\alpha}{\alpha_0} + 2\lambda_3'\left(\frac{\rho_0'}{a}\right)^2\frac{\delta\rho'}{\rho_0'}\right.$$
$$\left. + 2\lambda_4'\left(\frac{\alpha_0'}{a}\right)^2\frac{\delta\alpha'}{\alpha_0'} + \ldots\ldots\right\}$$
$$= \lambda_1\left(\frac{\rho_0}{A}\right)^2\frac{\delta\rho}{\rho_0} + \lambda_2\left(\frac{\alpha_0}{a}\right)^2\frac{\delta\alpha}{\alpha_0} + \lambda_3\left(\frac{\rho_0'}{a}\right)^2\frac{\delta\rho'}{\rho_0'} + \lambda_4\left(\frac{\alpha_0'}{a}\right)^2\frac{\delta\alpha'}{\alpha_0} + \ldots\ldots\ldots$$

where F_{t0} is the true value of F_t; ρ_0, α_0, ρ_0', α_0', are the true values of the half sectional dimensions, and F_t is the value of the force computed from the assumed dimensions $\rho = \rho_0 + \delta\rho$, etc.

Though it is impracticable to calculate with sufficient accuracy F from the series given, or to calculate the coefficients λ_1', etc., so as to derive F_t from F, yet for small values of $\frac{\delta\rho}{\rho}$, etc., we can without undue labor determine λ_1, etc., with sufficient accuracy to determine the effects of slight errors in the dimensions and to calculate the amount by which the force exerted by a coil of rectangular and nearly square section differs from that exerted by one of strictly square section.

This procedure is perfectly straightforward and is capable of application, but the determination of the coefficients λ_1, etc., necessitates somewhat tedious integration, so that it is expedient to adopt the method used in Maxwell §700, in which the coefficients are determined by differentiations. There it is shown that the mutual induction between two circular coils of dimensions $2\rho_0$, $2\alpha_0$, $2\rho_0'$, $2\alpha_0'$ is given by the expression

$$M = G_1 g_1 P_1(\theta) + G_2 g_2 P_2(\theta) + \ldots\ldots$$

where

$$G_n = G_{n0} + \frac{1}{6}\left(\rho_0^2\frac{\partial^2 G_{n0}}{\partial A^2} + \alpha_0^2\frac{\partial^2 G_{n0}}{\partial B^2}\right) + \frac{1}{120}\left(\rho_0^4\frac{\partial^4 G_{n0}}{\partial A^4} + \alpha_0^4\frac{\partial^4 G_{n0}}{\partial B^4}\right) + \ldots.$$

and an exactly similar equation connects g_n with g_{no}. The functions G_{no} and g_{no} are the coefficients of the solid zonal harmonic $r^n P_n(\theta)$ or $\frac{P_n(\theta)}{r^{n+1}}$ in the expansions of the solid angles subtended by the mean turns of the two coils at the point from which B and b are measured; θ is the angle between the axes of the coils. Expanding these functions in terms of the radii and the distances, and introducing the conditions that $\theta = 0$, and that the origin is to be taken at the center of the moving coil, we find as before

$$F_{t_0} = \frac{\partial M}{\partial b} = F + \lambda_1'\left(\frac{\rho_0}{A}\right)^2 + \lambda_2'\left(\frac{\alpha_0}{a}\right)^2 + \lambda_3'\left(\frac{\rho_0'}{a}\right)^2 + \lambda_4'\left(\frac{\alpha_0'}{a}\right)^2$$

where the λ's are functions of the radii of the coils and of their distance apart.

Writing

$$f_0 = 2 \cdot 3$$

$$f_1 = \frac{3 \cdot 5}{4} \cdot \frac{a^2(4B^2 - 3A^2)}{C^4}$$

$$f_2 = \frac{3 \cdot 5 \cdot 7}{4 \cdot 8} \cdot \frac{a^4(8B^4 - 20B^2A^2 + 5A^4}{C^8}$$

$$f_3 = \frac{5 \cdot 7 \cdot 9}{4 \cdot 8 \cdot 16} \cdot \frac{a^6(64B^6 - 336B^4A^2 + 280B^2A^4 - 35A^6)}{C^{12}}$$

the expressions for the various terms in F_{t_0} may be put in the form

$$F = \frac{\pi^2 a^2 A^2 B}{C^5}\left\{f_0 - f_1 + f_2 - f_3 \ldots\ldots\right\}$$

$$\lambda_1' = \frac{1}{6} \cdot \frac{\pi^2 a^2 A^2 B}{C^5}\left\{\frac{2 \cdot 3(2B^4 - 21B^2A^2 + 12A^4)}{C^4}\right.$$

$$- \frac{3 \cdot 5a^2(8B^6 - 200B^4A^2 + 395B^2A^4 - 90A^6)}{4C^8}$$

$$\left. + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8} \cdot \frac{a^4(16B^8 - 728B^6A^2 + 3066B^4A^4 - 2345B^2A^6 + 280A^8)}{C^{12}} + \ldots\right\}$$

$$\lambda_2' = \frac{1}{6} \cdot \frac{\pi^2 a^2 A^2 B}{C^5}\left\{2 \cdot 4 f_1 - 4 \cdot 6 f_2 + 6 \cdot 8 f_3 \ldots\right\}$$

$$\lambda_3' = \frac{1}{6} \cdot \frac{\pi^2 a^2 A^2 B}{C^5}\left\{1 \cdot 2 f_1 - 3 \cdot 4 f_2 + 5 \cdot 6 f_3 \ldots\right\}$$

$$\lambda_4' = \frac{1}{6} \cdot \frac{\pi^2 a^2 A^2 B}{C^5}\left\{2 \cdot 4 f_1 - 4 \cdot 6 f_2 + 6 \cdot 8 f_3 \ldots\right\}$$

As required by symmetry, we find that $\lambda_2' = \lambda_4'$.

These coefficients being of zero dimensions are determined solely by the ratio of the radii, provided that the distance B is always so chosen as to make the force a maximum.

Multiplying these coefficients by $\frac{2}{F_{i_0}}$ we obtain the coefficients in the variation formula. The values of these coefficients are given in Table XXI.

TABLE XXI

Coefficients in the Expression

$$\frac{\Delta F}{F}=\lambda_1\left(\frac{\rho_1}{A}\right)^2\frac{\delta\rho_1}{\rho_1}+\lambda_2\left(\frac{\alpha_1}{a}\right)^2\frac{\delta\alpha_1}{\alpha_1}+\lambda_3\left(\frac{\rho_2}{a}\right)^2\frac{\delta\rho_2}{\rho_2}+\lambda_4\left(\frac{\alpha_2}{a}\right)^2\frac{\delta\alpha_2}{\alpha_2}$$

$\frac{A}{a}$	λ_1	$\lambda_2=\lambda_4$	λ_3
2	+2.523	−0.8478	+1.704
2.5	+1.954	−0.4361	+1.201

TABLE XXII

Reduction of the Computed Forces to the Forces for Fixed Coils of 25 cm Radius and 2 cm Square Section, or of 20 cm Radius and 1.6 cm Square Section

Coils	Dimensions of fixed coil			Computed force	Reduced force
	Radius	2ρ	2α		
M3 L1	25.03121	2.035	2.027	3.138051	3.147097
M3 L2	25.03056	2.054	2.027	3.138331	96
M3 L3	24.99767	1.943	1.969	3.147744	98
M3 L4	25.00247	1.925	1.965	3.146282	98
M4 L3	Same as above			5.317906	5.316741
M4 L4				5.315175	42
M2 L3				5.355558	5.354384
M2 L4				5.352799	75
M3 S1	19.97510	1.528	1.580	5.417100	5.400429
M3 S2	19.96611	1.522	1.579	5.423282	06

By means of these coefficients and those in Table X we can reduce the calculated forces (Table XI) to what they would be

were the fixed coils, that are nominally equivalent, of exactly the same mean radius and of the same square section, thus obtaining a check upon the calculations. These reduced forces are given in Table XXII.

In the first four combinations, $M_3\ L_1$, $M_3\ L_2$, $M_3\ L_3$, and $M_3\ L_4$, the fixed coils are reduced to the same radius and cross section, hence the reduced forces should be the same. The variations are less than 1 in a million. The same reductions have been made for the other moving coils excepting that only two combinations can be compared. The maximum difference occurs in the comparison of the combinations $M_3\ S_1$, and $M_3\ S_2$, and amounts to 4 in a million in the force or 2 in a million in the resulting value obtained for the current. This may be due to a slight error of calculation or to the considerable range of the reduction.

4. RELATION BETWEEN THE OBSERVED AND THE TRUE MAXIMUM FORCE

In the body of this paper, p. 348, we have shown from physical considerations that the slope of the line representing the relation between the position of the moving coil and the difference in the forces exerted by the two fixed coils is a measure of the departure of the distance between the fixed coils from that which corresponds to the maximum force; that the observed difference in the forces for that position of the moving coil which corresponds to the axis of the parabola, which represents the relation between the sum of the forces and the position of the moving coil, is the value of the difference in the two true maximum forces; that the distance from the axis of this parabola to that point at which the tangent to the curve has the same slope as the line representing the difference in the forces is equal to one-half the amount by which the spacing of the fixed coils is in error; and that the value of the true maximum of the sum of the forces is as much above the vertex of the observed curve as the latter is above the point just mentioned. All of this is strictly true if the distances are so small that the variation in the force of either fixed coil alone can be regarded as strictly parabolic. It now remains to show mathematically that these statements are true and to see what modifica-

tions must be made if the distances are somewhat greater, so that the cube of the distance from the maximum must be considered.

Suppose that the fixed coils are very nearly identical, are coaxial with one another and with the moving coil, and are separated from one another by a greater distance than that corresponding to the maximum force for a given current. Let u and l denote quantities belonging to the upper and to the lower coil, respectively, let the dotted horizontal lines (Fig. 22) denote the

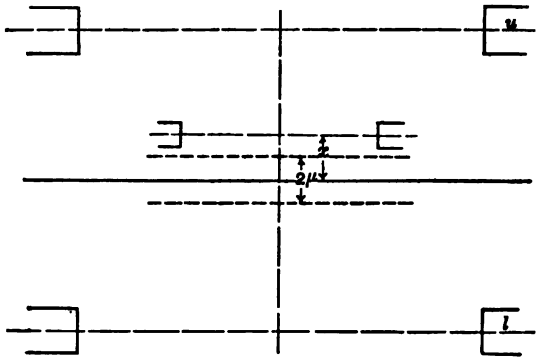

Fig. 22

positions of the moving coil for which the force exerted upon it by the respective fixed coils taken singly is a maximum. Denote the distance between these lines by 2μ. Let x denote the distance of the moving coil above the plane midway between these lines. Then, for any position of the moving coil the electromagnetic forces acting upon the moving coil will be

$$F_u = F_{uo}\{1 - \gamma(x-\mu)^2 - \delta(x-\mu)^3 \ldots\}$$

$$F_l = F_{lo}\{1 - \gamma(x+\mu)^2 + \delta(x+\mu)^3 \ldots\}$$

By hypothesis $F_{lo} - F_{uo}$ is a small quantity as compared with either force, and γ and δ will be essentially the same for both coils. Hence, the sum and the difference of the forces acting on the moving coil will be very approximately given by the expressions

$$\Sigma = F_u + F_l = \Sigma_o\{1 - \gamma(x^2+\mu^2) + \delta(3x^2\mu + \mu^3)\} + \Delta_o\{2\gamma\mu x - \delta(x^3 + 3x\mu^2)\}$$

$$\Delta = F_u - F_l = \Delta_o\{1 - \gamma(x^2+\mu^2) + \delta(3x^2\mu + \mu^3)\} + \Sigma_o\{2\gamma\mu x - \delta(x^3 + 3x\mu^2)\}$$

where $\Sigma_0 = F_{u0} + F_{l0}$ is the true maximum force, and Δ_0 is the difference between the two maxima.

Now γ and δ are small quantities (Table X, p. 330), μ can be made small, and x in this work never exceeded 2 or 3 mm. Hence, in practice, $\Delta_0\{2\gamma\mu x - \delta(x^3 + 3x\mu^2)\}$ will be very small as compared with Σ_0; likewise $-\gamma(x^3 + \mu^2) + \delta(3x^2\mu + \mu^3)$ is sufficiently small as compared with unity to be neglected where Δ is concerned. Consequently, we have to a very high order of accuracy

$$\Sigma = \Sigma_0\{1 - \gamma(x^2 + \mu^2) + \delta\mu(3x^2 + \mu^2)\}$$

$$\Delta = \Delta_0 + x\Sigma_0\{2\gamma\mu - \delta(x^2 + 3\mu^2)\}$$

Hence, the vertex of the (Σ, x) curve lies at $x = 0$, and at this point $\Delta = \Delta_0$. That is, the value Δ_0 is the value of Δ observed for that value of x which corresponds to the vertex of the observational (Σ, x) curve.

Also, if $\mu = 0$ the (Δ, x) curve will be a very flat cubic, the curvature being given by the very small coefficient δ.

Furthermore,

$$\frac{d\Sigma}{dx} = -\Sigma_0 x(2\gamma - 6\delta\mu)$$

$$\frac{d\Delta}{dx} = +\Sigma_0\mu\left(2\gamma - 3\delta\frac{\mu^2 + x^2}{\mu}\right)$$

Hence, for $x = -\mu$, $\frac{d\Sigma}{dx} = \frac{d\Delta}{dx}$.

That is, even when it is necessary to consider the cubic term, the conclusions stated at the beginning of this section are correct except for the fact that the (Δ, x) curve is not a straight line, but a very flat cubic. The latter fact involves the necessity of determining experimentally the slope of the (Δ, x) curve at that particular value of x for which this slope is the same as that for the (Σ, x) curve, if we desire to determine with extreme accuracy the error introduced by the error in the spacing of the coils.

This practically involves an experimental determination of the exact shape of the (Δ, x) curve through the region under consideration. This is impracticable, but another very approximate method can be developed from the recognition of the fact that the

values of Δ at the points $x = -x_1, o, +x_1$ all lie upon a straight line of which the slope is

$$S = \Sigma_0\{2\gamma\mu - \delta(x_1^2 + 3\mu^2)\}$$

But δ is always small as compared with γ, and x_1 can be made large as compared with μ, hence very approximately

$$\mu = \frac{S}{2\gamma\Sigma_0} + \frac{\delta x_1^2}{2\gamma}$$

All the quantities on the right being known, we can calculate μ and apply the proper correction to Σ. Or, by equating S to $\frac{d\Sigma}{dx}$ we find that the point at which the tangent to the (Σ, x) curve has the slope S is given by

$$x = -\mu + \frac{1}{2}\,\frac{\delta}{\gamma}\cdot\frac{x_1^2 - 3\mu^2}{1 - 3\frac{\delta}{\gamma}\mu}$$

$$= -\mu + \frac{1}{2}\,\frac{\delta}{\gamma}x_1^2 \qquad \text{approximately.}$$

This gives the best ocular estimate of the error in spacing.

Since in no case with which we are concerned does $\frac{\delta}{\gamma}$ exceed 0.1, and x_1 has never exceeded 0.3 cm, the absolute value of the second term on the right has never exceeded 0.045 mm. Hence, if μ is small, the method of procedure outlined for the simpler case will yield a sufficiently approximate value for the correction for the error in spacing. In the present work μ has always been very small, so that this approximate method has been of ample accuracy.

Since the force satisfies Laplace's equations, it will be a minimum with respect to horizontal displacements of the moving coil when the coils are properly adjusted. Hence, the above discussion applies in full to errors due to a noncoaxiality of the fixed coils, the planes of the latter being assumed horizontal. The only change being the multiplication of γ by the factor -0.5, and of δ by 0.0 (the values, for $\theta = 90°$, of the second and of the third zonal harmonics).

5. DIRECT MEASUREMENT OF THE COILS

As the coils were wound the mean diameter of each layer (of each third layer in the case of L_3 and L_4) was determined from the measurements of a number of diameters chosen so that their extremities were uniformly distributed around the circumference of the coil. These mean diameters, corrected to a common temperature, are given in Table XXIII and XXIV. The measurement of the individual diameters of the bottom (B) of the wire channel and of the top (T) of the outer layer are given in Table XXV. These numbers give an insight into the circularity of the forms and into the variations in the depth of the windings.

By referring to Table VIII, page 322, and comparing the direct and the electrical measurements of the radii, it will be seen that, though the direct measurements were made with great care, the accuracy attained is not sufficient to give an accuracy of even 1 in 10 000 in the current.

The only coils upon which such measurements were made as are capable of yielding information in regard to the compression of the forms by the wire wound upon them are the large coils L_3 and L_4 and the moving coil M_4. Unfortunately, the finish upon the latter was accidentally abraded during the winding, so that the value found for its compression is too great by an unknown amount. On the other hand, forms L_3 and L_4 have each been wound twice, the first winding having been damaged by a slight leak from the water channel in the forms; so that we have two measurements for the compression of them. These measurements are given in Table XXVI and show that the forms of the large fixed coils may be so compressed as to shorten their mean diameters by about 0.07 mm, but it is probable that the actual compression will not exceed a half of this. As now wound, L_3 and L_4 are each compressed by the same amount—0.027 mm. It is believed that no coil has been wound under a greater tension than L_3 when first wound, for then the wire fitted so tightly that it was difficult to draw into place the last turn of each layer. Also, M_4 was wound under a greater tension than any of the other moving coils, and even then its compression is less than that corresponding to 0.022 mm in the diameter.

TABLE XXIII

Diameters of Fixed Coils (Direct Measurement, mm)

Layer N	L1(t=20°0)		L2(t=25°0)		L3(t=20°0)		L4(t=20°0)		S1(t=25°0)		S2(t=25°0)	
	Micrometer setting S	Δ	Micrometer setting S	Δ	Micrometer setting S	Δ	Micrometer setting S	Δ	Micrometer setting S	Δ	Micrometer setting S	Δ
0	−19.879	1.075	−20.091	1.113	−19.275	1.021	−19.221	1.073	−15.897	1.107	−15.904	1.093
1	−18.804	1.056	−18.978	1.152	−18.254	.....	−18.150	.....	−14.790	1.073	−14.811	1.083
2	−17.748	1.184	−17.826	1.176	.....	1.047	.....	1.073	−13.717	1.093	−13.728	1.071
3	−16.564	1.229	−16.650	1.127	.....	.....	.....	.....	−12.624	1.093	−12.657	1.064
4	−15.335	1.131	−15.523	1.140	−15.112	.....	−14.932	1.053	−11.531	1.073	−11.593	1.042
5	−14.204	1.047	−14.383	1.122	.....	1.056	.....	.....	−10.458	1.084	−10.551	1.075
6	−13.157	1.090	−13.261	1.148	.....	.....	.....	.....	− 9.374	1.091	− 9.476	1.050
7	−12.067	1.122	−12.113	1.149	−11.943	.....	−11.774	.....	− 8.283	1.100	− 8.426	1.112
8	−10.945	1.219	−10.964	1.165	.....	1.083	.....	1.051	− 7.183	1.111	− 7.314	1.088
9	− 9.726	1.113	− 9.799	1.125	.....	.....	.....	.....	− 6.072	1.104	− 6.226	1.162
10	− 8.613	1.179	− 8.674	1.142	− 8.693	.....	− 8.622	.....	− 4.968	1.098	− 5.064	1.086
11	− 7.434	1.042	− 7.532	1.306	.....	1.060	.....	1.080	− 3.870	1.098	− 3.978	1.107
12	− 6.392	1.171	− 6.226	1.169	.....	.....	.....	.....	− 2.772	1.136	− 2.871	1.160
13	− 5.221	1.057	− 5.057	1.155	− 5.513	.....	− 5.381	.....	− 1.636	1.212	− 1.711	1.067
14	− 4.164	1.325	− 3.902	1.144	.....	1.073	.....	1.064	− 0.424	1.089	− 0.644	1.068
15	− 2.839	0.984	− 2.758	1.160	.....	.....	.....	.....	+ 0.665	1.211	+ 0.424	1.124
16	− 1.855	1.197	− 1.598	1.118	− 2.293	.....	− 2.189	1.068	+ 1.876	1.054	+ 1.548	1.050
17	− 0.658	1.122	− 0.480	1.111	.....	1.057	.....	.....	+ 2.930	1.097	+ 2.598	1.079
18	+ 0.464	1.101	+ 0.631	1.166	.....	.....	.....	.....	+ 4.027	1.074	+ 3.677	1.080
19	+ 1.565	1.252	+ 1.797	1.095	.....	.....	+ 1.019	.....	+ 5.101	1.049	+ 4.757	1.098
20	+ 2.817	1.012	+ 2.892	1.143	+ 1.936	1.063	.....	1.075	+ 6.150	1.082	+ 5.855	1.064
21	+ 3.829	1.146	+ 4.035	1.155	.....	.....	.....	.....	+ 7.232	1.078	+ 6.919	1.107
22	+ 4.975	1.059	+ 5.190	1.140	+ 4.062	.....	+ 4.243	.....	+ 8.310	1.067	+ 8.026	1.109
23	+ 6.084	1.086	+ 6.330	1.125	.....	1.129	.....	1.077	+ 9.377	1.056	+ 9.135	1.021
24	+ 7.120	1.104	+ 7.455	1.093	.....	.....	.....	.....	+10.433	1.067	+10.156	1.096
25	+ 8.224	1.239	+ 8.548	1.120	+ 7.449	.....	+ 7.473	.....	+11.500	1.044	+11.252	1.072
26	+ 9.463	1.134	+ 9.668	1.083	.....	1.159	.....	1.087	+12.544	1.082	+12.324	1.053
27	+10.597	1.130	+10.751	1.126	.....	.....	.....	.....	+13.626	1.046	+13.377	1.155
28	+11.727	1.108	+11.877	1.126	+10.925	.....	+10.733	.....	+14.672	.....	+14.532	.....
29	+12.835	1.251	+13.003	1.118	.....	1.054	.....	1.089	.....	.....	.....	.....
30	+14.086	0.967	+14.121	1.119	.....	.....	.....	.....	.....	.....	.....	.....
31	+15.053	1.096	+15.240	1.190	+14.087	.....	+13.999	.....	.....	.....	.....	.....
32	+16.149	1.054	+16.430	1.139	.....	1.073	.....	1.050	.....	.....	.....	.....
33	+17.203	1.104	+17.569	1.112	.....	.....	.....	.....	.....	.....	.....	.....
34	+18.307	1.127	+18.681	1.148	+17.305	1.093	+17.149	1.050	.....	.....	.....	.....
35	+19.434	1.337	+19.829	1.136	.....	.....	.....	.....	.....	.....	.....	.....
36	+20.771		+20.965		+19.492		+19.248		.....		.....	
Mean	$+0.973_0$		$+1.091_2$		$+1.034_2$		$+0.985_0$		$+0.026_5$		-0.159_6	

TABLE XXIII—Continued

	L1(t=20°0)	L2(t=25°0)	L3(t=20°0)	L4(t=20°0)	S1(t=25°0)	S2(t=25°0)
Mean, 1 to n.....	$+0.973_9$	$+1.091_2$	$+1.034_2$	$+0.985_8$	$+0.026_8$	-0.159_8
—wire.....	-0.587_5	-0.556_6	-1.021	-0.990_6	-0.546_6	-0.545_8
End standard.....	500.028	500.056	500.028	500.028	400.042	400.042
Cor to 22°00.....	+0.018	−0.027	+0.019	+0.019	-0.020_4	-0.021_1
$D_{22°00}$.....	500.432	500.564	500.060	500.042	399.502	399.316
$A_{22°00}$.....	25.0216 cm	25.0282 cm	25.0030 cm	25.0021 cm	19.9751 cm	19.9658_0 cm

N=The number of the layer on top of which the diameter is measured.

N=0 is the diameter of the bottom of the wire channel.

S=Diameter—End Standard, in mm and at the temperature t.

L_1, L_2, L_3, L_4, S_1, S_2, M_1, M_2, M_3, M_4, are the coils.

△=Difference between consecutive diameters.

Mean 1 to n=Mean of all measurements except the first.

Wire=Amount by which "Mean 1 to n" exceeds the mean diameter if uniformly wound.

=Diameter of wire if all diameters are measured.

=2 diameter of wire in case of L_3.

$=\frac{23}{14}$ diameter of wire in case of L_4.

$A_{22°00}$=Mean radius at 22°00.

TABLE XXIV

Diameters of Moving Coils (Direct Measurement, mm)

Layer N	M1 (t=20°0)		M2 (t=25°00)		M3 (t=23°50)		M4 (t=25°05)	
	Micrometer setting S	△	Micrometer setting S	△	Micrometer setting S	△	Micrometer setting S	△
0	−9.808	1.560	−9.539	1.526	− 9.592	1.394	−11.351	1.820
1	−8.248	1.498	−8.013	1.565	− 8.198	1.425	− 9.531	1.770
2	−6.750	1.511	−6.448	1.636	− 6.773	1.457	− 7.761	1.900
3	−5.239	1.706	−4.812	1.585	− 5.316	1.470	− 5.861	1.710
4	−3.533	1.506	−3.227	1.598	− 3.846	1.446	− 4.151	1.770
5	−2.027	1.494	−1.629	1.594	− 2.400	1.433	− 2.381	1.820
6	−0.533	1.455	−0.035	1.606	− 0.967	1.445	− 0.561	1.765
7	+0.922	1.384	+1.571	1.568	+ 0.478	1.461	+ 1.204	1.685
8	+2.306	1.590	+3.139	1.593	+ 1.939	1.510	+ 2.889	1.970
9	+3.896	1.494	+4.732	1.574	+ 3.449	1.543	+ 4.859	1.755
10	+5.390	1.514	+6.306	1.605	+ 4.992	1.567	+ 6.614	1.655
11	+6.904	1.465	+7.911	1.591	+ 6.559	1.450	+ 8.269	1.778
12	+8.369	.....	+9.502	.....	+ 8.009	1.483	+10.062	.....
13	.....	.....	.....	.....	+ 9.492	1.435	.....	.....
14	.....		.....		+10.927		.....	
Mean 1 to n	$+0.121_4$		$+0.749_8$		$+1.310_4$		$+0.304_2$	
—Wire	-0.765_9		-0.768_6		-0.701_9		-0.875_9	
End standard	200.004		250.011		200.012		250.011	
Cor. to 22°00	$+0.007_6$		-0.015_2		-0.005_9		-0.014_5	
$D_{22°00}$	199.368		249.977_1		200.616		249.426	
$A_{22°00}$	9.9684 cm		12.4988_6 cm		10.0308 cm		12.4713 cm	

TABLE XXV

Circularity of the Coils and Variations in the Depth of the Windings (Diameter Minus End Standard; for Bottom of Wire Channel and for Top of Outer Layer of Wire)

MOVING COILS

Coil M1			Coil M2			Coil M3			Coil M4		
Bottom	Top	T−B	Bottom	Top	T−B	Bottom	Top	T−B	Bottom	Top	T−B
mm	mm	mm	mm	mm	mm	mm	mm	mm	mm	mm	mm
−9.790	+8.976	18.766	−9.559	+9.566	19.125	−9.591	+10.944	20.535	−11.35	+10.09	21.44
			−9.527	+9.456	18.983	−9.607	+10.947	20.554	−11.39	+10.04	21.43
−9.845	+8.070	17.915	−9.524	+9.492	19.016	−9.586	+10.893	20.479	−11.38	+10.05	21.43
			−9.562	+9.555	19.117	−9.581	+10.924	20.505	−11.33	+10.03	21.36
−9.788	+8.062	17.850	−9.528	+9.472	19.000	−9.599	+10.938	20.537	−11.32	+10.03	21.35
			−9.535	+9.467	19.002	−9.588	+10.917	20.505	−11.33	+10.14	21.47
Mean	.....	18.177	.....	.....	19.040	.....	.....	20.519	.....	.....	21.41_2

FIXED COILS

Coil S1			Coil S2			Coil L1			Coil L2		
Bottom	Top	T−B	Bottom	Top	T−B	Bottom	Top	T−B	Bottom	Top	T−B
mm	mm	mm	mm	mm	mm	mm	mm	mm	mm	mm	mm
−15.905	+14.680	30.585	−15.937	+14.525	30.462	−19.877	+20.715	40.592	−20.102	+20.945	41.047
−15.899	+14.651	30.550	−15.907	+14.550	30.457	−19.887	+20.867	40.754	−20.083	+20.911	40.994
−15.886	+14.679	30.565	−15.860	+14.556	30.416	−19.879	+20.882	40.761	−20.076	+20.951	41.027
−15.898	+14.678	30.576	−15.913	+14.498	30.411	−19.873	+20.698	40.571	−20.095	+21.059	41.154
						−19.877	+20.768	40.645	−20.098	+20.958	41.056
Mean	.....	30.569	.....	.....	30.436	.....	.....	40.665[34]	.....	.....	41.056

[34] This value differs by 15μ from the value given in Table XXII owing to the latter value being based on a greater number of top measurements than is given here.

TABLE XXV—Continued

Coil L3			Coil L4		
Bottom	Top	T−B	Bottom	Top	T−B
mm	mm	mm	mm	mm	mm
−19.327	+19.577	38.904	−19.150	+19.398	38.548
−19.310	+19.458	38.768	−19.169	+19.268	38.437
−19.261	+19.560	38.821	−19.080	+19.518	38.598
−19.368	+19.697	39.065	−19.290	+19.366	38.656
−19.327	+19.435	38.762	−19.292	+19.366	38.658
−19.180	+19.535	38.715	−19.261	+19.098	38.359
−19.140	+19.595	38.735	−19.248	+19.160	38.408
−19.213	+19.393	38.606	−19.186	+19.162	38.348
−19.262	+19.621	38.883	−19.204	+19.248	38.452
−19.306	+19.631	38.937	−19.266	+19.101	38.367
−19.286	+19.494	38.780	−19.204	+19.094	38.298
−19.238	+19.481	38.719	−19.302	+19.040	38.342
−19.158	+19.374	38.532	−19.165	+19.260	38.425
−19.300	+19.313	38.613	−19.199	+19.402	38.601
−19.343	+19.193	38.536	−19.294	+19.239	38.533
−19.327	+19.278	38.605	−19.232	+19.386	38.618
−19.279	+19.730	39.009	−19.260	+19.106	38.366
−19.355	+19.485	38.840	−19.210	+19.220	38.430
Mean.....	.....	38.768	.....	.....	38.469

Since the forms for L_3 and L_4 are built up of rolled brass, it was feared that the pressure of the wire might cause the sides of the channel to spread. Consequently, the mean thicknesses of the forms were measured both before and after winding. The observations were so combined as to give the mean thicknesses on the three circles α, β, γ, Fig. 23. The results in millimeters are given in Table XXVII, and show that there is no appreciable spreading except in the case of the first winding of L_3, which, as stated, was very tight. The other fixed coils have forms of solid cast brass of about the same section, and consequently the spreading for them, though not measured, must be less than for L_3 and L_4.

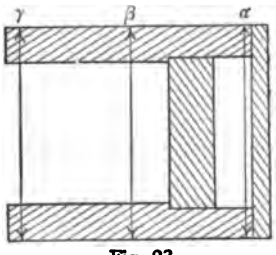

Fig. 23

Owing to the tapering of the sides, no accurate measurement of the spreading of the sides of the channels of the moving coils

has been obtained. It is, however, believed to be negligible, as there are relatively few layers on them, and the forms have been designed to secure stiffness.

TABLE XXVI

Compression of Forms (Diameter of Faces of the Forms on which the Wire was Wound)

	Coil L3		Coil L4		Coil M4
	Top	Bottom	Top	Bottom	Top
1st winding	cm	cm	cm	cm	cm
Start	52.4264	52.4238	52.4226	52.4254	26.2140
Finish	52.4201	52.4172	52.4201	52.4220	26.2118
Compression	0.0063	0.0066	0.0025	0.0034	0.0022
2d winding					
Start	52.4256	52.4230	52.4230	52.4256	.....
Finish	52.4229	52.4203	52.4203	52.4228	.....
Compression	0.0027	0.0027	0.0027	0.0028	.....

TABLE XXVII

Spreading of the Sides of the Channel (Thickness of Brass Form in mm at the points α, β, γ, Fig. 23)

	First winding			Second winding		
	α	β	γ	α	β	γ
Coil L3						
Before	30.037	30.052	30.063	30.038	30.052	30.065
After	30.040	30.057	30.080	30.037	30.049	30.062
Spreading	0.003	0.005	0.017	−0.001	−0.003	−0.003
Coil L4						
Before	30.059	30.073	30.081	30.058	30.072	30.080
After	30.057	30.072	30.087	30.059	30.072	30.086
Spreading	−0.002	−0.001	+0.006	+0.001	0.000	+0.006

The effective radial depth of the windings is shown (p. 375) to be ns where n is the number of layers and s is the mean distance between adjacent layers as measured to the axes of the wires.

If during the winding the diameter of the form is decreased by an amount c, and if the diameter of the wire is d, then $s = \frac{(T-B)+c-2d}{2(n-1)}$ where $(T-B)$ is the difference between the diameter measured to the outside of the outer layer, and the diameter of the bottom of the coil channel. The values of $(T-B)$ can be obtained from the tables just given. The computation of the radial depths (2ρ) is given in Table XXVIII.

TABLE XXVIII

Effective Radial Depths of the Windings

	M1	M2	M3	M4	S1
n	12	12	14.	12	28
T−B in mm	18.177	19.041	20.519	21.413	30.569
c in mm	.....	.....	.....	0.022	.....
2d in mm	1.530	1.537	1.402	1.750	1.093
2 (n−1) s in mm	16.647	17.504	19.117	19.685	29.476
2ρ in cm	0.9080	0.9548	1.0294	1.0737	1.5284

	S2	L1	L2	L3	L4
n	28	36	36	36	36
T−B in mm	30.436	40.650	41.056	38.767	38.469
c in mm	.....	.....	.....	0.027	0.027
2d in mm	1.091	1.075	1.113	1.021	1.073
2 (n−1) s in mm	29.345	39.575	39.943	37.773	37.423
2ρ in cm	1.5216	2.0353	2.0542	1.9426	1.9246

The effective axial breadth of the windings is shown (p. 375) to be ns where n is the number of spires in one layer and s is the axial distance of adjacent spires; this will be the breadth of the channel if the wires are uniformly spaced and the axes of the terminal spires are as far from the sides of the channel as half the distance between the axes of adjacent wires. If the wires are equally spaced, but the terminal spires touch the sides of the channel, then the effective breadth ns becomes $\frac{n}{n-1}(a-d)$ where a is the breadth of the channel and d is the diameter of the wire. These values are given in Table XXIX. It will be seen that the difference between these values is very small, only a few

hundredths of a millimeter. For reasons stated in the body of the paper, we believe the conditions are such that a is the correct value, except for $M4$, and this is the value we have used in the computations

TABLE XXIX

Effective Axial Breadth of the Windings, in Centimeters

Coils	M1	M2	M3	M4	S1
n	12	12	14	12	28
a	0.9850	0.9564	0.9967	1.0877	1.5803
d	0.0765	0.0768	0.0701	0.0875	0.0546
$\frac{n}{n-1}(a-d)$	0.991_1	0.959_6	0.997_9	1.091_1	1.582_3
$\frac{n}{n-1}(a-d)-a$	0.006_1	0.003_2	0.001_2	0.003_4	0.001_9

Coils	S2	L1	L2	L3	L4
n	28	36	36	35.97	35.97
a	1.5787	2.0267	2.027	1.969	1.965
d	0.0546	0.0538	0.0556	0.051	0.0536
$\frac{n}{n-1}(a-d)$	1.580_5	2.029_2	2.027_7	1.972_8	1.966_2
$\frac{n}{n-1}(a-d)-a$	0.001_8	0.002_6	0.000_7	0.003_8	0.001_3

6. TABLE OF LOG SIN $\gamma\{2F_\gamma-(1+SEC^2\gamma)E_\gamma\}$

TABLE XXX

Table of the logarithms to the base ten of sin $\gamma\{2F_\gamma-(1+\sec^2\gamma)E_\gamma\}$ from $\gamma = 55°$ to $\gamma = 70°$. Compiled by means of the Tables of Legendre and of Vega

γ	Log	Δ_1	Δ_2	γ	Log	Δ_1	Δ_2
°				°			
55.0	$\bar{1}$.91989049	517780	−117	59.0	0.12673397	518270	157
.1	.92506829	517663	111	.1	.13191667	518427	164
.2	.93024492	517552	106	.2	.13710094	518591	173
.3	.93542044	517446	97	.3	.14228685	518764	180
.4	.94059490	517349	93	.4	.14747449	518944	187
55.5	.94576839	517256	85	59.5	.15266393	519131	196
.6	.95094095	517171	80	.6	.15785524	519327	203
.7	.95611266	517091	72	.7	.16304851	519530	211
.8	.96128357	517019	66	.8	.16824381	519741	219
.9	.96645376	516953	60	.9	.17344122	519960	226
56.0	.97162329	516893	53	60.0	.17864082	520186	236
.1	.97679222	516840	47	.1	.18384268	520422	243
.2	.98196062	516793	40	.2	.18904690	520665	252
.3	.98712855	516753	33	.3	.19425355	520917	258
.4	.99229608	516720	27	.4	.19946272	521175	268
56.5	.99746328	516693	20	60.5	.20467447	521443	276
.6	0.00263021	516673	14	.6	.20988890	521719	284
.7	.00779694	516659	6	.7	.21510609	522003	293
.8	.01296353	516653	± 0	.8	.22032612	522296	301
.9	.01813006	516653	+ 6	.9	.22554908	522597	309
57.0	.02329659	516659	14	61.0	.23077505	522906	319
.1	.02846318	516673	21	.1	.23600411	523225	327
.2	.03362991	516694	27	.2	.24123636	523552	335
.3	.03879685	516721	35	.3	.24647188	523887	345
.4	.04396406	516756	40	.4	.25171075	524232	352
57.5	.04913162	516796	49	61.5	.25695307	524584	363
.6	.05429958	516845	55	.6	.26219891	524947	371
.7	.05946803	516900	62	.7	.26744838	525318	381
.8	.06463703	516962	69	.8	.27270156	525699	388
.9	.06980665	517031	76	.9	.27795855	526087	399
58.0	.07497696	517107	84	62.0	.28321942	526486	408
.1	.08014803	517191	91	.1	.28848428	526894	417
.2	.08531994	517282	98	.2	.29375322	527311	426
.3	.09049276	517380	104	.3	.29902633	527737	436
.4	.09566656	517484	113	.4	.30430370	528173	446
58.5	.10084140	517597	120	62.5	.30958543	528619	455
.6	.10601737	517717	127	.6	.31487162	529074	465
.7	.11119454	517844	135	.7	.32016236	529539	475
.8	.11637298	517979	141	.8	.32545775	530014	484
.9	.12155277	518120	150	.9	.33075789	530498	495

TABLE XXX—Continued

γ	Log	Δ_1	Δ_2	γ	Log	Δ_1	Δ_2
°				°			
63.0	0.33606287	530993	505	66.5	0.52563052	555378	924
.1	.34137280	531498	515	.6	.53118430	556902	940
.2	.34668778	532013	525	.7	.53674732	557242	955
.3	.35200791	532538	535	.8	.54231974	558197	968
.4	.35733329	533073	546	.9	.54790171	559165	984
63.5	.36266402	533619	556	67.0	.55349336	560149	999
.6	.36800021	534175	568	.1	.55909485	561148	1015
.7	.37334196	534743	577	.2	.56470633	562163	1029
.8	.37868939	535320	589	.3	.57032796	563192	1046
.9	.38404259	535909	599	.4	.57595988	564238	1061
64.0	.38940168	536508	610	67.5	.58160226	565299	1077
.1	.39476676	537118	623	.6	.58725525	566376	1094
.2	.40013794	537741	632	.7	.59291901	567470	1110
.3	.40551535	538373	644	.8	.59859371	568580	1126
.4	.41089908	539017	656	.9	.60427951	569706	1144
64.5	.41628925	539673	667	68.0	.60997657	570850	1160
.6	.42168598	540340	679	.1	.61568507	572010	1178
.7	.42708938	541019	691	.2	.62140517	573188	1195
.8	.43249957	541710	702	.3	.62713705	574383	1213
.9	.43791667	542412	715	.4	.63288088	575596	1231
65.0	.44334079	543127	726	68.5	.63863684	576827	1249
.1	.44877206	543853	740	.6	.64440511	578076	1268
.2	.45421059	544593	751	.7	.65018587	579344	1285
.3	.45965652	545344	764	.8	.65597931	580629	1305
.4	.46510996	546108	776	.9	.66178560	581934	1325
65.5	.47057104	546884	790	69.0	.66760494	583259	1343
.6	.47603988	547674	803	.1	.67343753	584602	1362
.7	.48151662	548477	814	.2	.67928355	585964	1384
.8	.48700139	549291	829	.3	.68514319	587348	1404
.9	.49249430	550120	842	.4	.69101667	588752	1423
66.0	.49799550	550962	856	69.5	.69690419	590175	1444
.1	.50350512	551818	869	.6	.70280594	591619	1466
.2	.50902330	552687	882	.7	.70872213	593085	1486
.3	.51455017	553569	897	.8	.71465298	594571	1509
.4	.52008586	554466	912	.9	.72059869	596080	1531
				70.0	.72655949	597611	1552

WASHINGTON, September 9, 1911.

DEFLECTION POTENTIOMETERS FOR CURRENT AND VOLTAGE MEASUREMENTS

By H. B. Brooks

1. INTRODUCTION

The deflection potentiometer differs from potentiometers hitherto used in one essential feature, namely, the use which it makes of the galvanometer. The latter has formerly been regarded as a mere indicator of the absence of a difference of potential, dials or slide wires being adjusted to give the condition of no current through the galvanometer. In the deflection potentiometer, as the name indicates, the normal condition of the galvanometer is that of deflection due to the passage of a current through it, and its reading gives several figures of the result.

In previous papers[1] the writer has given an outline of the principles on which have been based two forms of deflection potentiometer used by this Bureau. Both of these were constructed for voltage measurements only, one being in use in the photometric laboratory and one in electrical instrument testing.

It was early recognized[2] that current measurements were possible with this type of potentiometer, using a suitable shunt, and in view of the advantages already secured in voltage measurements by the use of the deflection potentiometer, efforts were made to secure the special high-grade pivoted galvanometer necessary for such a potentiometer for use with shunts. At first it was thought desirable to plan a potentiometer for current measurements only, but as the work progressed it became evident that one instrument

[1] This Bulletin, 2, p. 225; 1906 (Reprint No. 33). 4, p. 275; 1908 (Reprint No. 79).
[2] This Bulletin, 2, p. 237; 1906 (Reprint No. 33).

could be made to measure both current and voltage. The object of the present paper is to give a brief description of two recent forms of deflection potentiometer, which are quite similar in design, and are called for convenience Model 3 and Model 5, respectively. The theory of the use of current shunts with the deflection potentiometer will then be given, and the most suitable values for such shunts will be pointed out.

The need for electrical measuring instruments which combine accuracy and speed is continually becoming more apparent. It is not sufficient that one can get an accurate value, given steady sources of current and voltage, skilled help, and plenty of time. Instruments are wanted which will give the desired accuracy even on unsteady circuits, the result being quickly obtained; the manipulation must be simple and capable of being quickly comprehended by a workman of average intelligence. External disturbing influences (such as temperature changes and stray magnetic fields) must not cause errors in the result, as corrections for temperature are time-consuming, and stray magnetic fields can not always be avoided.[3]

2. OUTLINE OF POTENTIOMETER THEORY

In Fig. 1, (a) represents in outline a simple potentiometer for measuring voltages not exceeding that of the auxiliary battery. A battery e_1 sends a current through a resistance r_1r_2 which may be varied to enable the current to be kept constant as the emf of e_1 changes. E is a source (battery or dynamo) whose emf is to be measured. The positive poles of E and e_1 are directly opposed, and the negative pole of E is connected (through a galvanometer) to such a point on the resistance that the emf E is balanced by the fall of potential or "drop" in the portion r_1 of the resistance. This condition is shown by the absence of a deflection, and by Ohm's law we have:

[3] The need for this class of standard instrument is well stated by William Bradshaw in the following words (Electric Journal, 8, p. 394; 1906): "The refinement obtainable in electrical measurements by the use of [null] potentiometers is well known, but they are not adapted for measuring alternating current or unsteady direct current. The ideal instrument for this service is one that approaches the accuracy of the null potentiometer and allows the ease and quickness of manipulation necessary in the ordinary test room."

$$E = \frac{r_1}{r_1 + r_2} \times e_1, \text{ or } E = r_1 \times \frac{e_1}{r_1 + r_2}$$

Hence if the current $e_1/(r_1 + r_2)$ is adjusted to a convenient integral value, E may be read off from the value of r_1 which gives a balance. The usual and preferable method of adjusting the current to the proper value consists in replacing E by a standard cell, setting r_1 to a value proportional to the known value of the cell, and varying $r_1 + r_2$ until no deflection occurs. In most forms of potentiometers

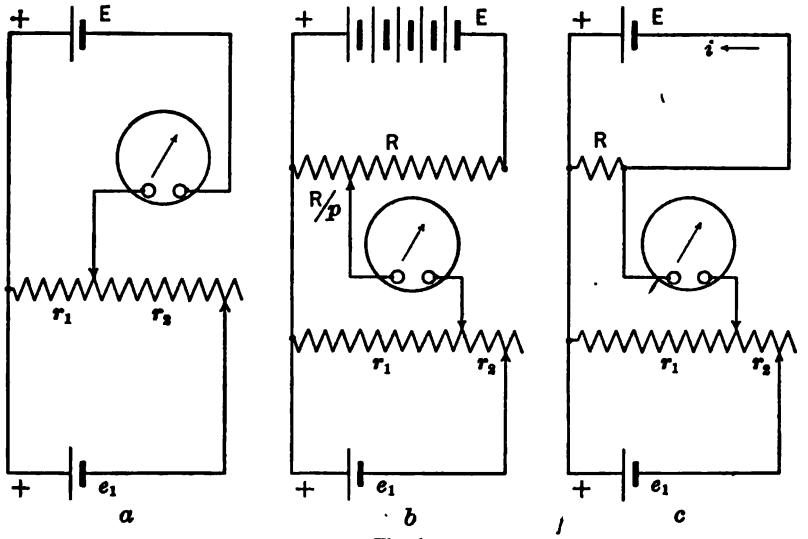

Fig. 1

the substitution of the standard cell for E is made by a suitable switch. Values of r_1 are frequently not marked in ohms, but in volts, on the assumption of a particular working current, or method of applying the standard cell.

When voltages much in excess of 2 volts are to be measured, it is convenient in most cases to use a volt box, as shown in Fig. 1 (b). R is a resistance high enough to be connected in parallel with the source E without appreciably changing the potential difference of the points to which it is connected. A suitable fraction of this resistance, R/p, is connected to the circuit $r_1 r_2$ just as the

source E was connected in Fig. 1 (a). When r_1 has been adjusted so that the galvanometer shows no deflection, we have:

$$\frac{1}{p}E=\frac{r_1}{r_1+r_2}e_1, \text{ or } E=r_1\times\frac{e_1}{r_1+r_2}\times p$$

where p is the ratio of the whole resistance R to the portion R/p; or p is the multiplying factor of the volt box.

The measurement of current is next to be considered. Fig. 1 (c) shows a source which passes the current i through a standard resistance or "shunt" R. When balance is obtained as before, we have

$$Ri=r_1\times\frac{e_1}{r_1+r_2}, \text{ or } i=\frac{1}{R}\times r_1\times\frac{e_1}{r_1+r_2}$$

Here the fall of potential over the shunt, as shown by the value of r_1, must be divided [4] by R, the resistance of the shunt.

In order to adjust r_1 to give no deflection of the galvanometer, it is necessary that the value E to be measured be quite steady. In some forms of potentiometer the resistance r_1 consists of a series of four or five dials, each of which has a value per step one-tenth that of a step on the dial above. The dials are set in succession, beginning with the highest, until balance is obtained. In other forms one dial is used, a calibrated slide wire taking the place of the three or four lower dials. In the deflection potentiometer two dials may be used, but it has been found desirable thus far to use but one dial, thus reducing to a minimum the manipulation necessary.

If we assume in Fig. 1 (a) that balance is not obtained, then the current i_g through the galvanometer is given by the expression

$$i_g=\frac{e_1\frac{r_1}{r_1+r_2}-E}{r_g+\frac{r_1r_2}{r_1+r_2}}$$

[4] It is convenient instead to multiply by $1/R$, the "conductance" of the shunt, especially for shunts of values below 1 ohm, as is generally the case.

This expression shows that the galvanometer current is equal to the difference between the voltage indicated by the dial or slide wire setting r_1 and the unknown emf E, divided by the total resistance in the galvanometer circuit. This total resistance is to be figured as if the sources E and e_1 contained no emf. This formula shows the possibility of reading any desired part of the result on the galvanometer, provided its scale is properly calibrated and the total resistance of the galvanometer circuit is kept at a proper value.

The use of a volt box, as shown in Fig. 1 (b), gives an expression[5] similar to the preceding. The value as read from the potentiometer dial and galvanometer scale is multiplied by the factor of the volt box used, to get the unknown voltage under measurement.

When a shunt is used, as in Fig. 1 (c), the current through the shunt is in general not equal to the line current which is to be measured, being greater or less than the line current by the amount of the galvanometer current. It has been found that a simple expedient will take this fact into account, and make the reading of the deflection potentiometer (when divided by R, the resistance of the shunt) give accurately the value of the line current i. It is only necessary to count in R as part of the galvanometer circuit, and provide means for keeping this circuit of constant resistance when different values of R are used. The proof of this is reserved for a later paragraph (see p. 410).

Reduced to its lowest terms, the principle of operation of the deflection potentiometer is very simple, as an illustration will show.

Fig. 2 is a repetition of Fig. 1 (a), except that the storage cell e_1 and the source E are not shown. The gap left in the lower circuit by the removal of the storage cell is closed by a wire of resistance equal to that of the cell, which is usually negligible. We thus have the equivalent of a potentiometer with its auxiliary current suppressed. It may be shown that if we regard what is left (galvanometer plus the resistances r_1 and r_2 in parallel) as a voltmeter, calibrating it to read the potential difference between

[5] This Bulletin, 2, p. 230; 1906 (Reprint No. 33). 4, p. 281; 1908 (Reprint No. 79).

the points marked + and −, we may then replace the auxiliary cell e_1 and make measurements either with the null method or with a combination of null and deflection methods. In the latter case the two actions are simply superposed, neither interfering with the other. The galvanometer having been calibrated with a definite resistance in circuit, namely, r_1 and r_2 in parallel, we have to provide the means (not shown in Fig. 2) for keeping

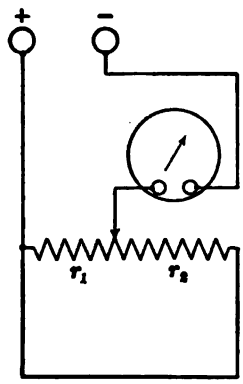

Fig. 2

the galvanometer circuit of constant resistance as r_1 and r_2 vary with the changes of the null part of the result.

3. ARRANGEMENTS OF CIRCUITS

A number of arrangements are possible in planning the circuits of a deflection potentiometer. It is not even necessary that the galvanometer circuit be of constant resistance; in fact, the first instrument constructed has a different value of galvanometer circuit resistance for each setting of the main dial. It is much more convenient, however, as regards facility of construction and subsequent checking, to use the plan of constant resistance. For detailed discussion of various possible plans of circuits the reader is referred to preceding articles.[6]

[6] This Bulletin, 2, pp. 230–234; 1906 (Reprint No. 33). 4, pp. 276–286; 1908 (Reprint No. 79).

The plan of circuits used in the instruments about to be described is shown diagrammatically in Fig. 3.

In this figure, E is the source whose voltage is to be measured, using a volt box of resistance R. The "potentiometer wire" AB is supplied with current from the auxiliary storage cell e_1, this current being regulated by the rheostat r_3 in series with the cell and the rheostat r_6 in shunt with the potentiometer wire. It will be noted that as r_3 is increased, r_6 is decreased; both of these changes tend to reduce the current flowing through the potentiometer wire AB. The values of r_3 and r_6 are so chosen that their resultant resistance in parallel is a constant; thus the resistance from the point B through the regulating rheostats and storage battery to the point A is constant for all settings of the rheostats. This simple expedient removed a very considerable difficulty in the design of convenient circuits for deflection potentiometers.

The resultant resistance in the potentiometer circuit proper thus consists of the resistance r_1 shunted by the sum of r_3 and the constant resistance of the regulating rheostats. This resultant resistance is zero when the sliding contact is at the point A, and increases as the slider moves to the right up to a point somewhat past the middle of AB, then decreases to a minimum value (greater than zero) at the point B. The resistance of the compensating rheostat r_4 is such that (for any setting) the sum of r_4 and the resultant resistance through the potentiometer is constant. The remainder of the resistance in the galvanometer circuit consists of the portion R/p of the volt box shunted by the rest of the volt box, the resultant resistance being figured as if the source E had zero resistance[7] and no emf. If a volt box is not used, the voltage to be measured is applied to the points which (in Fig. 3) are shown as joined by the coil R/p. If the galvanometer circuit was of correct resistance when the volt box

[7] It has been asked whether the deflection potentiometer would not thus be in error if the source E contains appreciable resistance. It has been shown (this Bulletin, 4, pp. 294–297 (Reprint No. 79)) that no error is introduced, whatever the internal resistance of the source. The quantity to be measured is (in such a case) not the internal or total emf of the source, but the difference of potential at the terminals of the volt box R, as in any potentiometer measurement with a volt box.

was used, it is necessary (when working without the volt box) to insert in series with the galvanometer a ballast resistance equal to the resultant of R/p shunted by the remainder of the volt box. The potentiometer will now measure the voltage at its terminals.[8] When current measurements are to be made, the terminals of the potentiometer (volt box having been removed) are connected to

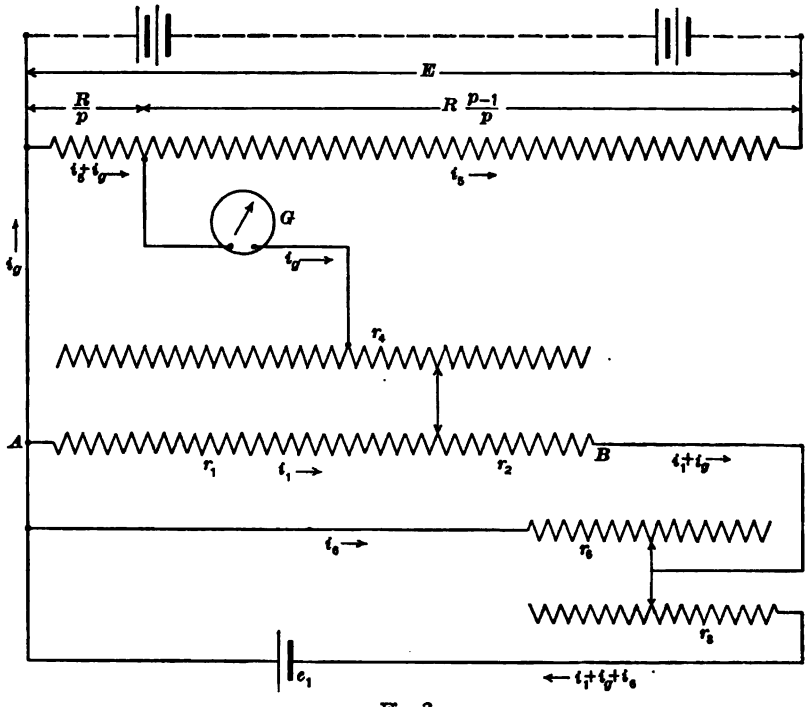

Fig. 3

the potential points of a suitable shunt, whose resistance is to be counted in as part of the required ballast resistance. If another shunt of different value is used, a suitable change must be made in the ballast resistance. By proper arrangement of the dials and switches, the values of resistance for the different circuits are

[8] If it were desired to run "pressure wires" of appreciable resistance from the potentiometer to the points where the voltage is to be measured, the resistance of these pressure wires should be counted as part of this ballast resistance.

obtained automatically; no attention on the part of the operator being required, except when changing shunts.

By providing a suitable switch, it is possible to throw the potentiometer terminals quickly from a current shunt to a volt box, or from one shunt to another, or one volt box to another. In such work as checking indicating wattmeters, or in watthour meter testing, it is convenient to go quickly from voltage to current, and vice versa.

A double-throw switch for this purpose[9] is provided in addition to a similar switch used for making the change from unknown quantity to standard cell.

4. DESCRIPTION OF INSTRUMENTS

Figure 4 shows a plan of circuits of the Model 3 potentiometer, which has been designed for general measurements of current and voltage in laboratories whose requirements include a reasonable degree of accuracy combined with speed of working.

The main dial has 30 steps of 5 ohms each. In series with it is a coil of 1.80 ohms and a standard-cell dial of 10 steps of 0.01 ohm each. The Weston portable *unsaturated* standard cell only is used; it is balanced around the last two-thirds (100 ohms) of the main dial, plus the 1.80-ohm coil and the standard-cell dial. Thus it is possible to use standard cells whose values are from 1.0180 to 1.0190 volts[10] inclusive. This covers the range of variation of these unsaturated cells sufficiently well, as cells may be bought with the specification that they shall fall within this range, and within several units of the lower end of the dial, to

[9] The writer is indebted for this and other valuable suggestions to Dr. C. H. Sharp. The use of a multiple-point switch for enabling measurements on any one of six circuits is a feature of the pioneer Crompton potentiometer. It was not thought desirable to carry this idea further than two circuits in the potentiometers here described. If occasion arises for quickly connecting a potentiometer to any one of three or more circuits, this can readily be done by combinations of double-throw switches, or by receptacles and plugs.

[10] The figures given in this paragraph refer to the value of the 1.80-ohm coil as it has been adjusted to meet the change which occurred January 1, 1911, in the value of the international volt. As at first made, the coil had 1.88 ohms resistance, and cells having values of 1.0188 to 1.0198 volts (on the old basis) were provided for.

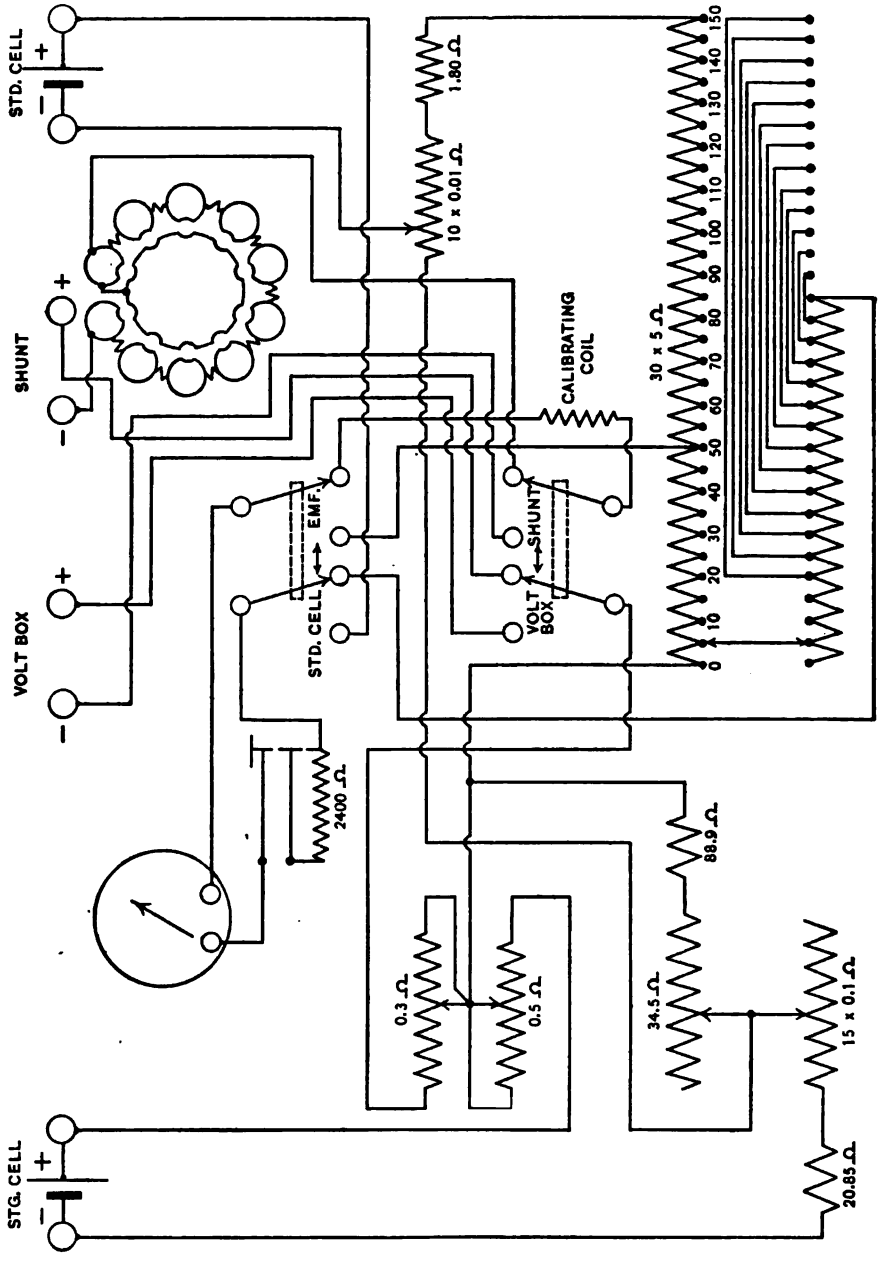

Fig. 4.—*Plan of Circuit, Model 3 Potentiometer.*

Fig. 5.—*Deflection Potentiometer, Model 3 (one-third size)*

provide for the slight decrease of emf to be expected in a period of years. The standard current through the main dial coils is thus 0.01 ampere; it is furnished by a storage cell. The series rheostat (r_3 of the preceding discussion) has a minimum value of 20.85 ohms, and increases by 15 steps of 0.1 ohm each. The shunt rheostat (r_4) has a minimum value of 88.9 ohms, and increases by 15 steps to a maximum of 123.4 ohms. A fine rheostat of 0.5 ohm in the battery circuit covers any step [11] of the coarse rheostat, and has a compensating resistance of 0.3 ohm in the galvanometer circuit. The circuits are so designed that the compensating resistance (r_4) of the main dial repeats at 90, 95, ... 150 the values for 80, 75, ... 20; hence a number of coils are saved by using cross connections.

The galvanometer key has a protective resistance of 2,400 ohms, which is in circuit on the first contact and is cut out on full depression. The total resistance in the galvanometer circuit under working conditions is 60 ohms between the binding posts marked "Volt Box," which with 40 ohms resultant resistance in the volt box makes up the normal total of 100 ohms. The total resistance measured between the binding posts marked "Shunt" is 100 ohms when the circular plug rheostat near these posts is plugged at the extreme right, and is less than this by amounts of 0.1, 0.2, 0.5 ..... 40 ohms when the plug is placed in succession toward the left. This allows the total resistance to be kept 100 ohms when using shunts of the values just given, the resistance of the shunt being counted in the total.

A view of this instrument is given in Fig. 5. The galvanometer is mounted below the hard-rubber top, giving an improvement in appearance over the preceding model. The scale of the galvanometer is made direct-reading by the method of marking [12] used.

The main dial is placed at the right so that the observer can use his right hand for setting it, and for recording results, while

[11] For the explanation of the large value which the fine rheostat must have in comparison with a step on the series rheostat, see this Bulletin, 4, pp. 284–285; 1908 (Reprint No. 79).

[12] This Bulletin, 4, pp. 288–289 (Reprint No. 79).

the left hand is used to operate the galvanometer key. The coarse and fine rheostats are at the left; these require attention only at very infrequent intervals as compared with the main dial. The standard-cell dial is above the main dial; its setting remains unchanged so long as a cell of one particular value is used. The two double-throw switches are at the left of the galvanometer. While slight changes of level produce no appreciable error in the reading of the galvanometer, a small circular level is provided to enable the instrument to be kept in a definite plane when desired.

The aim in producing this particular model has been to supply a deflection potentiometer for general use in electrical engineering laboratories and instrument testing rooms, and in view of the fact that 0 to 1.5 volts is a range very frequently used for potentiometers, and since 150-division scales are convenient and much used in portable instruments, the range of this potentiometer is made 0 to 1.5 volts,[13] the dial being marked from 0 to 150. The decimal point is not put in, since in most cases a potentiometer is used with a volt box or a shunt, introducing a multiplying factor. Ten divisions of the galvanometer scale equal one "dial unit" of 0.01 volt; hence one division corresponds to one millivolt. By estimation one can read tenths of a division, or parts in 15 000 of the total range.

The volt box used with this potentiometer is a special one, having 5 ranges[14] of 4.5, 15, 45, 150, and 300 volts. The only special point in the construction of this volt box is a plug switch, by means of which the resultant resistance introduced by the volt

[13] While the nominal range is 0 to 1.5 volts, the actual range is from −0.03 to +1.53 volts; this extension of the usual range is often very convenient. For example, 150-volt voltmeters often require a little more than 150 volts for full-scale deflection; with a null potentiometer this requires changing the range of the volt box for the full-scale point. With the Model 3 potentiometer, using the 150-volt range of the volt box, the upper limit is 153 volts, which enables one to check an instrument 2 per cent in error.

[14] It is believed that a more convenient set of ranges for the volt box is 3, 15, 30, 150, and 300. With these, the reading of the potentiometer is multiplied by 2 (except for the 15 and 150 ranges) and the decimal point located.

box into the galvanometer circuit is kept constant. The plan of circuits of this volt box is shown in Fig. 6.

The rubber-covered binding posts are located at the back of the potentiometer, and are mounted on a hard rubber apron extending down from the top, so that a cover may be placed over the potentiometer, when it is not in use, without disturbing connections.

The galvanometer requires a current of 10 microamperes per scale division. It has a coil resistance of about 10 ohms, and the

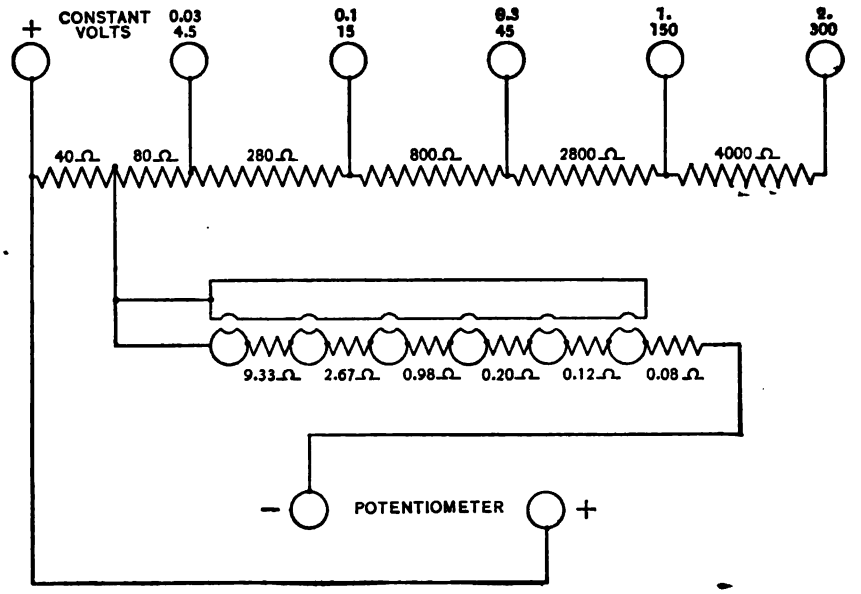

Fig. 6

total resistance in the galvanometer circuit for the condition of aperiodic motion (critical damping) is 100 ohms. On closing the circuit, the pointer comes to its deflected position (without passing it) in 1.2 seconds. The galvanometer was supplied by the Weston Electrical Instrument Co., of Newark, N. J., and the potentiometer was constructed and the galvanometer mounted by The Leeds & Northrup Co., of Philadelphia, Pa.

A view of the potentiometer with its accessories is shown in Fig. 7, which shows a wattmeter connected for test. The voltage

supply line is at the left, a slide resistance being used to set at the desired value. The current supply line (from a storage battery of 2 to 8 volts) is at the right. The current is controlled by a carbon rheostat, and flows through an oil-immersed [15] shunt whose potential terminals are connected to the "shunt" terminals of the potentiometer. The volt box, storage cell, and standard cell are back of the potentiometer. This outfit is convenient for such work, as one observer at the potentiometer can hold the voltage constant and also measure the successive values of current required to produce the desired deflections of the wattmeter. It is desirable to have reversing switches at the wattmeter, to enable two readings to be made at each point tested, reversing the current through the wattmeter for the second reading, to get a mean value in which no error due to (constant) local field enters. These switches have not been shown in the figure.

The special requirements of photometric work make it desirable to have a volt box of high resistance. To meet this condition a deflection potentiometer has been devised which is termed Model 5. It differs from Model 3 in the values of its coil resistances and in the constants of its galvanometer. In general appearance it is a duplicate of Model 3, Fig. 5, and the plan of circuits, Fig. 4, applies to Model 5, except as to values of the coils, and the point on the main dial at which the standard cell tap is taken off. The main dial has 30 steps of 20 ohms each, the last 100 ohms of the dial being in series with a 1.80-ohm coil [16] and a standard cell dial of 10 steps of 0.01 ohm each, providing for the same range of standard cell values (1.0180 to 1.0190 volts) as in Model 3. The coarse and fine rheostats have coils of 4 times the resistance of those of Model 3. The protective resistance in the galvanometer key is 96 000 ohms, being 24 times [17] the normal total resistance

[15] These are usually preferred in the work of the Bureau, as giving greater accuracy; but properly designed air-cooled shunts may be used in many cases.

[16] See note 10, p. 403.

[17] The normal maximum deflection of the galvanometer is 25 divisions, and with a protective resistance of 24 times the normal total, the deflection is to be brought to 1 division or less, by manipulating the main dial. If the key be now fully depressed, the reading will not be over 25 divisions.

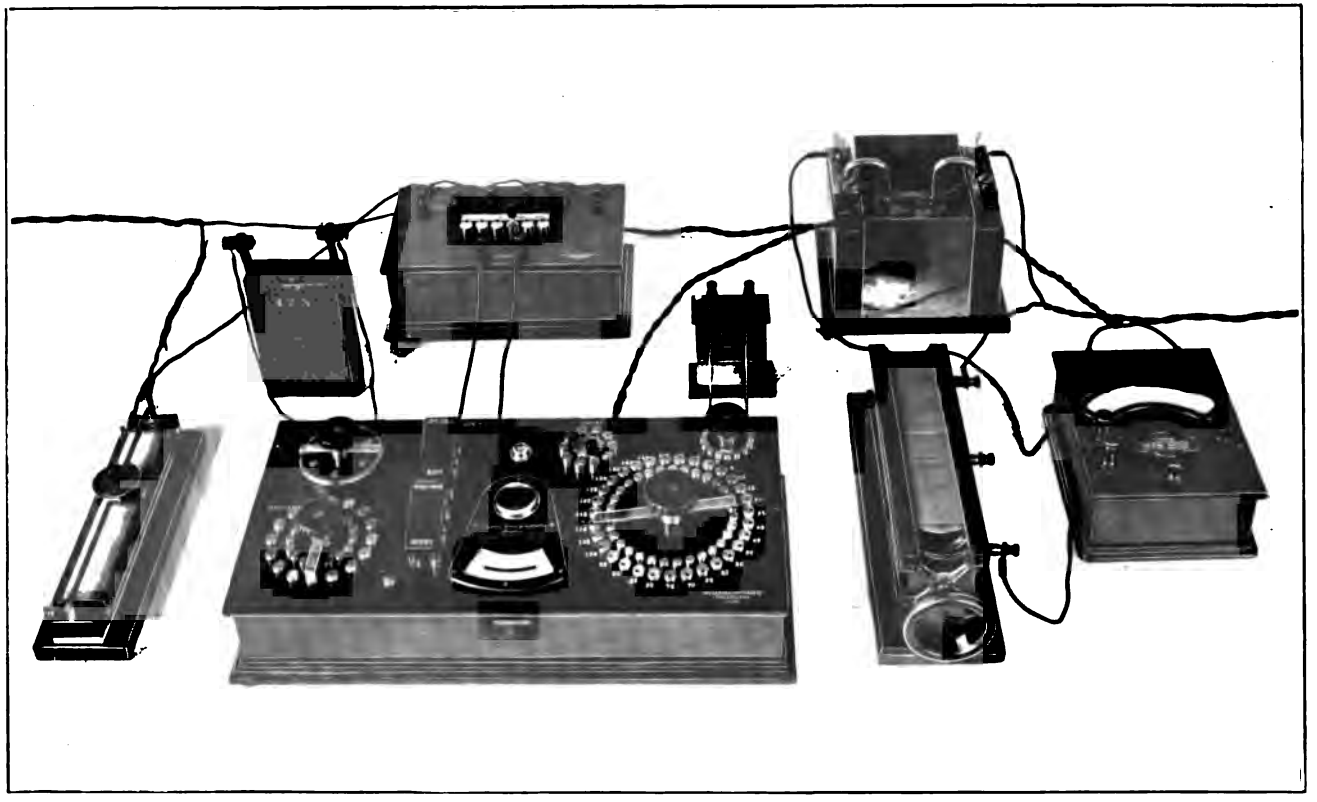

Fig. 7.—*Potentiometer and Accessories, with Wattmeter Connected for Test*

(4000 ohms) in the galvanometer circuit. The design is such that a large part (3000 ohms) of this total resistance is available for the resultant resistance of the volt box. The volt box is similar to the one already described (see Figs. 6 and 7), but has the ranges [18] 15, 45, 150, and 300 volts, the resistance of the 150-volt range being 75 000 ohms, and the other ranges in proportion. This high value reduces the power used by the volt box to 0.16 watt when 110 volts is applied to the 150-volt range. To secure this high volt-box resistance requires (in addition to a high-resistance galvanometer coil) the use of more voltage on the main dial, which has a range (when no volt box is used) of 0 to 6 volts instead of the usual 0 to 1.5 volts. The auxiliary current is supplied by a battery of four small storage cells in an oak case.

The Weston galvanometer used requires a current of 1 microampere per scale division. It has a coil resistance of about 500 ohms, and the total resistance for aperiodic [19] motion is 4000 ohms. The pointer comes to rest in 1 second after closing the circuit.

The high fundamental range of Model 5 potentiometer makes it unsuitable for the measurement of heavy currents, but as photometric work has little or no occasion to exceed 15 amperes, the power spent in the shunt is not important. For use in the laboratories of central stations, electrical engineering departments of colleges, instrument makers, public service commissions, municipal testing bureaus, and others who wish to make electrical measurements over usual commercial ranges with speed and accuracy, the Model 3 potentiometer is to be preferred.

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[18] It would probably be more convenient to have a 30-volt range instead of 45; see note 14, p. 406.

[19] It should be more generally recognized that a moving-coil galvanometer ought to be so proportioned to the circuit in which it is to be used that the condition of critical damping exists. If the external resistance is below the value required for this condition, the galvanometer coil will "creep" slowly to its position of rest, resulting in loss of time and greater liability of error. If the external resistance is too high, the motion is periodic, and time is lost while the observer waits for the oscillations to subside. If the condition of critical damping can not be had, it is better to have slight underdamping.

The accuracy of measurement of the two models is the same, and a conservative figure is 0.4 of a galvanometer scale division (0.04 dial unit), which is equivalent to one twenty-fifth of 1 per cent for measurements at two-thirds of the main dial. In the two instruments in use at this Bureau, the accuracy is about twice as good as the above.

5. USE OF A CURRENT SHUNT WITH THE DEFLECTION POTENTIOMETER

In referring to the use of a current shunt with the deflection potentiometer, it was stated that no error is caused by the abstraction of some of the line current from the shunt to operate the galvanometer, provided the resistance of the shunt is counted in as a part of the normal total resistance in the galvanometer circuit. The proof of this statement will now be outlined.

In Fig. 8 a source E provides the current i whose value is to be measured. With the potentiometer balanced so that the galvanometer current is zero, all of the current i flows through the standard resistance or shunt R. With a current i_g flowing through the galvanometer in the direction indicated, the current through R is less than i; with a current i_g in the opposite direction, the current in R is greater than i. For the sake of generality it is assumed that there is other resistance R' in the source circuit. The potentiometer consists of the circuit r_1+r_2 supplied with current from the auxiliary battery e_1. A sliding contact connected to one terminal of the galvanometer enables the value of r_1 to be varied while r_1+r_2 remains constant. For simplicity the regulating rheostats r_3 and r_6 of the preceding figure are not shown, since it may be seen that their resultant resistance may be thought of as part of r_2. The compensating resistance r_4 also is omitted, as it is in series with the galvanometer only, and may be thought of as a part of the galvanometer resistance.

From Kirchhoff's laws we have:

$$i_1r_1+(i_1-i_g)r_2-e_1=0$$

$$i_1r_1+i_gr_g-(i-i_g)R=0$$

and the solution for i gives

$$i=\frac{e_1\frac{r_1}{r_1+r_2}+i_g\left(r_g+\frac{r_1r_2}{r_1+r_2}+R\right)}{R} \tag{31}$$

The first term in the numerator is the emf indicated by the setting r_1 on the potentiometer wire r_1+r_2. The second term is

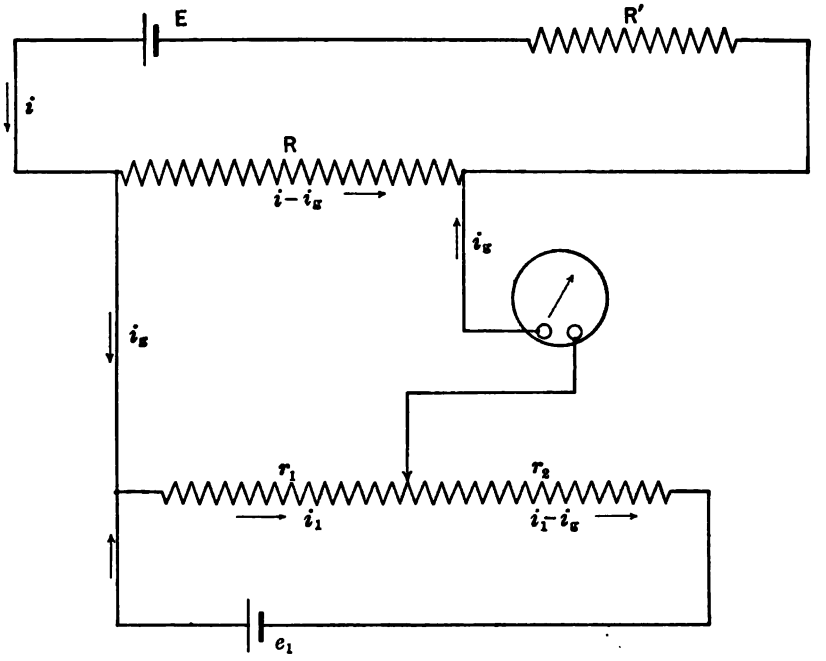

Fig. 8

the ohmic drop due to the galvanometer current i_g assumed to be flowing around the closed path through the potentiometer and through the shunt R; hence, if the galvanometer be calibrated so that it indicates correctly as a voltmeter when it has in series with it a resistance equal to $r_1r_2/(r_1+r_2)+R$, the sum of its reading and the null reading $e_1r_1/(r_1+r_2)$ will (when divided by R, the resistance of the shunt) give the value of the current i to be measured. The important thing about this expression for i is

that it does not contain[20] the terms E and R', the emf and the resistance in the source. Hence, all that is necessary is to provide means for keeping the galvanometer circuit of the proper constant resistance when using different shunts, to get a measurement of the line current i which shall be free from error, regardless of the amount of current taken from the shunt to operate the galvanometer.

If we assume that the shunt R is removed, and the object is to use the same potentiometer for voltage measurements, we find on solving for the emf of the source

$$E = e_1 \frac{r_1}{r_1 + r_2} + i_g \left(\frac{r_1 r_2}{r_1 + r_2} + r_g + R' \right) \tag{32}$$

Hence, to measure the total emf of the source, all that is necessary is to count in its internal resistance R' in providing the proper total galvanometer resistance. The preceding equation may be written in the form

$$E - i_g R' = e_1 \frac{r_1}{r_1 + r_2} + i_g \left(\frac{r_1 r_2}{r_1 + r_2} + r_g \right) \tag{33}$$

This shows that when the potentiometer is to measure the external difference of potential of a source whose resistance has any value R', the resistance in the galvanometer circuit should be complete without using R'.

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[20] The terms E and R' are really in the expression in disguise, as the galvanometer current i_g depends upon both. However, the value of i_g is known from the reading of the galvanometer, so that no knowledge of conditions in the source is required. This is analogous to the case of the deflection potentiometer used with a volt box for voltage measurements. In the latter case the current through the galvanometer depends upon the value of the emf and resistance of the source, but the value of the voltage at the terminals of the volt box (which is the quantity to be measured) is obtained without the necessity of knowing anything about the source.

By taking the upper part of the network in Fig. 8, we get the equation

$$iR' + (i - i_g) R - E = 0.$$

Using this with the two Kirchhoff equations for the rest of the network will enable a more general solution for i to be obtained, free from i_g and containing only e and r terms. This is of no value in potentiometer design, but is of interest in some special cases.

It will thus be seen from the foregoing equations that the potentiometer ought not to be narrowly regarded as a null instrument only, since in the general case of any current flowing through the galvanometer which the latter can measure, the principles of operation of the resulting deflection potentiometer (or "generalized potentiometer") are simple, and may be conveniently carried out in the construction of accurate and quick-working standard instruments for current and voltage measurements.

With the older forms of potentiometer it has been the common practice to use current shunts whose values are decimal multiples or submultiples of an ohm, as 10, 1, 0.1, 0.01 ohm, and so on. When used with a 1.5-volt potentiometer, these shunts are convenient for certain ranges; for example, a 0.1-ohm shunt is just what is desired for use in checking a 15-ampere ammeter of 150 divisions, or a 10-ampere ammeter of 100 divisions. However, for a 3-ampere or 7.5-ampere instrument such a shunt is not the most convenient, as the reading of the potentiometer requires (in addition to the location of the decimal point) the use of the factor 2 (or 5). This requires some calculation during the test to determine for the various readings of the ammeter the corresponding settings of the potentiometer in order not to waste time hunting for the balance. It also requires computation in working up the results and in scrutinizing the work for possible errors. Inasmuch as the speed of working of the deflection potentiometer is much greater than that of null potentiometers, it is well to avoid loss of time in auxiliary matters, so as to bring the whole system of measurement up to a high state of efficiency. It is easy to choose convenient shunt values that will be free from the foregoing objections.

Assuming that the fundamental range of the potentiometer is, say, 150 "dial units", for rapid and convenient work one scale division on an instrument under test should correspond to 1 dial unit. To accomplish this, the current per scale division of the ammeter, multiplied by R, the resistance of the shunt, must equal 1 dial unit; or

$$R \text{ of shunt} = \frac{\text{emf corresponding to 1 dial unit}}{\text{amperes per division of ammeter}}$$

Since in practice both numerator and denominator will have integral values, the shunt values called for are not abnormal, as examples will show. Take the case of Model 3 (or any 1.5-volt) potentiometer. One dial unit equals 0.01 volt; hence, for ammeters of 1, 2, 5, 10, and 20 amperes per scale division the current shunts required are 0.01, 0.005, 0.002, 0.001, and 0.0005 ohm, respectively. A single shunt will usually do for several ranges; thus the 0.01-ohm shunt is adapted for 100-ampere 100-division, 120-ampere 120-division, and 150-ampere 150-division instruments. The same shunts are equally convenient in testing wattmeters of corresponding current range; in this case 100 volts is applied to nominal 110-volt potential circuits.

Examination of the ammeter ranges given in a prominent maker's catalogue showed that over the whole range from 150 milliamperes to 10 000 amperes, the following shunt values would be required (neglecting the decimal point): 1, 1.25*, 2, 2.5*, 3*, 4*, 5. The values marked * will seldom be needed, as the values 1, 2, and 5 (neglecting the decimal point) will provide for over 80 per cent of the ranges.

With such shunts, the observer at the ammeter sets successively on 10, 20, 30.....divisions of the scale, calling out these numbers; the observer at the potentiometer sets the main dial to the same numbers, and depresses the key. The small deflection of the galvanometer gives the correction to be applied to the instrument under test, one division of the galvanometer corresponding to 0.1 division of the ammeter; deflections to the right are + corrections, to the left, −. The work may be plotted on the form [21] shown in Fig. 9, putting a pencil mark on the proper vertical line; if the galvanometer reads 2 divisions to the right, the current is greater than the reading of the ammeter, and the pencil mark is put two divisions below the zero line of the chart; if the galvanometer reads 1 division to the left, the mark is put one division above the zero line on the chart. The scale points may be checked several times if desired, and a smooth curve drawn through the

[21] This form is recommended in the Meter Code of the National Electric Light Association; see report of committee on meters, N. E. L. A., for 1910, p. 141.

pencil marks. Thus a correction curve may be quickly drawn without recording a single figure, and without any computation; much time may be saved and errors avoided.

In this connection may be mentioned a neat arrangement [22] of shunts which is used by a German manufacturer, and which would be convenient for use with the deflection potentiometer. As shown in Fig. 10, several shunts are soldered together, and their free ends are connected to a millivoltmeter. The line current enters at A, and may leave at B, C, or D. The section AB is of relatively low resistance and large carrying capacity; BC is

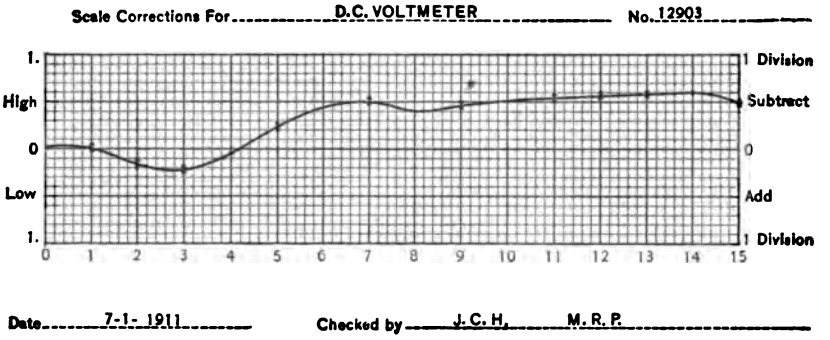

Fig. 6

of lower carrying capacity and higher resistance, and so on. The advantages of this method are as follows

1. In changing the range, but one connection has to be shifted.

2. The millivoltmeter circuit remaining closed, the connections may be well made, and errors due to dirty or loose contacts are avoided. Evidently, the millivoltmeter could have all connections soldered if the shunts were attached to it and did not need to be changed. The one movable contact is in the line, and variations of its resistance do not affect the accuracy of the result.

When applied to the deflection potentiometer, this set of shunts has the advantage that the plug rheostat ordinarily used for keeping constant galvanometer resistance for different shunts can be

[22] Hallo und Land; Elektrische und Magnetische Messungen und Messinstrumente, p. 254.

set to a value marked with the sum of the resistances of the set. Then the plug need not be changed, since no matter where the line current leaves the set, the resistance in the galvanometer circuit is constant.

6. CONCLUSION

The "Outline of Method of Design" given in a previous paper [22] has been revised and considerably extended, and will be published as a separate article. With it are included some notes on the design of moving-coil galvanometers, showing how to determine the important constants of an existing type of galvanometer, and what changes to make in spring strength, magnet

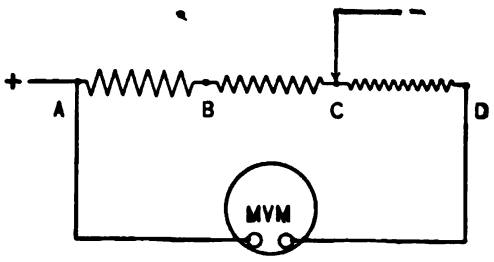

Fig. 10

strength, and size of wire in order to produce a galvanometer of certain desired constants.

Further progress in the development of the deflection potentiometer, such as the attainment of higher volt-box resistance, or lower fundamental ranges, will depend upon the improvements possible in the galvanometer. While the galvanometers used in the two potentiometers just described are of high grade, there is reason to look for still better performance due to certain changes of dimensions and material of coils. It is hoped that these improvements may be realized in the near future.

The deflection potentiometer combines, in suitable proportions, the accuracy and reliability of the (null) potentiometer with the quickness of working, ease of reading, and independence of fluctu-

[22] This Bulletin, 4, p. 298; 1908 (Reprint No. 79).

ations which characterize a well-damped deflection instrument. It makes the potentiometer no longer a time-consuming device, to be resorted to only at intervals when necessity demands, and avoided meanwhile by the use of "secondary standard" deflection instruments, which are but little better than the portable instruments they are used to check. With a deflection potentiometer in use, the manipulation and reading require but little more training, effort, or time than the reading of a portable instrument, and every day's work of instrument checking is done as it should be, namely, by reference to standard resistances and standard cells.

WASHINGTON, June 23, 1911.

OUTLINE OF DESIGN OF DEFLECTION POTENTIOMETERS, WITH NOTES ON THE DESIGN OF MOVING-COIL GALVANOMETERS

By H. B. Brooks

1. INTRODUCTION

This paper was originally planned as a continuation of an article[1] entitled, "Deflection Potentiometers for Current and Voltage Measurements," to which the reader is referred. The object of the present paper is to outline the principles on which deflection potentiometers are designed. It is thought that the matter here presented will be of interest not only to instrument makers and designers, but also to many users who are interested in the development of the potentiometer method. This method in its purely null form has done and is doing great service in the physical laboratory, where the highest precision must be had, regardless of the outlay of time and labor. It has also found some use in technical applications, such as the laboratories of the larger central stations. In such work, it has always labored under the disadvantages attending any balance method, namely, the large amount of time required, and the necessity for steady sources of current and voltage. The deflection potentiometer combines in any desired proportions the accuracy and reliability of the (null) potentiometer and the quickness of working and independence of fluctuations of a well-damped deflection instrument. It is therefore especially suitable for a large class of work which does not require extreme precision, but in which a high degree of reliability is necessary.

[1] This Bulletin, 8, p. 395; 1911 (Reprint No. 172).

While the essential principle of operation of the deflection potentiometer is very simple,[2] the requirements of speed of working, accuracy, and the use of a pivoted portable galvanometer demand considerable care in the design. The methods found most suitable by the writer will be considered in detail.

2. OUTLINE OF DESIGN OF DEFLECTION POTENTIOMETERS

In the "Outline of Method of Design" previously given,[3] the assumption was made that a volt box was to be used. In the more recent instruments it has been found more convenient to start with assumptions which apply directly to the potentiometer alone, and to consider the volt box later.

As emphasized in the preceding articles, the galvanometer is the central and important feature in the design; its constants determine what can be obtained in the way of sensitiveness of reading, speed of working, resistance of volt box, and resistance of the potentiometer circuits.

The quantity denoted by Σr in the preceding articles is the total resistance in the galvanometer circuit; it should have the proper value for the condition of critical damping.[4] This is a frequently neglected but very important galvanometer constant. With this value of total resistance the motion of the coil is "dead beat," and when a change of current takes place the coil moves to its new position without oscillation. This condition is assumed in the design of deflection potentiometers, in which the combination of accuracy and speed is the paramount feature. Σr thus denotes the sum of galvanometer resistance (including resistance of springs), the resultant resistance[5] $r_1 r_s/(r_1+r_s)$ of the potentiometer wire and rheostats, and the resistance which may be allowed in an accessory (volt box or current shunt). In a former article[6] it was shown that if $1/p$ is the "volt-box fraction," m the

[2] This Bulletin, 8, p. 399; 1911 (Reprint No. 172).

[3] This Bulletin, 4, p. 298; 1908 (Reprint No. 79).

[4] This is sometimes called the "total critical resistance."

[5] The symbol r_s is used here to denote the sum of r_2 and the constant resistance of the shunt rheostat r_6 in parallel with the series rheostat r_3. (See Fig. 1.)

[6] This Bulletin, 2, p. 233; 1906 (Reprint No. 33).

number of divisions deflection to be produced by 1 volt change in line voltage, and I the galvanometer current in amperes for a deflection of one division, then

$$\Sigma r = \frac{1}{pmI} \tag{11}$$

Putting $p = 1$ gives a special case in which the volt box becomes unnecessary, for if one were used it would be a single resistance across the source (whose voltage is to be measured) and in parallel with the potentiometer; it is therefore only a superfluous load on the source and does not affect the operation of the potentiometer. Omitting the volt box, the fundamental design equation becomes

$$\Sigma r = \frac{1}{nI}, \text{ or } I\Sigma r = \frac{1}{n} \tag{34}$$

in which n denotes the number of divisions deflection produced by a change of 1 volt at the potentiometer terminals. Equation (34) may be written in the form

$$\frac{1}{\Sigma r} = nI$$

That is, the *conductance* of the whole galvanometer circuit varies directly as the number of divisions to be given per volt, and as the current in amperes for one scale division. For any given galvanometer, Σr and I are fixed,[7] and hence n is determined. For commercial reasons one would usually try to adapt an existing type of galvanometer to the purpose, using it as a starting point and making changes until the proper values of Σr and I are obtained. In general, the total resistance for critical damping should be from 5 to 10 times the coil resistance, and the period should be short, say about 1 to 1.5 seconds. The combination of

[7] The resistance for critical damping is not sharply defined, and hence some range of values is permissible for Σr. However, it is better to have Σr slightly greater than the critical value, rather than the reverse, in order that the pointer may pass slightly beyond the position of rest and then return to it. With too low a value of Σr, creeping occurs and the impression is given that friction is present.

these two qualities calls for rather high flux density in the air gap.[8] The value of n, the number of divisions per volt, would usually be chosen arbitrarily. This fixes the value of the product $I\Sigma r$; the proposed galvanometer is then examined to see how nearly it meets this requirement. Further, it is necessary that Σr should be high enough to permit sufficient resistance in the potentiometer and its adjuncts; it is also necessary that the galvanometer shall have sufficient sensitiveness when used for checking the current in the potentiometer wire by reference to the standard cell.[9]

Assuming that the values of total resistance Σr and the product $I\Sigma r$ are satisfactory,[10] the upper limit of the potentiometer range may be chosen. A range in general use with null potentiometers is 0 to 1.5 volts. This has the advantages of requiring but one storage cell, and of being of the same order of magnitude as the voltage of the standard cell. The total resistance Σr is to be apportioned to the various parts in a somewhat arbitrary manner, according to the judgment of the designer and the requirements to be met. First, we may choose the external resistance; that is, the resistance of the portion of the volt box across which the potentiometer is to be connected. In general, with an average design, about 40 or 50 per cent of Σr would be allotted to this. The resistance assigned ought to have some even and usual value (such as 20, 40, 50, 100 ..... ohms) to avoid the expense of winding and adjusting for odd or special values. The value assigned to this purpose is also the value of the current shunt

[8] W. P. White, Physical Review **23**, p. 385 (1906), gives a table showing the approximate field strengths required for different periods, with various ratios of total resistance to coil resistance.

This high value of ratio (5 to 10) is necessary for two reasons: First, it makes the reading of the galvanometer (for a given voltage applied to Σr) practically independent of temperature; second, it gives the ability to use higher values of resistance in the potentiometer and in its accessories (volt box or current shunt). By reducing the value of H in the air gap of the galvanometer, Σr being reduced to retain the condition of critical damping, one can increase the volt sensibility, but at the expense of the two important features just mentioned.

[9] For a discussion of this requirement, see pp. 429–431.

[10] The question of changes to be made in the galvanometer, in case it does not have suitable constants, is treated in another part of this paper (p. 434).

of the highest resistance which can be used regularly; this is a minor consideration in most cases.

Having chosen the permissible external resistance, add to it the galvanometer resistance, and 5 to 10 per cent of Σr to be reserved [11] in a calibrating coil which can be varied if at some

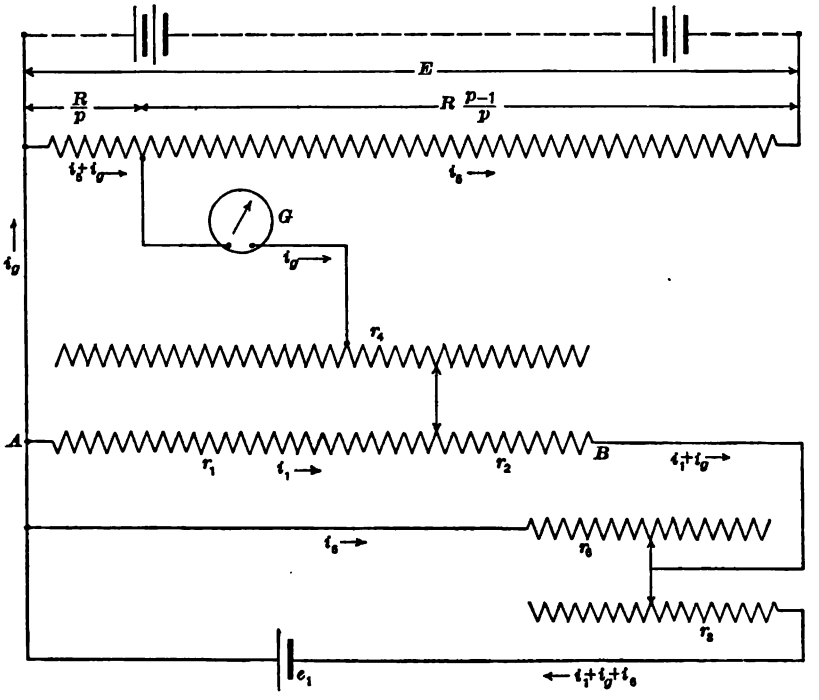

Fig. 1

future time the galvanometer shows a change of sensitiveness. This sum is to be deducted from Σr, and the remainder gives

[11] This margin in the value of Σr is convenient, making it unnecessary to have an exact value of I, the current per scale division, as the error in I can be compensated by varying Σr. It is important, however, that the scale of the galvanometer should be proportional. With careful workmanship, the departure from proportionality can be kept within one or two tenths of a division.

r_m, the internal resistance available for the potentiometer circuits; that is, the constant quantity[12] (see Fig. 1, p. 423).

$$\frac{r_1\left(r_2+\frac{r_3r_6}{r_3+r_6}\right)}{r_1+\left(r_2+\frac{r_3r_6}{r_3+r_6}\right)}+r_4$$

If the maximum dial setting is 1 volt or over, in a potentiometer supplied from a single (lead) storage cell, the slider of the main dial reaches or passes the center of the resistance denoted by the denominator of the fraction in the above expression; at the center the resultant resistance of the two halves in parallel is a maximum and is equal to one-fourth of the total. Hence this total may have four times the resistance r_m reserved for use in the internal circuits. As the value of the main dial resistance r_w is rather sharply limited in some cases by the restrictions due to the use of values of r_3 and r_6 which give constant resultant resistance, this point should now be investigated.[13] If the proposed upper limit of the potentiometer range is not enough below the voltage of the storage cell (or battery) intended to be used, the limits for r_w may be too narrow, and another cell should be added. Having a suitable margin, choose[14] a value for r_w such that each coil is of a convenient (and if possible, a usual) number of ohms, such as one of the following: 1, 2, 5, 10, 20, ... Having chosen r_w, see how much extension of this is necessary between the upper end of the dial and the rheostats r_3 and r_6, to provide for the deci-

[12] The significance of this quantity may be best seen by reference to this Bulletin, 4, p. 281 (Reprint No. 79). This quantity is part of the denominator in equation (17), and is discussed.

[13] This Bulletin, 4, pp. 283 and 284; 1908 (Reprint 79). See discussion of equation (20).

[14] In choosing r_m and r_w, if possible select values such that the maximum resultant resistance r_m is equal to an integral number of steps of the main dial, or to $(n+0.5)$ steps, where n is an integer. This enables a number of ballast coils for the main dial to be dispensed with, thus lowering the cost. For a numerical example see footnote 22, page 429.

As a first approximation, r_w may be taken as $3r_m$ to $3.6r_m$, for potentiometers of 1.5 volts and over. Below 1.5 volts in general a part of r_w will not be in the main dial, but will constitute an extension of this dial. The resistance r_m may then fall short of one-fourth of the resistance r_w+r_s; that is, r_w may have a value higher than is given by the general rule.

mal part of the emf of the standard cell. (See Fig. 2.) This increases r_w to an odd value which is used in getting the constant[15]

$$r_w(4r_m - r_w)/4ar_m$$

This constant, when multiplied by the successive values of e_1, the battery emf, gives the successive values of r_3, the series rheostat. Suitable upper and lower limits of e_1 (per cell) are 2.14 and 1.98 volts; using these, compute the two values of r_3, and take their difference. This difference is to be divided into a number of equal steps, the number depending on the degree of constancy of Σr desired, since the fine rheostat can not be perfectly compensated. The degree of constancy can, however, be made as perfect as desired. (See pp. 429 and 433.) In the deflection potentiometers so far designed for precision purposes, r_3 increases from its minimum value by 15 equal steps. The above limits for e_1 may be varied a little, so as to get a convenient value per step of r_3.

Having values of r_3 corresponding to each step, values of r_6, the resistance in parallel with the potentiometer wire, may be computed from the formula:

$$\begin{aligned} r_6 &= \frac{(4r_m - r_w)r_3}{r_3 - (4r_m - r_w)} \\ &= \frac{kr_3}{r_3 - k} \end{aligned} \tag{22}$$

If values of r_6 at first obtained are extreme, it may be necessary to revise the values of r_w or r_m so as to give workable values of r_6. In general, the best values are found for r_6 by taking a value for r_w near the mean value given by

$$r_w = \frac{2(a + e_1)r_m}{e_1} \tag{23}$$

where a and e_1 are the lower and upper limits of the battery emf.[16]

The ballast coils (r_4) of the main dial are given such values as will make the sum of the ballast resistance and the resultant resistance $r_1r_5/(r_1 + r_5)$, a constant, r_m.

[15] This Bulletin, 4, p. 283; 1908 (Reprint No. 79). See equation (21) and context.

[16] This Bulletin, 4, p. 284; 1908 (Reprint No. 79). It is sometimes convenient to solve formula (23) for r_m, and work from an assumed value of r_w which gives a desirable integral value of resistance per step of the main dial to determine whether the resulting value of r_m is attainable. It should be noted that (23) is an arbitrary relation only, being a mean between two impossible extremes.

The fine rheostat demands special consideration, as explained in a previous paper.[17] Since the coarse rheostat accomplishes a given change in the current through the potentiometer wire by the united effect of a change of the series resistance r_3 and a simultaneous change of the shunt resistance r_6, while the fine rheostat uses series resistance only, an amount of the latter greater than one step of the series rheostat must be used. In a previous article[17] a formula was developed for computing the value of the fine rheostat. In deriving this, the series rheostat r_3 was assumed to have its calculated value for each step, to which the fine rheostat could add any resistance from zero up to its full value. A better plan is to make the values of r_3 low by a constant amount, equal to one-half of the fine rheostat. Then the resultant resistance, $r_3r_6/(r_3+r_6)$, has its normal value when half of the fine rheostat is in circuit, and the departure of this resultant resistance from the normal value, due to adding or subtracting half of the fine rheostat, is but half as great as if r_3 had its normal value and the whole of the fine rheostat were added. This method of keeping down the deviation of the resultant resistance from its normal value was used in a previous form of deflection potentiometer.[18] The value of the fine rheostat is given with all required accuracy by the older formula, so long as the steps of the coarse rheostat are not large. However, a more accurate formula may be obtained, as follows. Assume that the battery voltage is at the upper limit for which the coarse and fine rheostats provide, the coarse rheostat r_3 requiring half of r_8, the fine rheostat, to give the normal value of resultant resistance of series and shunt rheostats in parallel. Now let the battery voltage drop so that the fine rheostat is all cut out; thus the series rheostat has the resistance $r_3-r_8/2$, and the shunt resistance has the value r_6. For this arrangement of circuits, a certain current will flow through the potentiometer wire. To be ready to take care of further drop in battery voltage, we must now cut out one step, Δr_3, of the coarse rheostat, and throw the fine rheostat all in; the series resistance now has the value $r_3-\Delta r_3+r_8/2$, and the shunt resistance has the value $r_6+\Delta r_6$.

[17] This Bulletin, 4, pp. 284–5; 1908 (Reprint No. 79).
[18] This Bulletin, 4, p. 291 (Reprint No. 79).

If r_8 has the correct resistance, the current will be the same as before. The general expression for the current through the potentiometer wire is (when no current flows through the galvanometer)

$$i=\frac{e_1}{r_w+\frac{r_w+r_6}{r_6}r_3} \tag{24}$$

Since e_1 is the same for the two cases just assumed, we may equate the corresponding expressions for the effective resistance, as given by the denominator of the right-hand member of (24); this gives

$$r_w+\frac{r_w+r_6}{r_6}\left(r_3-\frac{r_8}{2}\right)=r_w+\frac{r_w+r_6+\Delta r_6}{r_6+\Delta r_6}\left(r_3-\Delta r_3+\frac{r_8}{2}\right)$$

from which

$$r_8=\Delta r_3\frac{1}{1+\frac{r_w\Delta r_6}{2r_6\,(r_w+r_6+\Delta r_6)}}+\Delta r_6\frac{r_3r_w}{r_6(r_w+r_6+\Delta r_6)+\frac{r_w\Delta r_6}{2}} \tag{35}$$

This exact formula gives (for Model 3) values of r_8 about 0.7 per cent lower than are given by the older formula, equation (25). Since one would ordinarily allow at least 10 per cent margin above the computed value, in specifying the value to be aimed at by the maker, this small difference is insignificant. However, in special cases, with unusual values of series or shunt resistance, and with fewer steps in the main rheostat, it might be important to use the accurate formula (35). It will be found that different values of r_8 are required for different steps on the coarse rheostat. In the Model 3 potentiometer the value of r_8 required with all of the series rheostat r_3 in circuit is 0.355 ohm, and with but one step of r_3 in use, 0.421 ohm. The higher value should of course be used and a small margin allowed for the maker's convenience. The specifications for Model 3 call for a fine rheostat of nominally 0.5 ohm, the maximum and minimum permissible values being 0.55 and 0.45 ohm.

When the fine rheostat r_8 is in any but its mid-position the resultant resistance $r_3r_6/(r_3+r_6)$ does not have exactly its normal value. The small error in the value of Σr so introduced is greatest when the main dial setting is a maximum and the coarse rheostat

r_s is a minimum. In a previous paper[19] is given a numerical example of the calculation of the resulting maximum error in the galvanometer deflection (about 0.2 per cent), which would be below the usual limit of reading. This error is compensated (at settings of the main dial near the upper limit) by the use of a fine rheostat in the galvanometer circuit, as shown in the diagram of connections (Fig. 2, p. 430). These two rheostats are combined, a single contact lever running over both spirals.

A numerical example (given in the following seven pages) will illustrate the general method of design.

It is desired to have a fundamental range of 1.5 volts, which must be readable directly to a part in 1,500, and by estimation of tenths of a galvanometer division, to a part in 15,000. One scale division is thus to equal $1.5 \div 1{,}500 = 0.001$ volt; hence by (34)

$$I\Sigma r = 0.001$$

By modifying an existing type of galvanometer this value of $I\Sigma r$ was obtained. The value of I (current in amperes per scale division) was 0.00001; hence,

$$\Sigma r = \frac{0.001}{0.00001} = 100 \text{ ohms.}$$

The coil was critically damped for this value of the total resistance of the galvanometer circuit, and this total is over 10 times the coil resistance; hence the galvanometer fulfills the requirements. The period is about 1.2 seconds. The only improvement which would be desirable is that of a higher value of Σr, 100 ohms not being as high as is desirable.

Of the 100 ohms, 40 ohms will be assigned to "external" resistance (volt box or shunt); adding 10 ohms for galvanometer resistance and 10 ohms for the calibrating coil gives 60 ohms; $100 - 60 = 40$ ohms is thus available for "internal" resistance. The range being over 1 volt, and a single storage cell being evidently sufficient, the resistance of the potentiometer wire r_w plus the constant resultant of rheostats r_s and r_g in parallel will be $4 \times 40 = 160$ ohms. The resistance of the main dial will be from 3 to 3.6 times 40, or 120 to 144 ohms. This raises the question of the resistance of each coil of the main dial. The range of the galvanometer is 30 divisions on each side of the central zero; this covers a total of 0.060 volt. This scale is planned to provide for 0.050 volt, the other 0.005 at each end being left as margin, not ordinarily used. Thus the number of steps in the main dial is $1.5 \div 0.050 = 30$. The value of each coil of the main dial is thus 4 to 4.8 ohms. As 10 per cent is a liberal allowance for the calibrating coil, we may reduce it slightly, and allow more "internal" resistance; it may then be possible to allow 5 ohms per coil, or 150 ohms for the main dial. One hundred ohms

[19] This Bulletin, 4, p. 291; 1908 (Reprint No. 79).

of this dial would have 1 volt drop; to provide for the decimal part of the value of a Weston cell we must add 1.9 ohms to the dial; r_w is thus 151.9 ohms. This must be checked to see whether proper values of series and shunt rheostats (r_3 and r_6) can be obtained. First, use the arbitrary formula (23), solving for r_m; then substitute numerical values.

$$r_m = \frac{e_1}{2(a+e_1)} r_w = \frac{2.14}{2(1.519+2.14)} \times 151.9 = 44.4 \text{ ohms.}$$

This is close to the desired value (40 ohms), but requires further check. The value of the shunt resistance r_6 is given by the equation [20]

$$r_6 = \frac{e_1}{a} \frac{4r_m - r_w}{\frac{e_1}{a} - \frac{4r_m}{r_w}} \qquad (20)$$

Suitable upper and lower limits of e_1 for a lead storage cell are 2.14 and 1.98 volts; a, the drop on the potentiometer wire plus extension for standard cell, is 1.519 volts. The minimum value of r_6 occurs at the maximum value of e_1. Substituting,

$$r_6 = \frac{2.14}{1.519} \frac{(4 \times 40) - 151.9}{\frac{2.14}{1.519} - \frac{4 \times 40}{151.9}} = 32.1 \text{ ohms.}$$

The maximum value of r_6, for $e_1 = 1.98$ volts, $= 42.2$ ohms. These values are rather low, as with 32 ohms in parallel with the potentiometer wire of 151.9 ohms, the current through the shunt is nearly five times the useful current.[21] To improve this condition we may reduce the calibrating coil by 2.5 ohms, increasing r_m to 42.5 ohms.[22] With this value, the use of formula (20) gives 88 and 128 ohms, respectively, for the lower and upper values of r_6. These are satisfactory, the average current in the shunt circuit being 1.4 times the current in the potentiometer wire.

At this point we must investigate an important question, namely, the precision with which the auxiliary current through the potentiometer wire can be checked by reference to the standard cell. It is clear that any error in adjusting this current will enter into the null part of the result, which is the main portion. It has been sought (in the instruments so far designed) to have such a sensitiveness that the auxiliary current can be set to 1 part in 10 000. The electromotive force of the standard cell being a little over 1 volt, this requires that the galvanometer shall give a perceptible movement of the pointer for the current due to 0.0001 volt, the resistance in circuit being the sum of the internal resistance of the standard cell, the galvanometer resistance, and the resultant resistance

[20] This expression for r_6 should be used only at this preliminary stage; after values of r_3 have been determined, the values of r_6 are more conveniently found by using equation (22). See p. 425.

[21] A further disadvantage in having r_6 small as compared with r_w is that the fine rheostat must have a value unduly large compared with that of one step of the coarse rheostat r_3.

[22] The value 42.5 satisfies the condition given in footnote 14, p. 424, enabling 13 coils in the ballast resistance r_4 to be dispensed with.

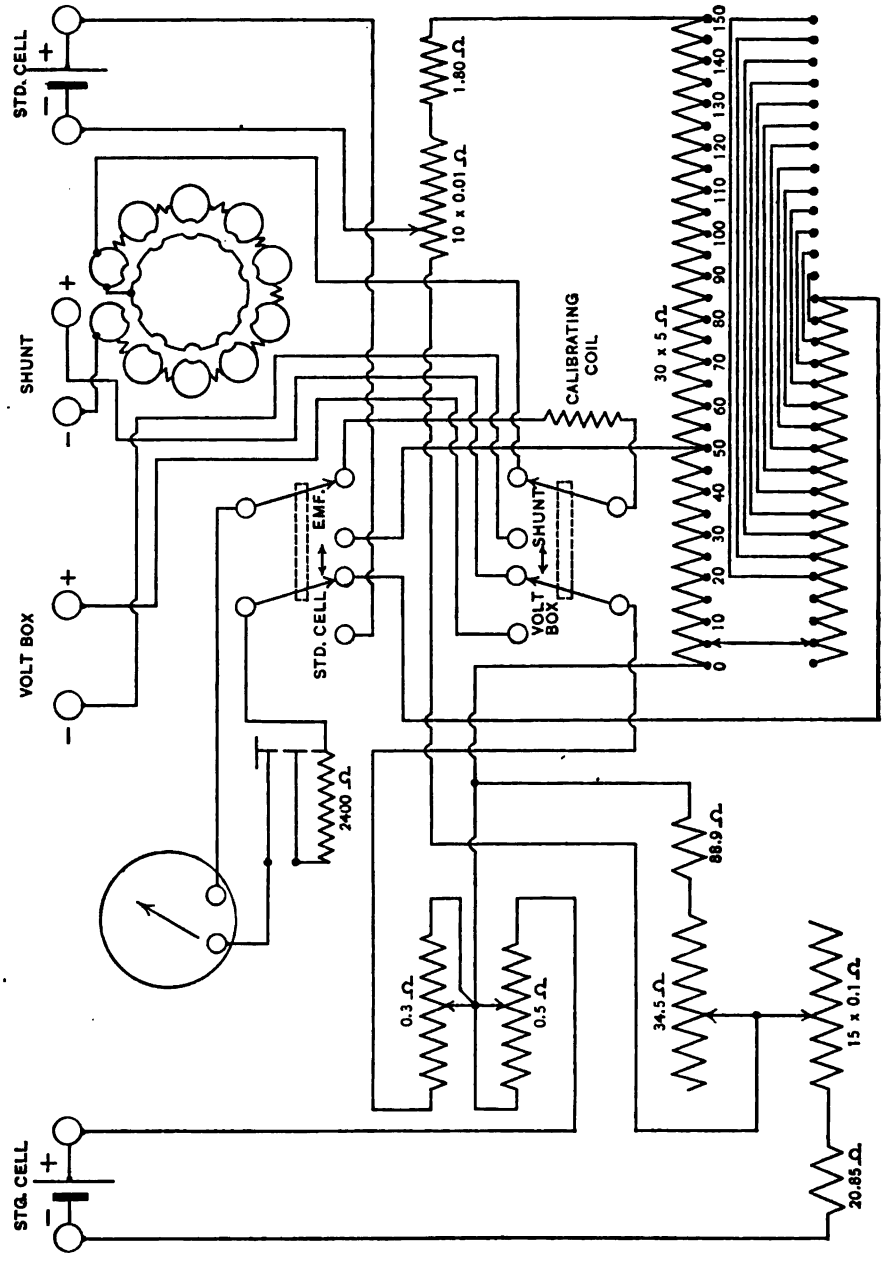

Fig. 2.—Plan of Circuits, Model 3 Potentiometer

of the part of the potentiometer wire around which the drop is taken, shunted by the rest of the potentiometer wire and the (constant) resistance through the rheostats and storage cell. The circuit in question will be apparent from Fig. 2, page 430. The standard cell in series with the galvanometer is connected across the last 100 ohms of the main dial, plus about 2 ohms. The (constant) resistance of the remainder of the main dial and the rheostats is $170-102=68$ ohms. The resultant resistance of 102 ohms in parallel with 68 ohms is about 41 ohms. Adding 10 ohms for the galvanometer and 200 ohms[23] for the standard cell gives a total resistance of 251 ohms. The current through this resistance due to 0.0001 volt is 0.4 microampere, and the resulting deflection of the galvanometer is 0.04 scale division. While the position of the index can not usually be estimated to better than 0.1 division, a movement of 0.04 division is perceptible when the index is directly over (or close to) a line. Hence a deviation of the auxiliary current from its normal value of 1 in 10 000 can be detected.

Owing to the caking together of the crystals of saturated standard cells, such cells often have very high internal resistance. *They should not be used with the deflection potentiometer.*

It is not always possible to have the condition of critical damping when using the galvanometer in the standard-cell circuit. Thus, in the example just given, the total resistance being 251 ohms, against 100 ohms for critical damping, the motion would be considerably underdamped. As the travel of the index (in this use of the galvanometer) is quite small, this is not of consequence. If (as occurs in Model 5 potentiometer) the total resistance is much below the critical value, when checking against the standard cell, it is advisable to use enough extra resistance in series to give nearly critical damping.

The accuracy in checking the auxiliary current, in the case previously mentioned, could be materially improved if necessary by the use of a special standard cell of larger size and consequently lower resistance. It should be possible to get a cell of, say, 40 or 50 ohms resistance, without undue increase in size. This would give 100 ohms total, and an error of 1 part in 10 000 in the auxiliary current would give 1 microampere through the galvanometer; this would cause a movement of 0.1 division. Except in special cases, however, one would not care to go to the trouble of making up a special standard cell.

Values of r_s may now be computed, using the formula

$$r_s=\frac{e_1}{a}\frac{r_w(4r_m-r_w)}{4r_m}=\text{constant}\times e_1 \qquad (21)$$

[23] The (European) Weston Co. gives the internal resistance of these cells as about 160 ohms. The values found by the Bureau of Standards range from 130 ohms for a relatively new cell to 260 ohms for one that had been in use for about five or six years. The internal resistance of a given cell can be roughly checked by setting the standard cell switch at the extreme right, adjusting battery current to give no deflection, then throwing the standard cell switch to the extreme left (a change of 0.0010 volt) and observing the deflection. This should be done with a new cell, and the value recorded.

Substituting

$$r_2 = \frac{151.9(170-151.9)}{1.519 \times 170} e_1$$

$$= 10.647 e_1$$

Using the values 1.98 and 2.14 for e_1, r_2 has the values 21.08 and 22.78 ohms; r_2 will thus consist (neglecting the fine rheostat for the present) of a fixed resistance of 21.08 ohms which is increased by equal steps to 22.78 ohms. The sum of the steps is 1.70 ohms, and as 15 steps have been found a suitable number, we may have 17 steps of 0.1 ohm each.[34] In the instrument which is here used for illustration, it was desired to use 15 steps to avoid change of patterns used in a previous model. This was done by narrowing the limits of e_1 to 1.99 and 2.13 volts. The fine rheostat is half in, for the corresponding values of r_2, and hence by inserting or removing the other half, the battery voltage may be carried a little beyond each of these limits. We may thus assign values for r_2 for all the steps of the coarse rheostat; the next thing is to compute a series of values of r_6 by formula (22), page 425. The lowest value of r_6 is a coil, to which the coarse rheostat adds 15 steps. These steps, unlike those of the series resistance r_2, will not be equal, the value of a step increasing with increase of r_6.

The values of r_4 (the "ballast" resistance) are such as will make up a total of 42.5 ohms, for any step of the main dial. When the sliding contact of the main dial is at the zero setting (the point A, Fig. 1) the current flowing down through the galvanometer meets no resistance in the potentiometer wire, but flows through r_4 only; for this point r_4 must have its maximum value, 42.5 ohms. When the slider is at the first step to the right of A, the resistance to the galvanometer current is the 5-ohm coil (r_1) shunted by the rest of the 170 ohms (main dial plus rheostats). Hence r_4 must be less than 42.5 ohms by the quantity

$$\frac{5\,(170-5)}{170} = 4.85 \text{ ohms.}$$

Other values of r_4 are found in the same way. At the seventeenth step of the main dial the resistance $r_1 = 85$ ohms, which equals the remainder of main dial plus the rheostats; the resultant of these two 85-ohm paths in parallel is 42.5 ohms, and the value of r_4 is zero. From this maximum point the resultant resistance will decrease as the slider takes positions farther to the right. The resultant resistance will repeat at the eighteenth, nineteenth, ... thirtieth steps the values it had at the sixteenth, fifteenth, ... fourth steps. Hence, instead of using a second set of r_4 coils, as shown in the elementary diagram, Fig. 1, we can save 13 coils by using cross connections, as shown in diagram of connections, Fig. 2.

[34] Since r_2 varies directly as the battery voltage, it is convenient to give r_2 equal steps, which should be of convenient (round) value.

The fine rheostat is next to be computed. Taking the upper value of r_3, 22.78 ohms, the corresponding value of r_6 is 88.05 ohms. If r_3 be reduced by 0.1 ohm (Δr_3) the corresponding increase (Δr_6) in r_6 is 1.58 ohms. Substituting in equation (35),

$$r_3 = \frac{0.1}{1 + \frac{151.9 \times 1.58}{2 \times 88.05(151.9 + 88.05 + 1.58)}} + \frac{1.58 \times 22.78 \times 151.9}{88.05(151.9 + 88.05 + 1.58) + \frac{151.9 \times 1.58}{2}}$$

$$= \frac{0.1}{1 + 0.0056} + \frac{5467}{21267 + 120}$$

$$= 0.0994 + 0.2556 = 0.355 \text{ ohm.}^{26}$$

If a similar calculation be made for the minimum value of r_3, it will be found that r_3 is larger, namely, 0.418 ohm. Some margin should be allowed beyond this value to avoid the unnecessary expense of adjusting the slide rheostat closely.

To compute the value of the fine rheostat in the galvanometer circuit (see p. 428), we may find the error in Σr for minimum r_3 and the maximum setting on the main dial. For this calculation assume that the battery fine rheostat r_3, just discussed, is given the value 0.5 ohm. With $r_3 = 21.08$ ohms, $r_6 = 128$ ohms (0.25 ohm of r_3 going to make up r_3), the resultant resistance of the rheostats is

$$\frac{21.08 \times 128}{21.08 + 128} = 18.10 \text{ ohms,}$$

the normal value. Now let the remaining half of r_3 be put in circuit, giving 21.33 ohms in place of 21.08, and increasing the resultant resistance to

$$\frac{21.33 \times 128}{21.33 + 128} = 18.28 \text{ ohms.}$$

In each case, the resistance r_w (151.9 ohms) is in parallel with the rheostats giving (for the two positions of the fine rheostat) final resultant resistances of 16.173 and 16.316 ohms. The difference is 0.14 ohm; a similar calculation for r_3 maximum gives 0.12 ohm. Hence, the galvanometer side of the fine rheostat may have a mean value of about 0.13

[26] The small quantities at the right in each of the denominators are absent in the formula given in a previous paper, equation (25), which did not take into account the practice of having the computed values of r_3 include half of the fine rheostat. See discussion of this point, p. 426, and following. It will be seen that for the present design these quantities are negligible.

ohm each side of its central position, or 0.26 ohm total. It may be noted that if this fine rheostat were not provided the error would be less than 0.2 per cent (0.14 in 100) of the deflection part of the result, or 0.04 of a scale division in the maximum deflection. This would be under the usual limit of reading. The fine rheostat is put in the galvanometer circuit on the general principle of avoiding small errors insignificant in themselves, which might add up to an appreciable amount.

The size of wire and length of coil composing the fine (battery) rheostat r_8 should be chosen so as to give a sufficient number of steps. For example, in the preceding design a step on the coarse rheostat corresponds to a change of 0.01 volt in the 2-volt storage cell, or 1 part in 200. If we make the fine rheostat of, say, 100 turns, each turn will be 1 part in 20 000, which is ample fineness of adjustment of the current. The steps on the galvanometer side of the fine rheostat may evidently be very much coarser if desired.

As it is not possible to get as perfect contacts on the fine rheostats as on the coarse, a safety connection is used. (See Fig. 2, p. 430.) This consists simply in connecting the free end of the spiral to the slider. With this connection, the failure of the slider to make contact can at most do no more than introduce the whole of the fine rheostat into the circuit. Without this connection, the failure of the slider to make contact would break the circuit. This is a device used in some dynamo rheostats; it should be more generally known and used. It is not used on the coarse dials of the potentiometer, partly because it is very much less needed, partly because it would have to be disconnected when checking the values of the coils.

The standard-cell dial may have 10 steps of 0.01 ohm each, thus providing for cells of from 1.0180 to 1.0190 volts. (See Fig. 2, p. 430.) The protective coil should be 24 times Σr, or 2400 ohms.[26]

The grouping of the various dials and the galvanometer should be such as will give speed and convenience of working. The reasons for the general arrangement used in the present models are briefly given in a preceding article.[27]

3. NOTES ON THE DESIGN OF MOVING-COIL GALVANOMETERS

The complete design of a moving-coil galvanometer of the high grade required for a deflection potentiometer is beyond the limits of the present article. However, some general considerations may be useful to those who wish to adapt an existing type of moving-coil instrument to the purpose.

In the deflection potentiometers so far designed at this Bureau, pivoted galvanometers only have been used. As much higher

[26] The normal maximum deflection of the galvanometer is 25 divisions, and with a preliminary external resistance of 24 times the normal total, the deflection is to be brought to 1 division or less, by manipulating the main dial. If the key be now fully depressed, the reading will not be over 25 divisions.

[27] This Bulletin, 8, p. 405, 1911 (Reprint No. 172).

sensibility can be had in the reflecting galvanometer with suspension strips or wires, the question may arise, Why are such galvanometers not used? Briefly, the reasons are as follows:

1. The reflecting galvanometer is harder to read, if a telescope is used, and causes much more fatigue to the operator. If lamp and scale are used, a semidarkening of the room is required.

2. Reflecting galvanometers are far inferior to good pivoted galvanometers in proportionality of scale, repeatability of reading for a given current, and general reliability. This is due largely to the smallness of the controlling force of the suspension in a reflecting galvanometer; disturbing forces, due to magnetic impurities in the coil and to electrostatic action, are relatively much larger and their effect much more pronounced than in good pivoted galvanometers.

3. If the level of a suspended-coil galvanometer is altered, the coil is brought into a different part of the field, and its deflection for the same current will be different. This necessity for accurate leveling makes the suspension galvanometer troublesome when apparatus must be moved.

4. The suspended-coil galvanometer is sensitive to vibration (as that due to running machinery) and will be put out of commission by ordinary handling such as would not affect a good pivoted instrument.

If the magnets of a moving-coil galvanometer be removed, or if we leave the coil on open circuit, no damping frame or coil being present, the coil will have but little damping, that which exists being principally due to air friction. If the coil be displaced from its position of rest, it will oscillate about this position with a period [28] T which is defined by the relation

$$T = 2\pi\sqrt{\frac{K}{U}}$$

where K is the moment of inertia of the coil and its fittings and U is the restoring couple of the spring for unit angle. This expression neglects the small damping due to air friction, which

[28] "Period" here refers to a complete oscillation; for example, starting from the position of rest, out to the right, back through the starting point to the left, then back to the starting point.

(in the type of galvanometer considered) does not affect the result appreciably. If K is expressed in $g\text{-}cm^2$ and U in centimeter-dynes per radian, T will be in seconds. The value of K will in general not be subject to modification, since changing the size of wire will not appreciably affect the mass of the coil, unless relatively coarse and fine wires are involved in two coils under consideration. In this case the relatively large space taken by the insulation of the fine wire may reduce the mass of the coil appreciably. However, the moment of inertia of the coil is only part of the total, as that of the pointer will be quite appreciable, due to its length. In general, then, one would regard K as fixed for the type, and would vary U, the spring strength, until a proper value of T (say 1 to 1.5 seconds) was obtained. The value of T may be conveniently found by using a stop watch to measure the time required for several complete oscillations. The value of U may be found [29] by turning the instrument so that the axis of rotation is horizontal, and the pointer is also horizontal when in its position of maximum deflection. The pointer is brought to this position by a small weight hung on it, such as a piece of wire. This weight is slid out along the pointer until it balances the torque of the spring and holds the pointer as described. Then the product of the mass of the wire in grams, its distance from the axis in centimeters, and the factor 981, will be the torque of the spring for full deflection, expressed in cm-dynes. The angle corresponding to the given deflection may be measured in any convenient way; the value of U, the torque for one radian (57°.3) may then be computed. The preceding method of torque measurement is subject to some errors (due to shifting of point of contact of pivots in jewels, etc.) which are relatively greater as the spring strength becomes smaller, for a given coil. A pendulum apparatus [30] has recently been used at this Bureau for the measurement of spring strength of instruments and the torque of integrating meters. This method has the advantage of keeping the moving coil in its normal position, with its axis of rotation vertical.

[29] This method has been used by the writer since 1900; it was described by Friedr. Janus in Elektrotechnische Zeitschrift, vol. 26, p. 561; June 15, 1905.

[30] P. G. Agnew, this Bulletin, 7, p. 45; 1911 (Reprint No. 145).

In order to avoid errors due to pivot friction, it is necessary that the ratio of torque (for a given angle) to the weight of the entire moving system shall not be allowed to fall below a certain value. According to Heinrich and Bercovitz,[21] the torque in cm-gm for 90° deflection should not be less than 5 per cent of the weight of the coil in grams. Janus[22] gives a considerably higher value, namely, 17 per cent, but does not state what angle; from the context, it is probable that 90° is intended. The quality of the pivots and jewels has much to do with the permissible limit for this ratio, lower values of ratio being allowable for more perfect pivots and jewels.

Assuming that a proper value of T has been obtained, using a spring which satisfies the requirements of torque-weight ratio, we may consider the other fundamental constants of the galvanometer.

With the magnets on the pole pieces, if the coil be connected to a very high external resistance the damping due to induced currents in the coil is inappreciable, and the motion is still periodic, though as at first, the amplitude continually grows less, due to air damping, pivot friction, etc. As the resistance external to the coil is decreased the damping becomes more noticeable and the coil comes to rest sooner. Finally, when the sum of coil resistance and external resistance reaches a particular value (the "critical resistance") the coil if displaced from its position of rest will return to it without oscillation. The time required for this return is roughly equal to the value of T as previously defined. This is the condition under which moving-coil galvanometers should be used in order to save time and to be able to work with unsteady current supply, catching the readings at momentary lulls. We may call this critical resistance R and the coil resistance R'.

The current required to give a deflection of one division may be determined in any convenient way. If this current I be multi-

[21] Handbuch der Elektrotechnik, vol. 2, pt. 5, p. 14. Published by Hirzel, Leipzig, 1908.

[22] Elektrotechnische Zeitschrift, 26, p. 560; 1905.

plied by the critical resistance R, the product IR is the voltage per scale division, and is one of the principal constants chosen at the start, in designing a deflection potentiometer. (See p. 421, eq. (34).) Another important constant of the galvanometer is H, the strength of the magnetic field in which the coil moves. A knowledge of the numerical value of H is not required if the following conditions hold, the value of T being 1 to 1.5 seconds, or, if necessary, being brought to this value by change of springs: (1) R must be from 5 to 10 times the coil resistance R'; (2) $R-R'$ must be high enough to allow the desired resistance in potentiometer circuits and external accessories; (3) IR must have the desired value. However, in general, one would not find all these conditions realized, so that it will be necessary to find the value of H and to consider how the galvanometer performance will be affected by change of H. A small test coil of n_1 turns, each inclosing a square centimeters area, may be inserted in the air gap. The test coil, a ballistic galvanometer and the secondary winding of a known mutual inductance are connected in series. The withdrawal of the test coil gives a ballistic throw, d_1. A direct current is now passed through the primary of the mutual inductance and is adjusted by trial to a value i such that the reversal of the current gives a throw d_2 nearly equal to that given by the withdrawal of the test coil. Then if M is the value of the mutual inductance in henrys, the value of H in the air gap is given by the expression

$$H=\frac{d_1}{d_2}\frac{2}{a}\frac{Mi}{n_1}\times 10^8 \tag{36}$$

If the current in the primary of M be simply broken to get d_2, the factor 2 must be omitted from the numerator.

If the ballistic galvanometer and mutual inductance are not available, H may be calculated from the dimensions of the coil, the number of turns, and the torque corresponding to a given deflection produced by a current i. Let N = number of turns in the coil, b the mean breadth of the coil (in the direction of the lines of force), and h the height of the coil in the magnetic field,

excluding dead wire at the ends. (Dimensions are to be in centimeters.) Then H is given by the formula [33]

$$H = \frac{10 \times \text{Torque in centimeter-dynes}}{Ni \times bh} \tag{37}$$

The denominator is the product of the ampere-turns by the area of the field enclosed by a mean turn.

To give the proper value of R/R', namely, 5 to 10, H must be equal to about 1500 to 2000, for the usual form of coil.[34] If the original value of H must be increased, a larger magnet [35] may be used, or two magnets may be used in parallel on the same pole pieces. If the area enclosed by the pole pieces is such as to give the usual 90° travel of the coil, the pole pieces may be reduced, as usually such a travel is not required for the present purpose. However, one should not go too far in this direction, as this will fail to give a proportionate increase in H, the leakage flux increasing rapidly after a certain point is reached. The length of the air gap may be reduced, if this can be done without sacrificing proper clearance for the coil. It should be kept in mind that the reduction of area of air gap will tend to impair the permanency of the magnet, unless at the same time the length of the air gap be decreased, or the length of the magnet increased. A formula for this matter is given by Heinrich and Bercovitz,[36] as follows: If q_m is the cross section of the magnet, l_m the length of the magnet, q_p the cross section of the air gap, and l_p the length of the air gap, then l_m/q_m must be greater than $100 l_p/q_p$. In the usual bipolar construction, l_p is equal to twice the length of each air gap, or, in general, it is the total length of air gap.

It remains to consider what effects are produced by varying H. The critical resistance R for a given coil and springs varies

[33] Friedr. Janus, Elektrotechnische Zeitschrift, **26**, p. 560; 1905. This article on "Die Berechnung von Drehspul-Messgeräthen" contains a discussion of the fundamental considerations in the calculation of moving coil measuring instruments, including the calculation of springs to provide a given torque.

[34] See note 8, p. 422.

[35] Concerning the design of permanent magnets, see J. Busch, Elektrotechnische Zeitschrift, **22**, p. 234; 1901.

[36] Handbuch der Elektrotechnik, vol. 2, pt. 5, p. 69.

directly as the square of H. The complete statement of this relation is given by White in the form [37]

$$H = A\sqrt{\frac{R}{R'T}}$$

where A is a nominal constant for a given form of coil, but varies slightly with the amount of inert matter in the coil. The current I per scale division varies inversely as H; from this fact and the preceding we see that the product IR varies directly as H. Hence if volt sensitiveness (smallness of IR per scale division) were the only requirement, a large value of H would be a disadvantage.

Taking R_2/R'_2 as the desired value of the resistance ratio (to have the value 10, if possible), we must change the original value H_1 to

$$H_2 = H_1\sqrt{\frac{R_2/R'_2}{R_1/R'_1}} = H_1\sqrt{\frac{R_2R'_1}{R'_2R_1}} \tag{38}$$

where R_1 and R'_1 are the original values. This will raise the original value I_1R_1 to

$$I_1R_1\sqrt{\frac{R_2R'_1}{R'_2R_1}}$$

In general, this value will not be what is desired, and we must now change the size of wire in the coil in accordance with the principle that IR varies directly as the number of turns.[38] This neglects difference of space factors of different sizes of wires, but will give a good first approximation, except in passing to sizes much finer or coarser than that of the existing coil. Let the desired value of IR be denoted by I_2R_2 and the number of turns of wire in the original coil by N_1; then a new coil must be substituted having N_2 turns, where

$$N_2 = \frac{I_2R_2}{I_1R_1}\sqrt{\frac{R'_2R_1}{R_2R'_1}} \times N_1 \tag{39}$$

[37] W. P. White, Every-Day Problems of the Moving Coil Galvanometer, Physical Review, 23, p. 384; 1906.

[38] If the coil of N turns is rewound to have kN turns, the critical resistance R becomes k^2R, the current per scale division becomes I/k, and their product IR becomes kIR.

If we call the diameter (over insulation) of the wire on the original coil D_1, then, since the diameter varies inversely as the square root of the number of turns in the coil, the new diameter will be

$$D_2 = \sqrt[2]{\frac{I_1R_1}{I_2R_2}}\sqrt[4]{\frac{R_2R'_1}{R'_2R_1}} \times D_1 \qquad (40)$$

Giving R_2/R_2' the desired value 10, we may use this formula to compute D_2; the computed value must be revised slightly to get the nearest available commercial size (taking account of thickness of insulation). Then substitute the actual number of turns the new coil will have as N_2 in (39), and solve for the value of R_2/R_2'. Substituting this value in (38) gives the numerical value of H_2 which will satisfy the requirements of the design. These computed values, D_2 and H_2, will give the data from which a coil may be made up and the field strength changed; tests may then be made of this "first approximation" coil from which as a new starting point any needed revision of values may be made. The space factor of fine wires varies greatly with the size, and hence the error due to the use of the design equations (38) to (40), which neglect variation of space factor for the sake of simplicity, is greatest when the finest sizes are concerned. Another practical limitation is encountered in the coarser sizes, due to the necessity for an integral number of layers. For example, the size of wire given by (40) may be such that 2.6 layers are required in the theoretical coil; in such a case one would be obliged to use two layers, or enlarge the space in which the coil moves, so as to use three layers.

As the error in using any approximate equation is lessened by reducing the relative amount of change which is to be made, it will be of advantage to compute H_2, then raise the existing value of H to the computed value, and determine the performance of the original coil in the new field. This performance may then be used to compute a revision of H_2, if necessary, when the performance may again be determined, and used as the basis for calculation of a new coil. It is easy to vary H by changing the size or number of magnets, and by magnetic shunting; or one may use as a trial magnet an electromagnet with suitable means for determining the value of H (see pp. 438–439).

One effect of the poor space factor of fine wires should be specially mentioned. The thickness of (silk) insulation can not be reduced below a certain point, and as the wire becomes finer the amount of metal becomes relatively much less than in a larger size which may have been taken as a starting point for design. Consequently the fine wire coil will have an abnormally high resistance, which will reduce the ratio R/R', and give less resistance for use in potentiometer and accessories than would be expected from the performance of the same galvanometer with a coarser coil.

It will be realized (by galvanometer makers, at least) that it would be a somewhat difficult and expensive procedure to make galvanometers in quantity, each to have exactly a specified critical resistance, current per scale division and length of scale division. For the deflection potentiometer *this is not necessary*, and we will now consider in what ways the maker can have the latitude which will expedite and cheapen the manufacture without sacrifice of quality.

1. Use tested springs of normal or slightly *under* normal strength.

2. Use the same type of magnet as is used for other standard products made in quantity, so that a good stock of magnets is available from which to select (by test) magnets of nearly the strength desired for galvanometers. Take a magnet whose strength is slightly above normal and reduce its strength by magnetic shunting until the galvanometer shows a critical resistance R equal to or above the normal and a drop per scale division IR within, say, 10 per cent above or below normal, using a trial scale having divisions of standard length.

3. Using the value of total resistance R which gives critical damping for the individual galvanometer, make a scale to fit it. For this purpose the galvanometer plus external resistance may be regarded as a voltmeter, and the scale may be laid out by applying 5, 10, 15 ..... times the standard voltage per division IR, to locate the 5, 10, 15 ..... division points. Care should be taken to make an accurately proportional scale, as there is no way to correct for lack of proportionality.

The reason for selecting the springs of normal strength or below and the magnets of slightly over normal strength will appear from

the following. Let the spring strength (U) be too great by the factor $1+e$, the moment of inertia of coil and fittings being assumed as not varying from standard.[30] To offset the excess of spring strength let the field strength H be changed to $H(1+f)$ which will give a value of total critical resistance $R(1+g)$, satisfying the condition of normal voltage IR per scale division. We have from White's equation (20 a) as quoted on page 440, for all quantities normal,

$$R=\frac{H^2}{A^2}R'T=\frac{H^2}{A^2}R'\times 2\pi\frac{\sqrt{K}}{\sqrt{U}}$$

$$=H^2U^{-\frac{1}{2}}\left(\frac{2\pi R'K^{\frac{1}{2}}}{A^2}\right)$$

$$C_1H^2U^{-\frac{1}{2}} \tag{41}$$

where C_1 is constant for a given coil and springs. For the case of U increased to $U(1+e)$, etc., as above, we have (making the usual approximations for terms of the form $1+$ a small quantity)

$$R(1+g)=C_1H^2(1+f)^2U^{-\frac{1}{2}}(1+e)^{-\frac{1}{2}}$$

$$=C_1H^2U^{-\frac{1}{2}}\left(1+2f-\frac{e}{2}\right)$$

$$R=C_1H^2U^{-\frac{1}{2}}\left(1+2f-\frac{e}{2}-g\right)$$

From this value of R and equation (41) we have

$$2f-\frac{e}{2}-g=0 \tag{42}$$

The change of spring strength and change of magnet strength will change the current per scale division to $I\ (1+e-f)$. The product of this current and $R(1+g)$ must give the normal value of IR; that is

$$I(1+e-f)R(1+g)=IR$$

[30] As far as the effect on the period is concerned, a given percentage excess in K can be counted as the same percentage deficit in U. However, a change of the latter affects the value of I and IR, while a change in K does not. Hence, in general, K and $1/U$ are not equivalent.

whence $1+e-f+g=1$, or $e-f+g=0$ (43)

Combining equations (42) and (43),

$$f=-\frac{e}{2}$$
$$g=-\frac{3e}{2} \tag{44}$$

Taking a numerical example: let the springs be 1 per cent too strong (e); then to keep IR normal, by (44) the magnet strength must be 0.5 per cent below normal $\left(-\frac{e}{2}\right)$, and the total critical resistance R must be 1.5 per cent low $\left(-\frac{3e}{2}\right)$. Hence the spring strength ought to be normal or below to keep R normal or above. The current I will be 1 per cent stronger because of the stronger springs, 0.5 per cent stronger because of the weaker magnet; I will thus be 1.5 per cent greater than normal. The product of this greater I by an R 1.5 per cent below normal gives the normal IR. Concerning the question of critical damping, we see from the equation

$$R=C_1H^2U^{-\frac{1}{2}} \tag{41}$$

that the right-hand member will be 1 per cent low due to the magnet strength H being 0.5 per cent low, and 0.5 per cent low due to spring strength U being 1 per cent high; hence R will be 1.5 per cent low, as required by the value of I 1.5 per cent greater than normal.

It is evident that other methods are available for adjusting the constants of the galvanometer, such as varying the moment of inertia of the coil by adjustable balance weights, or by adding auxiliary damping coils or rectangles. It is believed, however, that the general method outlined in the preceding discussion is likely to be most economical of time and most satisfactory in the results accomplished.

WASHINGTON, June 23, 1911.

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129. The Regulation of Potential Transformers and the Magnetizing Current . . . . . *M. G. Lloyd and P. G. Agnew*
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131. Selective Radiation from Various Solids, II . . . . . *W. W. Coblentz*
132. Luminous Efficiency of the Firefly . . *Herbert E. Ives and W. W. Coblentz*
133. Luminosity and Temperature . . . . . *P. G. Nutting*
134. A Theoretical and Experimental Study of the Vibration Galvanometer . . . . . *F. Wenner*
135. Specific Heat of Some Calcium Chloride Solutions between −35° C. and +20° C . . . . . *H. C. Dickinson, E. F. Mueller, and E. B. George*
136. On the Definition of the Ideal Gas . . . . . *Edgar Buckingham*
137. Mica Condensers as Standards of Capacity . . . . . *Harvey L. Curtis*
138. The Mutual Induction of Two Parallel Coaxial Circles in Terms of Hypergeometrical Series . . . . . *Frederick W. Grover*
139. A New Method for the Absolute Measurement of Electric Quantity . . . . . *Burton McCollum*
140. The Comparative Sensitiveness of Some Common Detectors of Electrical Oscillations . . . . . *Louis W. Austin*
141. Photometric Units and Nomenclature . . . . . *E. B. Rosa*
142. A Modified Method for the Determination of Relative Wave Lengths, especially adapted to the Establishment of Secondary Standards . . . . . *Irwin G. Priest*
143. Note on the Temperature Scale between 100 and 500° C. . . . . *C. W. Waidner and G. K. Burgess*
144. A New Form of Direct-Reading Candlepower Scale and Recording Device for Precision Photometers . . . . . *George W. Middlekauff*
145. A Device for Measuring the Torque of Electrical Instruments . . . . . *F. G. Agnew*
146. The Intensities of Some Hydrogen Argon and Helium Lines in Relation to Current and Pressure . . . *P. G. Nutting and Orin Tugman*
147. The Temperature Coefficient of Resistance of Copper . . . *J. H. Dellinger*
148. The Electrical Conductivity of Commercial Copper . . . . . *F. A. Wolff and J. H. Dellinger*
149. On the Constancy of the Sulphur Boiling Point . . . . . *C. W. Waidner and G. K. Burgess*
150. Note on Oscillatory Interference Bands and Some of their Practical Applications . . . . . *G. O. Squier and A. C. Crehore*

151. The Effect of Preliminary Heat Treatment upon the Drying of Clays . . . . . *A. V. Bleininger*
152. The Reflecting Power of Various Metals . . . . . *W. W. Coblentz*
153. The Action of Sunlight and Air upon Some Lubricating Oils . . *C. E. Waters*
154. The Visibility of Radiation. A Recalculation of Konig's Data . . *P. G. Nutting*
155. A Photometric Attachment for Spectroscopes . . . . . *P. G. Nutting*
156. Selective radiation from Various Substances, III . . . . . *W. W. Coblentz*
157. The Measurement of Electrical Oscillations in the Receiving Antenna . . . . . *L. W. Austin*
158. Some Experiments with Coupled-High-Frequency Circuits . . . *L. W. Austin*
159. Some Quantitative Experiments in Long Distance Radiotelegraphy . . . . . *L. W. Austin*
160. The Behavior of High-Boiling Oils on Heating in the Air . . . . *C. E. Waters*
161. The Determination of Vanadium in Vanadium and Chrome-Vanadium Steels . . . . . *J. R. Cain*
162. On the Computation of the Constant C_2 of Planck's Equation by an Extension of Paschen's Method of Equal Ordinates . . . . . *Edgar Buckingham and J. H. Dellinger*
163. A Comparison of American Direct Current Switchboard Voltmeters and Ammeters . . . . . *T. T. Fitch and C. J. Huber*
164. Study of the Current Transformer with Particular Reference to Iron Loss . . . . . *P. G. Agnew*
165. Thermodynamics of Concentration Cells . . . . . *Henry S. Carhart*
166. The Capacity and Phase Difference of Paraffined Paper Condensers as Functions of Temperature and Frequency . . . . . *Frederick W. Grover*
167. The Steam Expansion Line on the Mollier Diagram and a Short Method of Finding the Reheat Factor . . . . . *Edgar Buckingham*
168. Radiometric Investigation of Water of Crystallization, Light Filters, and Standard Absorption Bands . . . . . *W. W. Coblentz*
169. Formulas and Tables for the Calculation of Mutual and Self Induction (2d edition, revised and enlarged) . . . . *E. B. Rosa and Frederick W. Grover*
170. The Correction for Emergent Stem of a Mercurial Thermometer . . . . . *Edgar Buckingham*
171. Thermometric Lag . . . . . *D. R. Harper*
172. Deflection Potentiometers for Current and Voltage Measurements . . . . . *H. B. Brooks*
173. Outline of Design of Deflection Potentiometers with Notes on the Design of Moving-Coil Glavanometers . . . . . *H. B. Brooks*
174. The Determination of Total Sulphur in India Rubber . . . . . *C. E. Waters and J. B. Tuttle*

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MISCELLANEOUS.

International Metric System (chart).
Table of the Equivalents of Customary and Metric Weights and Measures.
The International Metric System of Weights and Measures. (Pamphlet.)
First Conference on the Weights and Measures of the United States.
Second Annual Conference on the Weights and Measures of the United States.
Fourth Annual Conference on the Weights and Measures of the United States.
Fifth Annual Conference on the Weights and Measures of the United States.

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DEPARTMENT OF COMMERCE AND LABOR

Vol. 8 No. 3

BULLETIN

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

NOVEMBER 1, 1912

CONTENTS

		Page
The Determination of Total Sulphur in India Rubber	*C. E. Waters and J. B. Tuttle*	443
The Measurement of the Inductances of Resistance Coils	*Frederick W. Grover and Harvey L. Curtis*	455
Luminous Properties of Electrically Conducting Helium Gas. II. Reproducibility	*P. G. Nutting*	487
Resistance Coils for Alternating Current Work	*H. L. Curtis and F. W. Grover*	495
The Hydrolysis of Sodium Oxalate and Its Influence upon the Test for Neutrality	*William Blum*	519
Wave-Lengths of Neon	*Irwin G. Priest*	539
On the Deduction of Wien's Displacement Law	*Edgar Buckingham*	545
The Four-Terminal Conductor and the Thomson Bridge	*Frank Wenner*	559

WASHINGTON
GOVERNMENT PRINTING OFFICE
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The Determination of Total Sulphur in India Rubber	*C. E. Waters and J. B. Tuttle*	445
The Measurement of the Inductances of Resistance Coils	*Frederick W. Grover and Harvey L. Curtis*	455
Luminous Properties of Electrically Conducting Helium Gas. II. Reproducibility	*P. G. Nutting*	487
Resistance Coils for Alternating Current Work	*H. L. Curtis and F. W. Grover*	495
The Hydrolysis of Sodium Oxalate and Its Influence upon the Test for Neutrality	*William Blum*	519
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THE DETERMINATION OF TOTAL SULPHUR IN INDIA RUBBER

By C. E. Waters and J. B. Tuttle

What may be still described as the usual method for the determination of total sulphur in india rubber is the one first published by Henriques.[1] The details of this method are too well known to require description here. In more recent years other methods have been advocated. Alexander[2] used sodium peroxide to decompose the nitrogen peroxide addition-product of rubber. In the same year Esch[3] recommended the use of Eschka's mixture and procedure for the determination of sulphur in coal. He also stated that the sodium-peroxide method gives good results.

Wagner[4] published a slight modification of the method of Henriques, stating that much sulphur is lost by volatilization. He, therefore, made the nitric-acid solution alkaline with sodium hydroxide, transferred to a nickel crucible, added sodium carbonate and then evaporated to dryness. The oxidation was carried to completion by heating in an air-bath.

Pontio[5] fused with manganese peroxide and a mixture of sodium and potassium carbonates. The results were about 0.1 per cent lower than by the method of Henriques. For the free sulphur[6] he extracted with absolute alcohol, distilled off the solvent, oxidized with alkaline hydrogen peroxide, evaporated to dryness and fused in a silver crucible.

[1] Z. angew. Chemie, 12, p. 902; 1899.
[2] Gummi-Ztg., 18, p. 729; Z. angew. Chemie, 17, p. 1799; 1904.
[3] Chem.-Ztg., 28, p. 200; 1904.
[4] Gummi-Ztg., 21, p. 552; Chem. Abstr., 1, p. 1327; 1907.
[5] Caoutchouc et Gutta-Percha, 6, p. 2751; Chem. Techn. Rep., 1900; 372.
[6] Caoutchouc et Gutta-Percha, 5, p. 2194; Chem. Abstr., 2, p. 3412; 1908.

A distinct departure from the usual methods is due to Hinrichsen,[7] who oxidizes electrolytically in the presence of concentrated or fuming nitric acid.

Finally, Hübener [8] devised a method intended to exclude insoluble mineral sulphates. The sample is boiled in a flask with concentrated nitric acid for some time, most of the acid is evaporated off on the steam-bath, and the oxidation completed by means of bromine and water.

One of the present writers, having frequent occasion to determine total sulphur in rubber, over a year ago made a number of comparative tests of different variations of the method of Henriques. The results obtained with two samples of rubber are given below (I–V). In all cases, 0.50 g of rubber was taken. All fusions were made over a flame of gasoline-air gas. The results are given as percentages of sulphur. All reagents were tested, and no determinations have been omitted.

I. Warmed 2½ hours in covered crucible with 25 cc conc. HNO_3, allowed to stand 36 hours, evaporated nearly to dryness, added $Na_2CO_3.KNO_3$ mixture and fused as usual.

Sample	1	1	2	2
Sulphur	3.39	3.44	3.26	3.22

II. Added HNO_3 and 1 cc Br, let stand 36 hours without preliminary heating, evaporated, etc., as usual.

Sample	1	1	2	2
Sulphur	3.40	3.47	3.39	3.27

III. The same as II, but allowed to stand only 1 hour, heated with cover for 2 hours, evaporated and fused as usual.

Sample	1	1	2	2
Sulphur	3.31	3.35	3.04	3.09

IV. Only HNO_3 added, digested at once on the steam-bath for 2 hours, evaporated and fused.

Sample	1	1	2	2
Sulphur	3.17	3.43	3.06	2.93

[7] Chem.-Ztg., **33**, p. 735; 1909.

[8] Gummi-Ztg., **24**, pp. 213–214; Analyst, **35**, pp. 266–267; 1910.

V. Treated with 1 cc Br and 5 cc H_2O, let stand over night without heating, next morning evaporated off the H_2O, added HNO_3, digested, evaporated and fused.

Sample	1	1	2	2
Sulphur	3.71	3.65	3.37	3.38

In an attempt to obtain satisfactory results without fusion, some determinations were made some months later, without a knowledge of Hübener's paper. Half-gram portions of a sample of medium hard rubber were digested with nitric acid in flasks covered with watch glasses. In some cases bromine was added after the digestion with acid and, after standing half an hour, water was added and the flasks heated on the steam-bath. Finally the volume was brought to about 175 cc, the solution heated, filtered, and a little sodium hydroxide added to the filtrate and wash-water. This was then evaporated to dryness, adding a little hydrochloric acid toward the end, taken up with very dilute hydrochloric acid, filtered and barium sulphate precipitated as usual. The results follow:

VI. Treated with HNO_3 alone.

Sulphur	7.76	7.51	7.68	7.96

VII. Treated with HNO_3, followed by Br.

Sulphur	7.62	7.51	7.93	7.87	7.76

All the precipitates obtained under VI and VII contained much lead.

After the method of Hübener was called to our attention some determinations were made on a sample of hard rubber containing no barium.

VIII. Hübener's method.

Sulphur	4.79	3.91	5.23	4.02	4.31	4.13

It is evident that widely different amounts of sulphur must have been retained in the insoluble residue in the form of lead sulphate.

IX. Total sulphur by method of Henriques.

Sulphur	8.65	8.70

X. Treated with HNO_3, followed by Br and H_2O and fused as usual.

Sulphur	8.63	8.62	8.77	8.80	8.72	8.80

It has recently been claimed by van't Kruys[9] that when an excess of calcium chloride over the amount of sulphuric acid is present, only calcium sulphate is carried down with the barium sulphate, and the calcium salt can be converted into barium sulphate by digestion with strong hydrochloric acid, or aqua regia, and barium chloride. Several determinations were made to test this suggestion.

XI. Preliminary treatment as under X, subsequent treatment as suggested by van't Kruys.

Sulphur 8.73 8.76 8.52 8.46 8.74 8.78 8.75 8.77

At this point, joint analyses of a fairly large sample of rubber were carried out by the present writers.

XII. Hübener's method. The sulphur in the insoluble residue was determined by fusion with soda-saltpeter mixture, extracting the melt with water, etc., as usual:

S in original filtrate	0.91	0.91	0.68	0.79	0.82	0.99	1.52	1.15	1.24
S in insoluble residue	2.35	2.43	2.54	2.69	2.53	2.36	1.95	2.31	2.21
Total sulphur found	3.26	3.34	3.22	3.48	3.35	3.35	3.47	3.46	3.45

All the precipitates of barium sulphate from the original filtrates were found to contain lead when tested with dilute ammonium sulphide.

XIII. Treated with HNO_3, allowed to stand over night, the acid driven off on the steam-bath, 1 cc Br and 10 cc H_2O added; then the H_2O and excess of Br driven off by heating. The residue was mixed with soda and saltpeter and fused as usual.

Sulphur 3.41 3.21

XIV. The same as XIII, but the HNO_3 not driven off before adding Br.

Sulphur 3.60 3.63 3.58 3.63 3.57

XV. The same as XIII, but treatment with Br omitted.

Sulphur 3.29 3.49 3.35 3.38 3.55 3.43 3.36 3.58 3.56

XVI. Treated first with bromine and water, allowed to stand over night without heating, then Br and H_2O driven off on steam-bath, treated with HNO_3, etc., and fused.

Sulphur 3.45 3.47 3.48 3.53 3.49

XVII. The same as XVI, but excess Br and H_2O not driven off before adding HNO_3.

Sulphur 3.59 3.47 3.64

[9] Zs. anal. Chem., 49, p. 393; 1910.

XVIII. The method of Henriques, except that the HNO_3 was saturated with Br.

Sulphur 3.66 3.65 3.62 3.73 3.68 3.65 3.69 3.63 3.66 3.71 3.62

XIX. The same as XVIII, but followed by the treatment suggested by van't Kruys.

Sulphur 3.69 3.73 3.76 3.75

In order to obtain a definite idea of the variations caused by differences in the preliminary treatment and in the conditions under which the barium sulphate is precipitated, a very dilute solution of sulphuric acid was made. In each of the following determinations, a 25-cc portion was taken. The weights of the barium sulphate found were calculated as percentages of sulphur in 0.50 g of rubber, in order that the results might be more readily compared with the determinations above.

XX. Direct precipitation with $BaCl_2$. The last two determinations were made with the addition of 2 cc of 1 : 1 HCl, the first six without adding HCl.

Sulphur 3.11 3.11 3.11 3.11 3.11 3.11 3.10 3.11

XXI. Evaporated off the water from 25 cc of the dilute H_2SO_4, added the soda-saltpeter mixture and fused as usual.

Sulphur 3.12 3.15 3.12

XXII. Like XXI, but added $CaCl_2$ to the solution of the melt before precipitating $BaSO_4$, and treated the latter according to van't Kruys.

Sulphur 3.22 3.18 3.19

XXIII. Like XXII, but did not digest the precipitated $BaSO_4$ nor evaporate the filtrate to recover traces of dissolved $BaSO_4$.

Sulphur 3.13 3.12 3.11

XXIV. Like XXI, but did not fuse. The solution was acidified with HCl.

Sulphur 3.17 3.19 3.16

XXV. Added 250 cc H_2O and 10 cc conc. HCl to 25 cc dilute H_2SO_4, added $BaCl_2$, digested two hours, poured off the supernatant liquid, digested the $BaSO_4$ with 1 cc of 10 per cent $BaCl_2$ and 15 cc HCl (1:1). Diluted, filtered, evaporated the combined filtrates to dryness in platinum, took up with 50 cc of slightly acidified H_2O and collected the slight residue on the same filter.

Sulphur 3.12 3.10 3.11

XXVI. Exactly neutralized NaOH solution with the dilute H_2SO_4, using phenolphthalein as indicator. After each addition of acid, the solution was heated in a platinum dish until the pink color no longer reappeared. Then evaporated to dryness, ignited gently, and weighed the Na_2SO_4.

H_2SO_4 used (cc)	75.56	75.55
Na_2SO_4 found (g)	.2082	.2080
$BaSO_4$ equivalent to Na_2SO_4 (g)	.34197	.34164
$BaSO_4$ equivalent to 25 cc H_2SO_4 (g)	.1131	.1130
Sulphur (calc. on 0.5 g rubber)	3.11	3.11

XXVII. The same as XXVI, but used $NaHCO_3$ instead of NaOH.

H_2SO_4 used (cc)	45.01	43.48
Na_2SO_4 found (g)	.1240	.1194
$BaSO_4$ equivalent to Na_2SO_4 (g)	.20367	.19612
$BaSO_4$ equivalent to 25 cc H_2SO_4 (g)	.1131	.1128
Sulphur (calc. on 0.5 g rubber)	3.11	3.10

In order to test the completeness of the oxidation of sulphur by means of the nitric acid-bromine mixture, the following determinations were carried out.

XXVIII. Powdered sulphur crystals, digested in the cold with 20 cc HNO_3 and an excess of Br. Finally added 20 cc H_2O and heated on the steam-bath for about two hours. Then evaporated nearly to dryness, took up with water, and precipitated with $BaCl_2$.

Sulphur taken (g)	0.0483	0.0395	0.0561
Sulphur found (g)	.0481	.0399	.0563

XXIX. Powdered sulphur crystals treated at the same time as some of the samples of rubber. The exact methods are referred to in the table, the Roman numerals indicating the method employed.

Method	I	II	III	IV	V
Sulphur taken (g)	0.0528	0.0619	0.0494	0.0646	0.0411
Sulphur found (g)	.0485	.0590	.0479	.0595	.0414
Sulphur, per cent	91.84	95.33	96.91	92.13	100.67

In the determinations by methods I to IV, part of the sulphur was not attacked by the nitric acid, nor by the sodium carbonate added before making the fusion. Part, at least, of this unattacked sulphur was seen to burn when the fusion was made.

As stated above (XII), the barium sulphate precipitates representing soluble sulphates, etc., in the Hübener method, were found to contain lead. Lead sulphate dissolves slightly, and is, besides, partially decomposed by water, hydrobromic and nitric acids, etc.[10] In order to get an idea of the amount of barium sulphate to be expected to result from the decomposition and solution of lead sulphate under the conditions of the Hübener method, some determinations were made. Lead sulphate was first prepared by precipitation from a hot, dilute nitric-acid solution of lead nitrate by means of a hot, dilute solution of sulphuric acid. It settled rapidly as a coarse-grained powder, which was washed by decantation with hot water, then in a Gooch crucible with hot water, followed by strong alcohol. It was then dried in an air-bath.

In the first experiments it was treated with hot water, and the amount of barium sulphate precipitated from the filtrate was calculated as percentage of sulphur in 0.50 g rubber.

XXX. Washed 0.200-gram portions of $PbSO_4$ on filters. Each time 250 cc hot water was used. The filtrates were slightly acidified with HCl, and precipitated with $BaCl_2$.

Sulphur 0.24 0.25

These precipitates contained only traces of lead.

XXXI. Treated 0.200-gram portions of $PbSO_4$ according to Hübener's method, slightly modified. Treated with 13 cc conc. HNO_3, evaporated practically to dryness on the steam-bath, added 50 cc H_2O, and 0.5 cc Br and 2 cc of dilute HNO_3 (1:4). Heated, filtered, and washed with about 200 cc of hot water. Then precipitated with $BaCl_2$.

Sulphur 0.73 0.80 0.57

These precipitates contained a little lead.

[10] Kolb: Dingl. pol. J., 209, p. 268. Ditte: Ann. Chim. Phys. [5], 14, p. 190.

From these determinations it seems quite certain that the larger part of the sulphur found as soluble sulphate under XII must have come from the solution and decomposition of lead sulphate first formed when the rubber was attacked by nitric acid.

At the suggestion of Dr. Hillebrand, four determinations were made of the amount of lead carried down with the barium sulphate precipitated in the usual way from the aqueous extract of the fusion mass. In spite of the presence of a large excess of sodium carbonate, some lead goes into solution. The preliminary treatment was according to XVIII, and 2 g of rubber instead of 0.50 g was taken each time.

XXXII. After fusion, the melts were dissolved in water. To each of the first two there was added 2 g of sodium bicarbonate in order to decompose any alkali plumbate. The solutions were heated on the steam-bath for one and one-half hours and then filtered from the insoluble. After acidifying with hydrochloric acid, barium sulphate was precipitated in the usual way.

	1	2	3	4
$BaSO_4$ found (g)	0.5379	0.5395	0.5437	0.5417
Sulphur (per cent)	3.69	3.71	3.73	3.72

The barium sulphate precipitates were then mixed with soda and potash and fused. The melts were dissolved in water, filtered, and the residues washed with hot, very dilute sodium carbonate solution. The residues of barium carbonate and lead oxide were then dissolved in dilute nitric acid and the lead precipitated from the cold solutions by hydrogen sulphide. After standing over night in stoppered flasks, the precipitates of lead sulphide were filtered off, washed, dissolved in nitric acid, and finally converted into sulphate by evaporating down in porcelain crucibles with sulphuric acid and gently igniting.

	1	2	3	4
$PbSO_4$ found (g)	0.0086	0.0071	0.0040	0.0045
Equivalent to $BaSO_4$ (g)	0.0066	0.0055	0.0031	0.0035
Corrected $BaSO_4$ (g)	0.5359	0.5379	0.5428	0.5407
Corrected sulphur (per cent)	3.68	3.69	3.73	3.71

It is quite evident from these figures that although notable quantities of lead sulphate are carried down with the barium sulphate, the correction in the percentage of sulphur is negligible.

The filtrates from the original precipitates were treated with hydrogen sulphide and gave slight precipitates. The alkaline filtrates from the barium carbonate and lead oxide were tested with ammonium sulphide and became brown. The next day there was a slight film of a dark color on the bottom of each of the beakers in which these solutions were tested. This was probably a mixture of small amounts of lead and iron sulphides. In all the solutions tested, as well as in the actual determinations of lead sulphate, greater amounts of lead were found in 1 and 2, which had been treated with bicarbonate. Apparently at the temperature of the steam-bath the lead bicarbonate probably formed was not decomposed.

CONCLUSIONS

Treatment of the rubber with nitric acid alone gives low results. (Compare XV with XVIII.) This is probably largely due to loss of free sulphur, since nitric acid alone does not completely oxidize sulphur to sulphuric acid in the length of time ordinarily taken for a determination.

The Hübener method can not be employed in the presence of mineral fillers which tend to form insoluble sulphates. This applies especially to barium carbonate and litharge.

A comparison of XX to XXVII shows that the fusion method gives results very close to those obtained by direct precipitation and by neutralization. The van't Kruys method gives high results.

The best results seem to be obtained by the use of method XVIII, according to which the rubber is decomposed by means of nitric acid saturated with bromine.

WASHINGTON, July 19, 1911.

THE MEASUREMENT OF THE INDUCTANCES OF RESISTANCE COILS

By Frederick W. Grover and Harvey L. Curtis

CONTENTS.

	Page.
I. Introduction	456
Preliminary notions	456
II. Historical	458
Potentiometer methods	459
Resonance methods	459
Dynamometer methods	459
Bridge methods	460
III. Measurement of the Inductance of Coils of Low Resistance	461
1. Methods used	461
(a) Anderson's method	461
(b) Use of a variable capacity	462
(c) Use of a variable inductance	463
2. Apparatus	463
3. Calibration of the variable small resistance	464
4. Standards of reference	467
5. Experimental results	468
IV. Measurement of the Inductance of Coils of High Resistance	470
1. Effect of the capacity to earth on the standard	470
2. Adjustment of the potentials	474
3. Experimental results	479
4. Effect of the capacity to earth on the inductance to be measured	479
V. Less Precise Methods for High Resistance Coils	481
1. Use of a bridge with unequal arms	481
2. Use of a standard in parallel with the unknown	482
3. Experimental results	484
VI. Summary	484

I. INTRODUCTION

In the choice of a bridge method for the measurement of a given inductance it is necessary to take into account not only the order of magnitude of the inductance in question but also the value of the associated resistance. That is, the determining factor is the ratio of the inductance to the resistance, or, as it is generally designated, the *time constant* of the coil, rather than the value of the inductance alone.

The use of Anderson's method for the precise measurement of the inductance of coils having a time constant of the order of 0.001 second or greater has been treated at length in a previous article of this Bulletin.[1] Owing to the increasing precision demanded in electrical standardization, it is becoming more and more important to be able to measure the inductance of coils having time constants of much smaller value. For example, there may be cited the measurement of the small residual inductances of so-called noninductive resistance coils or of multipliers for voltmeter circuits. For such measurements the ordinary methods for the determination of inductance are more or less inapplicable or require extensive modification.

A good deal of time has accordingly been devoted to this subject at the Bureau of Standards, with the result that methods have been developed which are capable of giving results of a good degree of precision for time constants as small as 10^{-7} second, such, for example, as that of a resistance of 100 ohms having an inductance of a hundredth of a millihenry. It is hoped that a presentation of these methods and a discussion of the sources of error will be useful in stimulating interest in the subject and in calling attention to the errors which may arise from the use, without test, of apparatus supposed to be free from inductance.

PRELIMINARY NOTIONS

In addition to the effect of the inductance in causing the current in a given coil to lag behind the electromotive force impressed at its terminals, there has to be taken into account the influence of the capacity between the adjacent turns of wire and the capacity of the various parts of the coil with respect to the earth. These

[1] Rosa and Grover, this Bulletin, 1, p. 291; 1905. Reprint No. 14.

capacity effects give rise to a phase angle in the opposite direction to that occasioned by the inductance. The resulting phase angle will, therefore, be in one or the other direction, according as the effect of the inductance or that of the capacity preponderates.

It is convenient, in so far as its effect on the phase angle is concerned, to regard the capacity as equivalent to a negative inductance. For the ideal case of a capacity concentrated between the terminals of a coil, it is easy to show that the phase angle between current and impressed electromotive force is proportional to $L-CR^2$ (where R, L, and C are, respectively, the resistance, inductance, and capacity of the coil), and the current lags or leads according as this quantity is positive or negative.

In the case of a simple bifilar winding, the capacity between the wires is uniformly distributed, and the resultant phase angle depends on the value of $L-\frac{1}{3}CR^2$, where C is the capacity which would be measured between the wires if they were entirely disconnected from one another.

In an actual coil the effect of the capacity between the wires and to earth can not in general be calculated. We may, however, speak of the *effective inductance* of the coil, meaning thereby that inductance which would, at the same frequency, produce the observed phase angle. According to this definition the effective inductance L' is connected with the measured phase angle φ (which may be positive or negative) by the equation $\tan \varphi = \frac{pL'}{R}$, where $p = 2\pi$ times the frequency, L' being taken positive when the current lags behind the impressed electromotive force.

It is also often convenient, when the resistance is large, to regard the actual coil as replaced by an equivalent bifilar winding, having the same resistance, zero inductance, and a distributed capacity C' of such a value that the phase angle, calculated by the formula $\tan \varphi = -pC'R/3$, is equal to the observed phase angle of the coil.

In the case of low-resistance coils, the measured phase angle is positive and it is easy to show that the capacity effect is negligible. Since, however, the change in phase angle, due to a given capacity, is, as shown above, proportional to the square of the resistance associated with it, it is easy to understand that in coils of high resistance the capacity becomes the predominating factor.

Hence, the problem of the measurement of the effective inductance of a 1-ohm coil, having a time constant of 10^{-6} second, is quite different from that of the determination of the effective inductance of a 10 000-ohm coil having the same value of time constant. In the case of the 1-ohm coil, the inductance of all lead wires must be allowed for or eliminated, and the mutual inductance of one bridge arm on another must be shown to be negligible, or a correction applied. In the measurement of the 10 000-ohm coil, however, although the inductances of all ordinary lead wires and the mutually inductive effects of the various bridge arms may be regarded as of negligible importance, all capacities between the various parts of the bridge and to earth, even though no greater than a few millionths of a microfarad, may produce a measurable effect on the balance of the bridge.

The work, therefore, divides itself logically into two sections, which will be separately treated—first, methods for the measurement of the effective inductance of coils of small time constant and small resistance (1 ohm or thereabouts); and, second, methods adapted to coils having small time constant, and resistances greater than about 1000 ohms. Coils of intermediate values of resistance may be treated by modifications (which will readily suggest themselves) of these methods for extreme values.

II. HISTORICAL

The various methods which have been previously employed for the measurement of the inductances of coils having small time constants, may be roughly grouped into four classes, which will be briefly reviewed below. However, the papers mentioned by no means exhaust the list which has been consulted.

Potentiometer methods.—In 1909 Orlich[2] published an electrometer method for determining the difference in inductance of two conductors, carrying the same current, the standard of reference being a rectangular bar, whose inductance was calculated from its dimensions. This method, which is especially well adapted to small resistances of high current-carrying capacity, and, in particular, those provided with potential terminals, gave an accuracy of about 1 per cent with coils having a resistance of 0.001 ohm and inductances of the order of 10 cm.

Resonance methods.—Taylor[3] measured very small inductances by a substitution method in a resonating circuit. Two parallel wires with a movable contact served as a standard, the inductance being calculated from the length and diameter of the wires and their distance apart. He confined himself entirely to coils of small resistances, an accuracy of a few per cent being obtained with time constants of about 10^{-5} second.

Dynamometer methods.—A number of electrodynamometer methods have been devised for the measurement of small inductances, their essential differences lying, for the most part, in the means employed for compensating the change in phase produced by the insertion of the unknown coil into the circuit.

For example, in the method of Blondel[4], who used a simple electrodynamometer and two phase currents, the inductance to be measured is made to depend on the value of a known inductance or a known resistance, or on both.

Martienssen[5] employed a special electrodynamometer with a suspended metal cylinder as moving system, and restored the original adjustment of the circuits for zero torque on the cylinder (disturbed by the introduction of the unknown coil) by varying the resistance in the circuit of an auxiliary field coil. The inductance of the coil to be measured was thus made to depend on a known mutual inductance and a ratio of two resistances.

In none of these dynamometer methods, however, does any

[2] Zs. für Instk., **29**, p. 241; 1909.
[3] Phys. Rev., **19**, p. 273; 1904.
[4] Ecl. Elect., **21**, p. 139; 1899.
[5] Ann. der Phys., **67**, p. 95; 1899.

high degree of accuracy seem to have been obtained with coils of very small time constant.

Bridge methods.—Essentially two bridge methods have been employed, namely, (*a*) Maxwell's method for the comparison of two inductances[6], and (*b*) Maxwell's method for the comparison of an inductance with a capacity[7] together with Anderson's modification[8] of the same. In most cases alternating current of frequencies ranging from 256 to 3000 cycles have been used, although La Rosa[9] obtained a satisfactory sensibility using direct current.

Wien and Prerauer[10], using the method for the comparison of two inductances, measured inductances as small as 500 cm by indirect comparison with an absolute standard of one millihenry, through auxiliary standards of intermediate values, obtaining an accuracy of about 1 per cent with time constants of about 10^{-5} second.

In a low-inductance bridge such as was used by them, the simple relation derived by Maxwell no longer holds, but the inductances of all the arms of the bridge must be taken into account, and the method becomes increasingly inconvenient and subject to error as the time constant is smaller.

To obviate this difficulty, an important modification was introduced by Giebe[11], who showed that, if relatively large inductances be inserted in the ratio arms of the bridge, the formula of Maxwell is subject to only a slight correction, which may be determined experimentally. Thus he was able to measure, with a good degree of accuracy, inductances of the order of 500 cm and with time constants as small as 10^{-8} second. Here, again, the inductance to be measured is made to depend on a standard, whose value can be calculated from its dimensions.

The method of comparing an inductance with a capacity has been used to measure the effective inductance of resistance coils of the order of 1000 ohms, by Taylor and Williams[12] and by Brown.[13]

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[6] Elect. and Mag., vol. 2, art. 757.
[7] Elect. and Mag., vol. 2, art. 778.
[8] Phil. Mag., **31**, p. 329; 1891.
[9] Atti della R. Accad. dei Linc.; June 3, 1905.
[10] Wied. Ann., **53**, p. 772; 1894.
[11] Ann. der Phys., **24**, p. 941; 1907.
[12] Phys. Rev., **26**, p. 417; 1908.
[13] Phys. Rev., **29**, p. 369; 1909.

In both cases the unknown coil was compared, by substitution, with a standard of the same nominal resistance. The 1000-ohm standard, employed by Brown, was made by stretching two German silver wires parallel to one another, and at such a distance apart that the calculated inductance and capacity effects were balanced.

III. MEASUREMENT OF THE INDUCTANCE OF COILS OF LOW RESISTANCE

Attention has already been called the fact that the direct measurement of the inductance of coils of small resistance is rendered difficult by the necessity for knowing the inductances of all lead wires and auxiliary resistance coils and the mutual inductances between the various arms of the bridge. In the case of coils having relatively large time constants these difficulties may in part be avoided by the use of a method of difference. The bridge is successively balanced with the unknown coil in place and with the coil removed, the change in balance giving a measure of the required inductance. For the success of this method it is necessary to take into account the inductance of the resistance coils, which must be added to compensate for the resistance of the coil, as well as the inductance of the conductor used for closing the gap left by the coil on its removal.

The ideal method of procedure is to measure the unknown coil by a method of substitution, using a standard of exactly the same resistance as the unknown, but of such a form as to allow its inductance to be calculated from its dimensions, and in so far as was found possible the methods here employed were made to conform to this condition. The preparation of absolute standards for these measurements and the method of measuring the inductance of the small variable resistance used in balancing the bridge will be separately treated below.

1. METHODS USED

a. Anderson's method.—The standard having been inserted in the Q arm of the bridge, Fig. 1, the bridge is balanced by alternately varying the resistance r and the resistance Q' in the Q arm, external to the standard. To obtain balance with a convenient value of r, it may be necessary to place a small auxiliary inductance

in series with the standard. Balance having been obtained, the unknown is substituted for the standard, and, everything else being left untouched, the bridge is again balanced by varying r

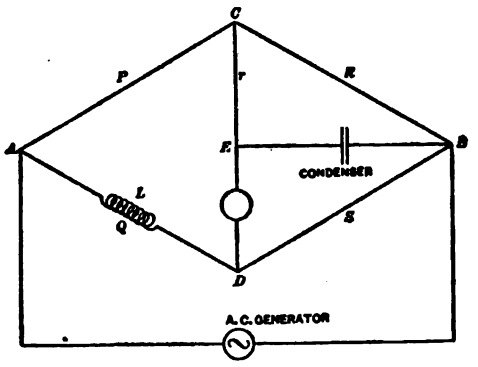

Fig. 1 —*Anderson's Bridge*

and Q'. If the observed changes in r and Q' be denoted, respectively, by Δr and ΔQ, and if Δl be the change in inductance corresponding to ΔQ, the difference in inductance, ΔL, between the unknown and the standard is given by the relation

$$\Delta L = 2CS \,.\, \Delta r - \Delta l \qquad (1)$$

Since Δl is a correction, it should be made as small as possible, which is the reason for using a standard whose resistance is adjusted to be closely equal to that of the unknown. Care must, of course, be taken to take into account the algebraic signs of Δr and Δl.

In order that Δr may be as large as possible in any given case, C and S should be made small. Reducing C, however, tends to reduce the sensitiveness of the setting of the bridge. There is, however, rather a wide latitude allowable in the choice of the bridge constants.

b. Use of variable capacity.—For coils having a very small inductance, it is often advantageous to omit the resistance r in the bridge and to obtain a balance by adjusting the capacity instead. This is a special modification of Maxwell's method for comparing an inductance with a capacity.

If ΔC denote the change in the capacity necessary to obtain a

balance, when the unknown coil is substituted for the standard, the difference in inductance between the unknown and the standard is given by the equation

$$\Delta L = PS \cdot \Delta C - \Delta l. \tag{2}$$

The variation of C is conveniently obtained by means of a calibrated variable air condenser.

In case the two above methods are combined, it is about as easy to obtain ΔL from the two values of L as to use the formula for ΔL. (See Example 1 below.)

c. Use of a variable inductance.—If a calibrated variable inductance of suitable range is at hand, it may be used, in series with the standard of reference, in one of the arms of a simple Maxwell's bridge for comparing two inductances. Then

$$\Delta L = L_0 - L_1 - \Delta l \tag{3}$$

where L_0 is the reading of the variable inductance with the standard in the bridge and L_1 the reading with the unknown.

For the calibration of the variable inductance one of the foregoing methods was used.

2. APPARATUS

The current for the measurements at low frequencies was taken from a one-half kilowatt generator, driven by a motor which was supplied from a storage battery. The frequency of the current was about 100 cycles per second, the exact value being determined by the natural period of the Ruben's vibration galvanometer, used to determine the point of balance of the bridge. The impedance of the galvanometer being considerably larger than the impedance of the other arms of the bridge, it was found advantageous to use a step-up transformer in the galvanometer circuit. With a 3:1 ratio on a 600-watt transformer the sensitiveness of the bridge setting was a little more than doubled.

The high frequencies, 200 to 3000 cycles, were obtained from an alternator with two stationary armatures and revolving fields, capable of giving 4.5 amperes at 110 volts at frequencies of 600 or 3000 cycles per second. By varying the speed of the driving motor a range of frequencies from 200 to 3000 cycles may be

obtained. In the absence of a vibration galvanometer adapted to these frequencies, a low-resistance telephone served as indicating instrument in balancing the bridge, a procedure which was allowable in the present instance, since the conditions for balance, in the bridge methods used, are sensibly independent of the frequency.

Inasmuch as the change in the total impedance due to a given small change of inductance becomes more important with increasing frequency, the sensibility of the bridge should be greater at the higher frequencies than at the lower. On the other hand, the sensibility of the ear begins to decrease for sounds of frequencies much above 1000 per second, and, in addition, the natural periods of vibration of the diaphragm of the telephone have an appreciable effect on the position of the observed frequency for maximum sensitiveness. Thus, for example, with one of the telephones used, the maximum sensitiveness of the bridge was reached with a frequency of about 1500 and had fallen off greatly at 3000. With the other telephone, on the contrary, the maximum sensitiveness probably lies somewhat above 3000 cycles per second. In both cases the telephone became silent when the balance point of the bridge was reached, no trouble being experienced from higher harmonics. This does not, however, necessarily point to any very high degree of purity of the emf wave employed, since the effect of the reduced sensibility of the ear for these very high frequencies is to largely diminish the effective amplitude of the harmonics relatively to that of the fundamental.

3. CALIBRATION OF THE SMALL VARIABLE RESISTANCE

In the ideal substitution method the resistances of the unknown and the standard would be exactly equal, and it would be merely necessary to provide means in the bridge for varying the inductance without any accompanying change in the resistance. Since, however, the desired degree of approximation of the resistances of unknown and standard is often difficult of attainment, it is desirable to devise means for changing the resistance without, at the same time, appreciably changing the inductance, in order that the

correction Δl in equations 1, 2, and 3 may be kept as small as possible. The problem of obtaining change in inductance with a constant resistance has been successfully solved in a number of existing forms of variable inductances. The reverse problem of obtaining variation of resistance with a constant inductance is not so easy of solution.

An obvious method for doing this is to prepare resistances of simple forms, having small but calculable inductances, one of the units being provided with a sliding contact to allow of a continuous variation of resistance in balancing the bridge. It is, however, difficult, if not impossible, to construct a slide wire with a contact resistance sufficiently constant to be satisfactorily employed in one arm of a low-resistance bridge. Further, the variation of the resistance in steps, with a negligible resultant change in the inductance, may be carried out more simply as follows:

Links are prepared of two metals of widely different specific conductivities (for example, copper and manganin), the wires being drawn through the same die so as to have the same diameter, and cut and bent so as to have, as closely as possible, the same form and dimensions. It is easy to so determine the length that the resistances of the copper and the manganin links shall differ by some convenient value, such as 0.1 ohm. Mercury cups are arranged so that any link may be placed in the circuit in the same, perfectly definite, position as any other, and with the ends of the wire dipping into the mercury just sufficiently to make contact. If now, for example, one of the copper links be replaced by a manganin link, the resistance in circuit is increased by 0.1 ohm, but since one link has exactly the same dimensions and occupies exactly the same position in the circuit as the other, the change of inductance, even with a considerable skin effect, is entirely negligible.

To provide the necessary means for obtaining a continuous variation of the resistance over a range of one-tenth ohm, a slight modification of the form of variable mercury resistance[14] designed

[14] Phys. Rev., **32**, p. 614, 1911; and **33**, p. 215, 1911.

by Dr. F. Wenner and Mr. J. H. Dellinger of the Bureau of Standards was employed. Two straight glass tubes, one having approximately ten times the cross section of the other, were joined together as a U-tube and mounted vertically. The system was then filled with mercury up to the copper terminals, which extended into small glass cups blown on the end of each tube, to prevent any overflow of the mercury. To obtain the variation of resistance two copper wires are selected, of such diameter that each shall closely fill the cross section of its respective tube. Owing to the greatly superior conductivity of the copper, it effectively short circuits that portion of the mercury column into which it dips. The larger tube serves for the finer adjustment of the resistance.

This form of variable resistance has shown itself eminently satisfactory for the present purpose. Whereas, with a carefully constructed slide-wire resistance in series with a total resistance of 1 ohm in the bridge arm, the fluctuations of the current were so large and erratic that no close settings of the bridge could be made, the mercury variable was, on the contrary, free from all fluctuations of resistance as great as one hundred-thousandth of an ohm, only a small steady drift of resistance, due to temperature changes, being noted.

The mercury variable offers a further valuable advantage, in that its inductance changes only very slightly as the resistance is varied. If the copper wires could be made exactly to fill the cross section of their tubes, the inductance would, with currents of low frequency, be sensibly independent of the position of the wires. Since, however, this condition is only approximately realized, it is necessary to calibrate the instrument. This was accomplished by means of copper and manganin links of suitable lengths as described above. As a result of this calibration it was found that the change of inductance is practically linear, the changes in inductance corresponding to the entire ranges of the fine and coarse adjustments, being only about 12 and 7×10^{-9} henrys, respectively. Since it is often unnecessary to make any considerable changes in the fine adjustment when balancing the bridge, the more considerable source of variation of the inductance may usually be avoided.

It may be noted in passing that the use of such a variable is not confined to currents of low frequencies, since it is possible by calculation to correct the calibration with all the accuracy necessary to take into account the skin effect.

4. STANDARDS OF REFERENCE

The forms of standards for use in the substitution method which most readily suggest themselves are parallel wires, rectangles, and circles.

The inductance of two parallel wires of permeability μ, radius of wire ρ, length l (of a single wire) and distance apart d, is given with sufficient accuracy by the formula

$$L = 4l\left[\log_e \frac{d}{\rho} + \frac{\mu}{4} - \frac{d}{l}\right] \tag{4}$$

provided l is greater than 10 d and d is greater than 10 ρ.

If the wires are close together or, in any case, if the resistance is large, it may be necessary to take into account their capacity. The effective inductance is given by

$$L' = L - \frac{CR^2}{3} \tag{5}$$

where R is the resistance of the wires and C is the capacity which would be measured between the wires if they were disconnected from one another. The calculation of C is treated later.

For circles of wire of circular cross section

$$L = 4\pi a\left[\left(1 + \frac{\rho^2}{8a^2}\right)\log_e \frac{8a}{\rho} + \frac{\rho^2}{24a^2} - 1.75\right] \tag{6}$$

where a is the radius of the circle, and ρ is the radius of the cross section.

The principal source of uncertainty in the use of such standards of reference lies in the difficulty of estimating the inductance and mutual inductive effects at the terminals. The use of large binding posts is to be avoided. That no appreciable inconsistency resulted from these causes in the standards here used is shown by the following examples.

5. EXPERIMENTAL RESULTS

EXAMPLE 1

Two standards of 1-ohm resistance were prepared, both of manganin wire 0.4 mm in diameter, the wire in one being arranged in a loop with the two legs 1 cm apart, while in the other it was bent into a circle of 5.80 cm radius with a gap of 1.8 cm between the ends. Each standard was provided with binding posts, in one case set perpendicular to the plane of the parallel wires, and in the other perpendicular to the plane of the circle. The self inductance of the binding posts was calculated and the mutual inductance with respect to the wire was assumed to be negligible.

The inductances were calculated, giving the following values:

Parallel wires,	Circle.
279	437
8 End	−22 Gap correction
4 Binding posts	4 Binding posts
291 cm	419 cm.

The calculated difference was therefore 128 cm. The difference actually measured by a combination of methods 1 and 2, above, was 128 cm, thus agreeing to the nearest cm. The details of this measurement are given below.

Referring to Fig. 1, the P and R arms each included a 1-ohm sealed resistance standard. In the S arm, in addition to a 1-ohm standard, was placed a coil consisting of three or four turns of copper wire wound to a radius of about 4 cm. The Q arm was made up of the unknown (or standard) to which was added the variable mercury resistance already described. The 1-ohm standards were hung between the mercury cups of a bridge designed for precision resistance measurements. All other connections were made to the binding posts of this bridge, and thus was avoided any indefiniteness in the relative positions of the parts of the bridge, a very necessary precaution with such small inductances. The small inductance inserted in the S arm was to insure obtaining a balance of the bridge. A small resistance was used in r for convenience, and for the capacity a 0.1-μf condenser, shunted by a 0.005-μf variable air condenser was employed.

The latter could be set to about 1 degree of the scale which corresponds to about 22×10^{-12} microfarad, or 2 parts in 10 000 of the whole capacity. The measured value of the resistance S was 1.056 ohms. The remaining data are as follows:

	r	C	CS(2r+P)	Corrected	L
	ohms	μf	cm	cm	cm
Circle	3.1	0.10217	778	778	(419)
Parallel wires	3.7	0.10178	905	906	291
Coil 19a	4.9	0.10217	1167	1167	29
Coil 20a	4.9	0.10180	1161	1161	35

For the order of accuracy aimed at (1 in 1000) the change in r could be taken as equal to the difference in the nominal values. The values in the fifth column are corrected for the inductance of the mercury resistance. The last column gives the absolute values of the inductances, assuming for the circular standard the value calculated above. Coils 19a and 20a are specially wound coils.

(NOTE.—The capacity was connected in this bridge not, as in Fig. 1, to the junction of R and S, but to the junction of P and Q. The reading of the bridge increases, therefore, for a decrease in the inductance in Q.)

EXAMPLE 2

The accuracy of the methods was also tested for resistances of larger values. For this purpose parallel wire standards of manganin wire, stretched on vulcanite spacing blocks attached to a long board, were prepared, of such values as to allow a step up from lower values to higher, and the observed differences in inductance could thus be compared with the calculated.

It was found convenient to employ method 3 in the measurements, making use of a previously calibrated variable inductance of about 12-microhenrys range. The other three arms of the bridge were made up of resistance standards hung in mercury cups, and to take care of the small differences in resistance, the tenth ohms of copper and manganin links and the variable mercury resistance were used. Care was taken, when using two or more parallel wire standards, to place them perpendicular to one another, to avoid effects of mutual induction.

The following table gives an idea of the consistency of the results obtained, and some estimate can be made as to how thoroughly constant errors have been eliminated.

Coils compared	Observed difference	Calculated difference
	Microhenrys	Microhenrys
100_1-100_2	0.16	0.17
100_1-100_3	0.16	0.17
$(200+100_3)-300$	2.79	2.71
$(100_1+100_2+100_3)-300$	7.83	8.08
$(300+200)-500_1$	3.67	3.52
$(300+200)-500_2$	4.02	3.59
$(500+300+100_2+100_3)-1000$	11.4	11.5
$(500_1+500_2)-1000$	2.92	3.04

IV. MEASUREMENT OF THE INDUCTANCE OF COILS OF HIGH RESISTANCE

In the previous section it has been shown that, over a wide range of resistances, the simple substitution method may be depended upon to give reliable values of the inductance. However, in the measurement of coils whose resistance is of the order of 1000 ohms, or greater, where a small capacity can produce an appreciable change of the phase angle, special care must be taken to avoid serious errors. The effect of the capacity between the windings has already been discussed and has been shown to be proportional to the square of the resistance around which the capacity is shunted. There remains yet to be considered the action of the capacity between the various parts of the coil and the earth, and it will be shown that this effect, which is negligible in the case of low resistances, may be important with coils of no greater resistance than a few thousand ohms.

It is convenient to consider separately in this connection (*a*) the earth capacity of the standard and (*b*) that of the coil to be measured.

1. EFFECT OF THE CAPACITY TO EARTH ON THE STANDARD

If the two parallel wires represented in section in Fig. 2 be brought, respectively, to potentials v_1 and v_2, the quantities of

electricity q_1 and q_2 on the wires will be given by

$$q_1 = c_1 v_1 + c_{12} (v_1 - v_2)$$
$$q_2 = c_2 v_2 - c_{12} (v_1 - v_2)$$

where c_1, c_2, and c_{12} are positive constants depending upon the configuration of the system. These are the classical Maxwell

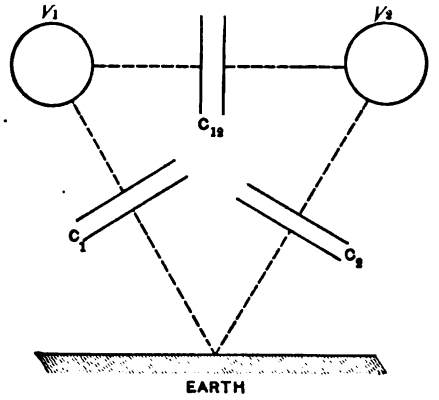

Fig. 2.—Capacities of Two Parallel Wires Above the Earth

equations, for a system of charged conductors, put into a modified and more convenient form.[15] According to these equations, we may regard the quantity of electricity on 1 as composed of two parts, viz, a displacement between the earth and 1, and a displacement between 2 and 1. Similarly, the quantity q_2 on 2 is the resultant of a displacement between the earth and 2 and a displacement between 1 and 2. That is, if we consider the lines of electrostatic force, then $c_1 v_1$, $c_2 v_2$ and $c_1 (v_1 - v_2)$ are respectively proportional to the numbers of lines which join 1 and the earth, 2 and the earth and 1 and 2. The system 1, 2, earth, is therefore equivalent to a network of three simple capacities arranged as in Fig. 2.

If, instead of two wires, we consider the case of two plates whose distance apart is small compared with their dimensions, as in an ordinary condenser, the coefficient c_{12} is very large compared with c_1 and c_2, and we may, without sensible error, speak of the *capacity* of the condenser without specifying the surroundings of

[15] Orlich: Kapazität und Induktivität (Friedrich Vieweg u. Sohn, Braunschweig), pp. 20–22.

the latter. Where, however, c_1 and c_2 become relatively more important as in the case of the wires it is evident that the quantities of electricity q_1 and q_2 will in general be different and will depend on the potentials v_1 and v_2. If an alternating potential difference be impressed upon the two wires, the charging current flowing on to one wire, at any instant, will not, in general, be equal to that flowing away from the other, since the displacement current between one wire and the earth will not be equal to that flowing between the other wire and the earth.

It is evident, therefore, that one can not speak of the capacity of the wires in the usual sense. The phase difference of the parallel wire standard depends not only on the size of the wires and their distance apart, but also on their absolute potentials, and can not usually be simply calculated.

However, in the important special case that $q_1 = -q_2$ the general equations above take the following simple forms:

$$c_1 v_1 = -c_2 v_2$$

$$C = \frac{q}{v_1 - v_2} = c_{12} + \frac{c_1 c_2}{c_1 + c_2} \tag{7}$$

The quantity C here is the capacity, in the usual sense, between the two wires, and is seen to be equal to the joint capacity of the condenser c_{12} joined in parallel with two condensers c_1 and c_2 in series.

In the case of two equal parallel wires $c_1 = c_2$ and the equations become

$$v_1 = -v_2 \text{ and } C = c_{12} + \frac{c_1}{2} \tag{8}$$

Therefore, if the absolute potentials of the two wires be so adjusted that, at every moment, the potential of one is just as far above the earth potential as the other is below it, not only will the quantities of electricity on the two wires be equal and of opposite sign, but they will be proportional to the potential difference between the wires. That is, the displacement current is the same as would flow in a single condenser (capacity to earth negligible) having a capacity $C = c_{12} + \frac{c_1}{2}$, subjected to the same voltage.

This capacity C is what is ordinarily designated (without, however, a clear statement of the conditions to be fulfilled by the potentials) as the "capacity" of the system. Orlich, however, whose method of treatment has been followed here, very clearly states the conditions (loc. cit., p. 26) and recommends for the capacity C the name "Betriebskapazität" or "working capacity," a term which will be employed below.

The working capacity of two parallel wires arranged at equal distances from the earth is [16]

$$C=\frac{kl}{4\log_e\frac{2h}{\rho}-2\log_e\left(1+\frac{4h^2}{d^2}\right)}\text{ electrostatic units} \tag{9}$$

where

k = dielectric constant of the medium
l = length of each wire
ρ = radius of wire
d = distance between the centers of the wires
h = height of the wires above the earth.

This may be thrown into the form

$$C=\frac{kl}{3.6\left[\log_e\frac{d}{\rho}-\log_e\left(1+\frac{d^2}{4h^2}\right)\right]}\times 10^{-6}\text{ microfarads} \tag{10}$$

$$=\frac{kl}{3.6\left[\log_e\frac{d}{\rho}-\left(\frac{d^2}{4h^2}+\frac{d^4}{32h^4}+\ \ldots\right)\right]}\times 10^{-6}\ \mu\text{f.} \tag{11}$$

The correction terms in $\frac{d}{h}$ are, however, negligible in most practical cases. For example, in the case of two wires for which $l=1300$ cm, $d=10$ cm, $\rho=0.0025$ cm; the difference of the capacity as calculated with and without the terms in $\frac{d}{h}$ for a value of h as small as 10 cm, was only 3 per cent; and with $h=100$ cm the difference was smaller than the errors of observation.

The working capacity of a parallel wire standard can be made,

[16] Russell: Alternating Currents, vol. 1, p. 133.

therefore, perfectly definite and can be accurately calculated by equation (11); provided, of course, *that the absolute potentials of the wires satisfy the condition* $v_1 = -v_2$, *as discussed above.*

This condition is, however, of prime importance, as was shown by experiment. For example, with an equal arm bridge, insulated from the earth, the observed value of the capacity of the parallel wires, whose dimensions are given above, was twice as great as that calculated. With small resistances this source of error is not important. On the other hand, with the 5000-ohm standard, which was made by joining the further ends of these wires, the phase difference depends almost entirely on the capacity. In order, therefore, to use the calculated constants of the standard, means must be provided for the adjustment of its potential so as to satisfy the condition $v_1 = -v_2$, or, what is equivalent, to adjust the potential of the middle of the coil to zero potential.

2. ADJUSTMENT OF THE POTENTIALS

Fig. 3 represents a bridge to one arm of which have been connected two parallel wires A and B. In accordance with Fig. 2, we may represent their impedances due to their capacities to earth by α and β, their connections being drawn in dotted lines to distinguish them from the actual conductors of the bridge. The capacity c_{12}, Fig. 2, is not indicated, but is included in the impedance R.

The desired condition which the potentials v_1 and v_2 of A and B are to satisfy is that v_1 shall be as much above earth potential as v_2 is below it, or vice versa. The supply voltage may be taken from a transformer or from a rheostat B placed across the terminals of a well-insulated generator. In either case, if the entire system is insulated from the earth, the center will be close to earth potential. The potentials of A and B will not, therefore, normally fulfill the desired condition, and means must be provided to arbitrarily ground the supply at the correct point. In figure 3 this is done by means of a variable contact D, grounded through an impedance M. The total voltage E of the supply is thus divided into two parts $K\,E$ and $(K-1)\,E$, one being above and the other below earth potential. We will next consider what value K must have in order that the potentials of the wires A and B

shall have values symmetrically above and below the earth potential.

Indicate the impedances of the different arms by the symbols used in Fig. 3 and designate by x, y, z, u, and v the currents in

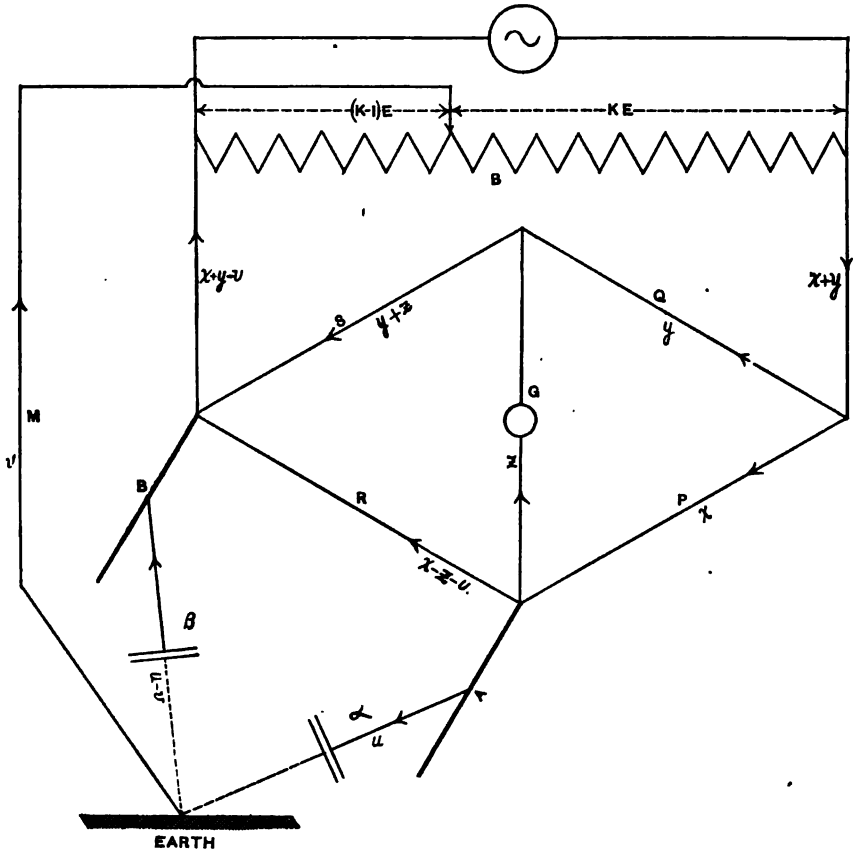

Fig. 3.—*Special Bridge in which the Impedances and Currents are Indicated*

the impedances P, Q, G, L, and M, respectively. Then by Kirchhoff's first law, the currents through the various impedances will be as follows:

P,	Q,	R,	S,	G,	α	β	KB,	$(K-1)B$,	M
x,	y,	$(x-z-u)$,	$(y+z)$,	z,	u,	$(u-v)$,	$(x+y)$,	$(x+y-v)$,	v.

Applying Kirchhoff's second law to five of the closed circuits of the network, the following equations are deduced:

$$Px-Qy+Gz=o$$
$$R\,(x-z-u)-S\,(y+z)-Gz=o$$
$$R\,(x-z-u)-\beta\,(u-v)-\alpha u=o$$
$$KB\,(x+y)+Px+\alpha u+Mv=KE$$
$$KB\,(x+y)+Qy+S\,(y+z)+(K-1)\,B\,(x+y-v)=E.$$

The general condition which must be satisfied that the current in the galvanometer arm may be zero ($z=o$) is as follows:

$$\begin{vmatrix} P & -Q & o & o & o \\ R & -S & -R & o & o \\ R & o & -(R+\alpha+\beta) & o & \beta \\ KB+P & KB & \alpha & KE & M \\ B & B+Q+S & o & E & (K-1)B \end{vmatrix}=o \qquad (12)$$

or

$$[K\,(K-1)\,BE-EM]\begin{vmatrix} P & -Q & o \\ R & -S & -R \\ R & o & -(R+\alpha+\beta) \end{vmatrix}$$

$$+E\beta\begin{vmatrix} P & -Q & o \\ R & -S & -R \\ KB+P & KB & \alpha \end{vmatrix}-KE\beta\begin{vmatrix} P & -Q & o \\ R & -S & -R \\ B & (B+Q+S) & o \end{vmatrix}=o \qquad (13)$$

For correct adjustment of the potentials, there will be no resultant displacement from the wires A and B to earth; that is, the displacement current u from A to earth is equal to the displacement current $(u-v)$ from the earth to B. This means that v must equal zero, or there will be no current in M.

Therefore, for simultaneous balance of the bridge and correct adjustment of the potentials of A and B two conditions must be fulfilled, namely, $z=o$ and $v=o$. The desired value of K may be obtained from the solution of these two simultaneous equations. The work may, however, be simplified from the following consideration:

When the current v is zero, the potential difference on the impedance M must also be zero, and since that portion of the current z, which is due to the presence of M, must vanish when the potential difference on M is zero, the coefficient of M in equa-

tion (13) must become zero when $v=o$. That is—

$$\begin{vmatrix} P & -Q & o \\ R & -S & -R \\ R & o & -(R+\alpha+\beta) \end{vmatrix} = o$$

which is equivalent to

$$PRS+(\alpha+\beta)(PS-RQ)=o \tag{14}$$

The remaining two determinants in (13) reduce to the simple relation

$$PQR-KPR(Q+S)-\alpha(PS-RQ)=o \tag{15}$$

which, on elimination of $(PS-RQ)$ by means of the preceding equation, gives finally

$$K=\frac{Q+\frac{\alpha}{\alpha+\beta}S}{Q+S} \tag{16}$$

or for two wires of the same dimensions, since $\alpha=\beta$

$$K=\frac{Q+\frac{S}{2}}{Q+S} \tag{17}$$

To check the work we have also expanded the determinant for v and shown that if the conditions (14) and (15) are satisfied it reduces identically to zero.

The preceding discussion leads at once to a simple method for adjusting the potentials of the wires A and B (see Fig. 4). With wires of equal size, which is the only case with which we are concerned, the adjustment of the potentials of A and B will be correctly attained if the potential of the middle point W of the resistance R' be made zero. The supply voltage to the bridge is taken from the terminals of a rheostat B, on which is a movable contact D, permanently connected to earth through the wire F. The key N being open, the bridge is balanced as usual. Then, with N closed, the position of the contact D is changed until no current flows in the telephone T. The balance of the bridge will usually be somewhat disturbed by the latter adjustment, and should be restored and the position of D again adjusted. The balance of the bridge

is not appreciably affected by a small lack of adjustment (1 per cent) of the potential of W, so that this second approximation should be ample. Nevertheless, the adjustment of D having been accomplished, the circuit through T should be left open while the measurements of inductance are being made, since any small current in T, due to imperfect adjustment of the potentials, has to flow in part through the galvanometer arm.

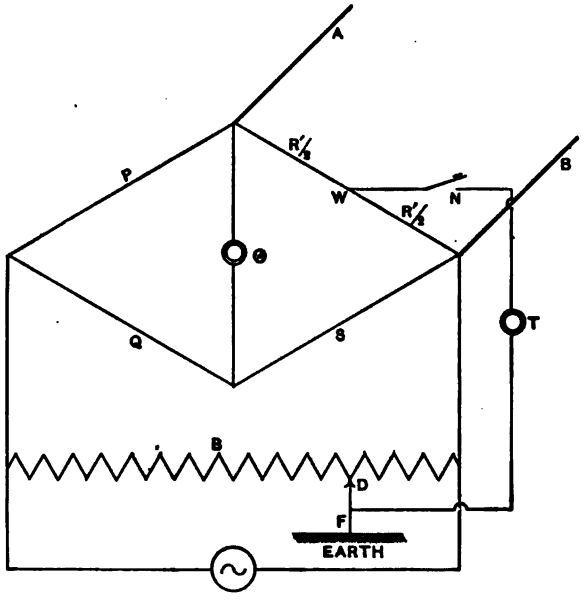

Fig. 4.—*Method of Adjusting the Potentials*

It should be noted that, in the analytical discussion of the bridge, it was tacitly assumed that the time constant of every part of the impedance B is the same, so that the phase angle of the portions KB and $(K-1)B$ are the same. Since this condition will rarely be exactly realized in practice, it will usually be necessary to insert a small variable inductance in B, at one or the other end of the circuit, in order to bring the telephone T to silence. Usually, however, the adjustment of the potentials can be made with sufficient accuracy by setting D to give a minimum sound in the telephone, and the inductance may be dispensed with.

3. EXPERIMENTAL RESULTS

As a check on the sufficiency of the adjustment of the potentials by the method just described, measurements were made on a 5000-ohm parallel wire standard of 0.05-mm manganin wire. The bridge having been balanced with the wires 10 cm apart, this distance was successively reduced to 5 cm and 2 cm, and the change in inductance noted. In every case the observed difference agreed well with the computed, as the following table will show:

Distances between the wires in centimeters	Difference of Inductance in microhenrys	
	Observed	Computed
10-5	49	49
10-2	114	110

4. EFFECT OF EARTH CAPACITY ON INDUCTANCE TO BE MEASURED

It has been shown in the preceding sections that if the potentials of a parallel wire standard forming one of the arms of an alternating current bridge be so adjusted that the potential of one wire is as far above earth potential as the other is below it, the capacity of the standard may be calculated. The inductance and capacity being both calculable, the phase angle of the standard is known, and the substitution method, whose application to coils of 1000 ohms or less has already been considered, may also be used for higher resistances, at least up to 10 000 ohms.

It must be emphasized, however, that the phase angle of a coil thus measured is the value holding only for the case that its center is at earth potential. Unless the capacity of the coil to earth is negligible, the phase angle of the coil will vary with any deviation from this distribution of potential, and this effect may be very important. Evidently, for this reason, the capacity of the coil with respect to the earth should be kept as small as possible.

This has special application to the connection of resistance coils in series. If, for example, we know the effective inductance of each of two coils, the effective inductance of the two coils in series

can not in general be taken as the sum of the individual values. For if the phase angle of each coil has been measured with the center of the coil at zero potential, this condition can no longer hold for the combination in series. If the phase angle of the coils in series be measured by the substitution method, the point of zero potential will be the junction of the two coils. A further source of error lies in the fact that the mutual capacity effect of two adjacent coils in a resistance box is often appreciable.

Campbell [17] suggested that each coil, or group of coils, in a resistance box be surrounded by a metal shield, joined to some definite point of the circuit. This procedure undoubtedly gives a perfectly definite capacity of the coil with respect to its surroundings, although the actual values of the capacity are thereby increased and need to be determined with correspondingly increased percentage accuracy.

If such a shielded resistance box is always used under exactly the same conditions which held during calibration, it undoubtedly may be depended upon to give a perfectly definite and known phase angle. If, on the contrary, the resistances are to be used for a variety of purposes, it will not, in general, be possible to keep the point of zero potential always at the same place in the system, and although the earth capacities of the coils are still perfectly definite, the values of phase angle will no longer be those of the calibration. In fact, the error in the phase angle, due to a given variation in the position of zero potential, will usually be much larger than in the case of well-designed and well-mounted unshielded coils. Just as has been shown for parallel wires, the capacity of a coil with respect to its surroundings varies only slowly with change of position, except in the immediate vicinity of extended metallic surfaces or other coils carrying current, and the effect of such changes will be reduced by keeping the capacity of the coil small. By experiment we find that with coils of small capacity the effect of changes of capacity with the position of the coil are only appreciable for distances between the coil and other coils and conductors less than about 5 cm.

[17] Elect. World, 44, p. 728, 1904.

It is accordingly best, at least in the case of coils to be used for a variety of purposes, to make the capacity of the coils as small as possible, and to mount each so that it is at least several centimeters from any other coil. The phase angle of each coil and of each desired combination should be measured by the substitution method, and, in so far as is possible, in using the coils the position of zero potential should be adjusted to conform with that for which the phase angles were measured. With properly designed coils of small capacity, not too close to one another, all extended metal surfaces being avoided in the neighborhood of the coils, the errors in phase angle owing to unavoidable deviations from the ideal positions of the point of zero potential will not be appreciable except in work of high precision.

V. LESS PRECISE METHODS FOR HIGH RESISTANCE COILS

When the effective inductances of resistance coils of large value have to be measured, the use of the substitution method is attended by the difficulty of preparing a standard. For example, each wire of the 5000-ohm parallel wire standard of 0.05-mm manganin wire, used in the present work, was 13 meters long. It may often, therefore, be convenient, where the highest precision is not desired, as in the case of coils of large capacity, to use one of the following methods.

1. USE OF BRIDGE WITH UNEQUAL ARMS

The unknown coil is joined in one of the arms of a Wheatstone bridge, for example R, Fig. 5. The other arms are composed of resistances at least 10 times smaller than R and of known effective inductances. The potential of the middle point of R having been adjusted to that of the earth, the bridge is balanced by varying the resistance and inductance in R by means of an auxiliary resistance of known effective inductance and a calibrated variable inductance in series with the unknown.

The condition for balance is

$$l+l'_3=\frac{S}{Q}l_1+\frac{P}{Q}l_4-\frac{R}{Q}l_2$$

l being the unknown inductance and l'_3 the total inductance in the

R arm external to the unknown. This will usually be sensibly equal to the reading of the variable inductance, since the effective inductance of the auxiliary resistance coils will be relatively negligible.

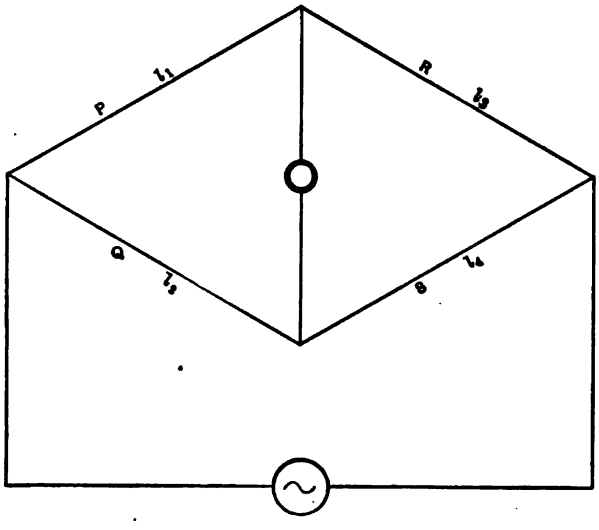

Fig. 5.—*Wheatstone's Bridge*

The coils P, Q, and S being of relatively small value, their effective inductances will not be affected appreciably by their capacity to earth. It is, however, difficult, if not impossible, to avoid errors due to the earth capacity of the auxiliary apparatus in the R arm in series with the unknown.

2. USE OF A STANDARD IN PARALLEL WITH THE UNKNOWN

In this method, see Fig. 6, a coil of known resistance R_1 (much less than the unknown) and effective inductance l_1 is joined in series with a variable inductance L_2 of convenient value together with a variable resistance, the total resistance of this part of the arm being R_2. The bridge having been balanced by variation of R_2 and L_2 (a suitable fixed inductance L_4 being inserted in S) the unknown coil is joined as a shunt around the standard and the bridge is again balanced by varying R_2 and L_2.

If ΔL_2 be the observed change in the variable inductance when the unknown is introduced, we have, neglecting terms which are inappreciable at ordinary frequencies,

$$l_0 = -(\Delta L_2 + \Delta l_2)\left(\frac{R_1+R_0}{R_1}\right)^2 + l_1\left(1 + \frac{2R_0}{R_1}\right)$$

where Δl_2 is the change in the effective inductance of the resistance coils of R_2 when the unknown is introduced. Unfortunately, when

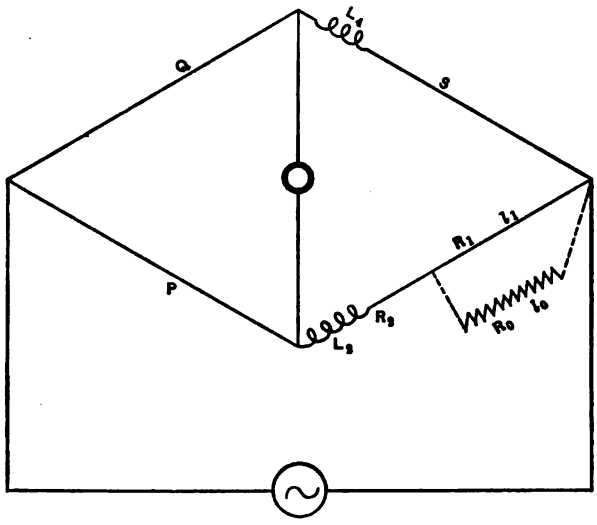

Fig. 6.—*Diagram of Shunt Method for Measuring the Effective Inductance of a High Resistance*

the effective inductance of the unknown is small, all of the terms are important, and Δl_2 and l_1 must be known with a good deal of accuracy. In this method, as in the preceding, the potential of the center of the unknown should be made zero, which may be done by so choosing the standard that connection may be made to its center.

In order that ΔL_2 may be large, the resistances R_1 and R_0 should be of the same order of magnitude. This is, however, opposed to the essential advantage of the method—that the effective inductance of a large resistance is to be made to depend on that of a convenient standard of smaller value.

3. EXPERIMENTAL RESULTS

The accuracy which may be easily attained by the use of these methods is shown in the following table:

Measured Effective Inductance in Millihenrys

Coil	Method 1	Method 2	Substitution
Multiplier 10 000 ohms	− 1.4	− 0.5	−1.8
Wolff 2438 10 000 No. 1	−111.3	−112.3	
Wolff 2438 10 000 No. 2	−113.3	−114.1	
Wolff 2438 Sum of 1 and 2	−231.2	−228.2	

VI. SUMMARY

1. In this paper is treated the measurement of the effective inductance of "noninductive" resistance coils and multipliers for use in alternating current bridges and in potential circuits where it is desired to know the phase angle of the circuit. The measurement of the inductances of very low resistance conductors, such as shunts with potential terminals, is not included.

2. The phase angle of a resistance coil is proportional to the resultant effects of its inductance and the capacities between the windings and between the various parts of the coil and the earth. The *effective inductance* is defined as that value of inductance, positive or negative, which would produce the observed phase angle.

3. In a low-resistance coil the effect of the inductance is most important; in a high-resistance coil the capacity is the determining factor, and the effect of the capacity of the coil to earth must be carefully taken into consideration. The method of treatment of the problem, therefore, varies according to the order of magnitude of the resistance of the coil.

4. The measurements with coils of low resistance are best made by means of a substitution method, using as a standard of reference a resistance of the same nominal value as the unknown, but of such a form that its inductance may be calculated from its dimensions. In this connection both parallel wires and circles were used.

5. To obtain small variations of resistance, with negligible change in inductance, wires of equal dimensions but of different conduc-

tivities were substituted one for the other. A continuous variation of resistance over a small range was obtained by varying the effective length of a mercury column. The accompanying change of inductance was measured, but was practically negligible.

6. For resistances of the order of 1000 ohms and greater the capacity of a parallel wire standard with respect to the earth becomes important. Attention is called to the fact that the usual formula for the capacity of such a standard applies only when the potential of one wire is as much above the earth potential as the other is below it. Failure to make this adjustment may easily give rise to capacity effects twice as great as those calculated by the formula. A method, developed from analytical discussion of the phenomenon, is given for adjusting the potentials of the bridge to their correct values.

7. This adjustment of the potentials having been accomplished, the substitution method gives accurate results with resistances at least as high as 10 000 ohms, the limit to its applicability being set only by the difficulty of preparing a suitable standard of reference.

8. The errors due to earth capacities in the case of resistance coils in series is discussed, together with the relative advantages of shielded and unshielded coils.

9. Two methods are given for the measurements of the effective inductances of coils in those cases where it is impracticable to construct primary standards. The results obtained by the use of these methods are not, however, as precise as those obtained by the method of substitution. In the case of coils having large capacity, where the phase angle is largely dependent on the temperature and other external conditions, these methods are sufficient.

WASHINGTON, September 1, 1911.

LUMINOUS PROPERTIES OF ELECTRICALLY CONDUCTING HELIUM GAS. II. REPRODUCIBILITY

By P. G. Nutting

Since the publication of a preliminary report [1] under the above title, a set of about 50 helium tubes has been tested at the Bureau of Standards, the tubes being intercompared both as regards total light and line by line. The results of this reproducibility test are of interest in connection with the possible use of the helium tube as a primary photometric standard as well as of general interest. These tubes were prepared and operated in accordance with specifications shown by preliminary study to be most suitable; except for slight unavoidable differences, all were alike and were operated alike.

Preliminary work on the variation of light emitted by helium tubes with the current through the tube, the potential gradient in the capillary, gas density, frequency of alternation of current, and diameter of bore of capillary was done in 1906–7 and described in this bulletin (loc. cit.). This investigation showed that the most suitable bore was 2 mm and the best current 25 milliamperes. Under these conditions the ratio of light to current is very nearly constant, and the light emitted is quite independent of gas density over a range of pressure from 4 to 7 mm, falling off rapidly at pressures both below 3 and above 8 mm. This independence of gas density is of vital importance, since this gas density is the only uncontrollable variable in the finished tube and for the

[1] B. S. Bulletin, 4, pp. 511–523, Dec., 1907.

practical reason that gas densities are not easily measured with precision during filling. The light emitted was further shown to be the same whether the tube was operated on direct current or on alternating current of 60 or of 900 cycles.

This preliminary work also showed that there was great room for improvement in the wall of the capillary portion of the tubes. Variations in thickness produced such an unequal distribution of light that results even with rotating tubes were uncertain from this cause. Accordingly, capillary tubing in 10 cm lengths was ordered from Baudin, of Paris, for the new set of tubes.

This new tubing was very uniform in bore (1.99–2.07 mm) and in wall thickness, and fairly free from striations. The mean bore of the middle 7 cm was determined with a mercury thread. The tubes were made up and filled with pure helium from Tyrer in the spring of 1909. The electrodes used were aluminum disks 1 mm thick and 25 mm diameter; the bulbs were spherical, 35 mm in diameter.

The filling was done with a good Geryk pump without subjecting the tubes or gas to contamination by either mercury or rubber tubing. The exhaustion was done with a heavy current passing through the tube and was carried to a nonconducting stage in about 15 minutes. No attempt was made to eliminate all the hydrogen in the electrodes, but so little was left that what appeared (if any) on standing six months disappeared in a few seconds on starting the tubes. No attempt was made to adjust the helium pressure to exactly 5 mm, but care was taken not to deviate more than 1 mm from this standard.

In July, 1909, two months after filling, each tube was run a few minutes at 25 m.a. for inspection in preparation for the photometric test. About half of the tubes showed hydrogen, which disappeared in 2 to 10 seconds.

In September, 1910, the tubes, having lain idle for 14 months, were intercompared line by line with a spectrophotometer, necessitating a run of about 15 minutes per tube. Each pair of tubes was operated on the same circuit in series. The yellow lines, 587, were set at equality, then the relative intensities of four other lines (red 668, green 501, blue 471, and violet 447) were determined. These observations and their reduction were made by Dr. Orin Tugman, then of this Bureau.

The results of chief interest in this mass of data are the average deviations from the mean, given below for 46 tubes:

Line	668	587	501	471	447
Color	Red	Yellow	Green	Blue	Violet
Average deviation	0.016	0.00	0.016	0.018	0.015

With tubes exactly alike, the maximum deviations to be expected would be twice the difference limen or about 0.04, the limen entering once in setting the yellow comparison lines to equality and a second time in setting on the pair to be observed. The average deviations to be expected are half this, or 0.02, since readings differing from the actual value by less than the limen are equally probable. The observed average deviations are for every line less than those to be expected if the observations had been on tubes identical in every respect; hence we may conclude that these 46 tubes show spectral reproducibility; that is, they are all of the same color to within differences much too small to determine by visual methods.

Variations of line intensity with current were studied over a wide range of current (0–450 m.a. in a 3 mm capillary) by the writer in collaboration with Dr. Tugman, using specially constructed tubes.[2] All the eight prominent lines studied showed a variation with current having the same characteristic form as the curve of total light as a function of current published in the first paper (loc. cit.). The five lines 471, 492, 504, 587, and 668 showed nearly the same variation with current, 447 and 501 a more rapid and 706 a less rapid variation than these. A wide variation in current will therefore produce a slight variation in the color of a helium tube, but the change in a small range at low current densities, say from 20 to 30 m.a., would not be measurable.

Seven of the eight lines studied (loc. cit.) showed but slight relative variations with gas density over a range from 2 to 10 mm, but the faint green line 504 is relatively much brighter at low pressures. This accounts for the relatively greenish hue exhibited in tubes containing helium at very low pressures.

A full photometric test of 38 tubes was made in May, 1911, in the photometric section of this Bureau. There were four observers M, A, J, and N, all experienced in photometry. The tubes were rotated at a speed of about 1.2 revolutions per second during

[2] B. S. Bulletin, 7, pp. 49–70; 1911.

observation. The current used was 0.025 ± 0.0001 ampere, from a 5000-volt 60-cycle transformer, using current from a special generator run by a battery.

The results are given in the following table:

I, Tube number	II, Hydrogen (sec)	III, Bore (mm)	IV, Length (cm)	V, Flux at 37 cm	VI, Mean deviation reading	VII, cp/cm	VIII, cp/cm 2 mm bore	IX, Tube deviation
1	4	2.062	5.038	1.555	0.033	0.3088	0.3119	−0.0028
2	2	50	4.996	560	15	3122	3147	− 00
3	5	24	5.166	632	13	3159	3171	+ 24
5	0	10	5.056	592	15	3148	3153	+ 6
6	10	70	5.129	585	03	3090	3125	− 22
10	4	2.064	5.135	1.582	0.009	0.3082	0.3114	−0.0033
11	0	44	5.120	580	20	3086	3108	− 39
12	2	22	4.976	562	12	3139	3150	+ 3
15	0	50	5.059	589	15	3141	3166	+ 19
16	5	00	5.121	612	15	3148	3148	+ 1
17	0	2.052	5.157	1.630	0.015	0.3161	0.3187	+0.0040
18	2	32	5.067	566	15	3089	3105	− 42
19	0	60	5.114	546	8	3025	3055	− 92
20	3	46	5.103	574	17	3084	3107	− 40
21	4	40	5.068	602	18	3161	3181	+ 34
23	0	2.024	5.075	1.552	0.019	0.3058	0.3070	−0.0077
24	2	32	5.065	594	17	3148	3164	+ 17
25	2	26	4.970	570	28	3160	3173	+ 26
27	6	14	5.139	642	15	3196	3203	+ 56
28	2	64	5.121	605	07	3135	3167	+ 20
29	0	2.000	5.093	1.604	0.016	0.3151	0.3151	+0.0004
31	3	70	5.037	593	16	3164	3199	+ 52
32	8	66	5.167	604	28	3105	3138	− 9
33	5	60	5.016	558	11	3107	3137	− 10
34	0	08	5.100	591	13	3120	3124	− 23
35	3	2.022	4.997	1.577	0.024	0.3156	0.3167	+0.0020
36	0	42	5.016	601	21	3192	3213	+ 66
38	0	38	5.138	561	35	3037	3056	− 91
39	5	00	5.115	621	22	3169	3169	+ 22
41	0	54	4.855	542	16	3177	3204	+ 57
42	2	2.054	4.916	1.545	0.009	0.3145	0.3172	+0.0025
43	5	80	5.043	565	12	3103	3143	− 6
45	0	04	5.152	646	13	3196	3198	+ 51
46	0	1.992	5.113	596	05	3125	3121	− 26
47	5	2.034	5.003	519	15	3035	3052	− 95
49	5	2.090	5.134	1.592	0.013	0.3101	0.3146	−0.0001
54	2	62	4.998	558	07	3150	3181	+ 34
55	0	54	5.117	618	18	3162	3189	+ 42
Means					0.015		0.3147 ±0.0005	0.0033 −1.15%

I. Tube numbers.

II. The time of disappearance of the hydrogen showing (if at all) after the current was started, the full 25 m.a. being turned on in each case. The tubes had not been operated for eight months and over half of them showed some hydrogen on first starting. There appears to be no relation between the presence of this transient hydrogen and the subsequent light emission.

III. The mean bore of the capillary as determined with a mercury thread before the tubes were made up.

IV. The length of capillary between the screening clips at the ends.

V. The luminous flux at about 37 cm from the tube center approximately in international candles. These values are the means for the four observers, each observer taking from 10 to 20 readings by an automatic recording device. The individual variations from the mean are not reproduced here but are summarized below. Each tube read high at the start but reached a steady value after running less than two minutes.

VI. The average deviation from the mean reading (Column V) due to the observers; that is, the average of the four separate deviations of the four observers. This shows how the mean reading of each observer differs, on the average, from the mean of all.

VII. The luminous flux (V) divided by the length of capillary (IV) radiating. The anti tangent correction, applicable on account of the source being linear, was not applied, as it is very small and very nearly the same for all tubes.

VIII. The same quantity as VII, but corrected for the slight differences in bore. In the earlier paper (1908, loc. cit., p. 521) it was shown that the bore correction for bores near 2 mm is very nearly linear and is -0.0050 cp for $+0.10$ mm difference in bore. This correction is here applied to reduce all tubes to the equivalent of 2 mm bore. The mean value of all the tubes is 0.3147 ± 0.0005 candles per cm length of capillary per 2 mm bore, the uncertainty indicated being the computed probable error.

IX. The deviation of each tube from the mean of all the tubes; this is the measure of reproducibility. The average deviation is 0.0033, or about 1.15 per cent, the maximum deviation (Nos. 19, 23, 38, 45, 47) less than 3 per cent. These larger deviations show

no definite relation to deviations in readings, amount of initial hydrogen in the tubes, nor to abnormal gas density or bore of capillary. Tubes 20, 21, and 47 showed pressures as low (estimated by width of dark space) as 3 mm, 10 and 27 as high as 7 mm, all others being apparently between 4 and 6 mm.

Readings on 15, 27, 32, and 33 were repeated after 2 days as a photometric test.

Tube	cp/cm first	cp/cm second	Diff.	Mean deviation. (cp/cm)
15	0.3146	0.3196	+0.0050	0.0026
27	0.3182	0.3177	−0.0005	
32	0.3116	0.3152	+0.0036	
33	0.3117	0.3132	+0.0015	

Readings, on 27 and 45 were taken at 20.2 25.0 and 29.9 m. a. current.

Current	20.2	25.0	29.9 m. a.
27	1.320	1.651	1.945
45	1.331	1.646	1.962

Life tests were made on a number of the tubes at 25 m.a. All finally developed hydrogen, the amount of hydrogen increasing about in proportion to the amount of metal deposit on the wall of the bulb. None of the seven tubes showed any definite increase or decrease in light emission up to the time the hydrogen began to show distinctly. This occurred after 5 to 15 hours in different tubes. Tubes would probably show a longer life if constructed with larger bulbs, say 50 mm in diameter instead of 35 mm.

An analysis of the readings of the individual observers shows that they read on an average as follows:

Observer.....	M	A	J	N
Average reading	0.019 low	0.002 high	0.024 high	0.009 low

showing the tendency to estimate the intensity differently. For example, J read high throughout, his reading being in every case higher than the average for the four observers. Taking the devia-

tion of each observer's reading from the mean reading of the four observers and finding the average deviation of these deviations from their mean value we have:

Observer.....	M	A	J	N
Deviation of deviation.....	0.010	0.006	0.010	0.008

This is an indication of each observer's consistency in reading, a measure of his tendency to form a criterion of equality and hold to it throughout the series.

Finally the mean deviation for all observers on all tubes was 0.015 or 0.0030 candles per cm. Eliminating this error of observation from the mean tube deviation (0.0033) leaves $((0.0033)^2 - (0.0030)^2)^{1/2} = 0.0014$ cp/cm = 0.4 per cent for the residual uncertainty in the final value of the cp/cm due to accidental variations in the tubes.

SUMMARY

A number of helium tubes were constructed, having glass capillaries very approximately of 2 mm bore and 7 cm long, with spherical end bulbs of 3.5 cm diameter, provided with aluminum electrodes 2.5 cm in diameter. The candlepower per centimeter length of tube (of the middle 5 cm of the capillary tube), when a current of 25 milliamperes was passing through the tube, was found to be 0.315 international candles. The effect of small variations in the frequency and strength of current, in the diameter of the capillary, in the pressure of the helium gas, etc., and the life of the tubes, are discussed.

The mean deviation in candlepower per cm length of tube from the mean of all the 38 tubes was 1.15 per cent. The maximum deviation of any one tube was 3 per cent.

The four observers read slightly differently, due probably to different color sensibilities. M read 1.2 per cent low, A 0.1 per cent high, J 1.5 per cent high, and N 0.5 per cent low on an average, compared with their mean.

Each individual observer varied somewhat in his criterion of equality. The deviations in the deviations from the mean readings amounting to, on an average, in the case of M 0.6 per cent, A 0.4 per cent, J 0.6 per cent, and N 0.5 per cent.

The chief improvement to be made in the next set of tubes is in using capillary tubing freer from striations. These long drawn out bubbles are in effect cylindrical lenses and with a rotating tube the sharp flicker produced by them is more troublesome in taking readings than the color difference. To secure a longer life either the bulbs will be made larger (40 mm) or the electrodes smaller (20 mm) in diameter.

WASHINGTON, August 25, 1911.

RESISTANCE COILS FOR ALTERNATING CURRENT WORK

By H. L. Curtis and F. W. Grover

CONTENTS

	Page.
I. Introduction	495
II. Historical review	496
III. Theoretical discussion	497
1. "Skin effect"	497
2. Capacity effects	498
(a) Effect of temperature upon the capacity	502
(b) Effect of humidity upon the capacity	502
3. Effect of absorption in the dielectric between the wires	503
4. Conclusions from theoretical work	505
IV. Applications	505
1. Hundred-ohm coils	507
2. Ten-ohm coils	507
3. One-ohm coils	508
4. Tenth-ohm coils	509
5. Thousand-ohm coils	509
6. Five-thousand-ohm coils	511
7. Ten-thousand-ohm coils	513
8. Comparison of new coils with ordinary "noninductive" coils	513
9. Methods of connecting coils in boxes	514
V. Summary	517

I. INTRODUCTION

The increase in the precision demanded in alternating-current work has made desirable an investigation into the behavior of resistance coils when subjected to an alternating-current voltage. As many of the commercial coils which give excellent results with direct currents have in the course of these experiments been found to be quite unsatisfactory for alternating-current measurements, specifications have been prepared for the construction of coils which will be suitable for this kind of work. The range of resistances covered is from 0.1 to 10000 ohms, though in most cases

only coils whose resistance is a power of 10 have been studied. The range of frequencies covered is from 0 to 3000 cycles, though the greater part of the work has been done in the interval from 1200 to 1500 cycles.

In an ideal resistance coil for alternating-current work two requirements need to be fulfilled, (1) the resistance must be independent of the frequency and (2) the phase angle must be zero. The design of coils which will fulfill these conditions as nearly as possible presents several interesting problems and the solution in any given case will depend upon the magnitude of the resistance of the coil. In order that some of the difficulties may be appreciated, a theoretical discussion of the factors which must be considered in the design of such a coil will be given. This will be followed by a description of the designs which have been found satisfactory, together with some suggestions as to proper methods of connecting the coils when they are to be used in a resistance box.

II. HISTORICAL

Some work along the lines indicated has already been done. When Kohlrausch began to make [1] alternating-current measurements, he found that the coils which were in use at that time were not satisfactory. In his later work [2] he avoided the use of coils larger than 2000 ohms. He attributed the difficulty with the larger-valued coils to capacity between the windings. Acting on this suggestion Chaperon [3] devised his well known winding. This consists in winding each layer inductively, but reversing the direction of winding of alternate layers, and thus necessitates a multiple-layer coil. This construction reduces the capacity without increasing the inductance materially. A slight improvement was claimed by Cauro [4] who carried the wire back across the coil, thus starting each layer at the same point but winding alternate layers in opposite directions.

A different method of making a high resistance with small effective inductance is due to Rowland, [5] who wound a fine wire inductively on a thin sheet of mica. While not of a convenient

[1] Pogg. Ann., 138, p. 280; 1869.
[2] Ann. d. Phys., 26, p. 161; 1885.
[3] Comptes Rendus, 108, p. 799; 1889.
[4] Comptes Rendus, 120, p. 308; 1895.
[5] Amer. J. Sci., (4) 8, p. 35; 1899.

form for some kinds of work, such resistances have been extensively used in the multipliers of voltmeters and wattmeters. Recently Orlich [6] has designed resistances in which the time constant (according to an approximate calculation) is zero. It consists of an inductive winding on a slab of slate, with a second winding held at the calculated distance outside of this by caps on the ends of the slab. In this way he obtained a time constant of 4×10^{-7} second with a coil of 25 000 ohms.

The above work was all done with high resistances. Within a few years there has been a demand for low resistances having a small time constant for use as shunts to alternating-current ammeters. Campbell,[7] Orlich,[8] and Drysdale [9] have each designed resistances for this purpose. A consideration of these lies outside the purpose of this paper.

Some measurements of the effective inductance of commercial coils have been made by Taylor and Williams [10] and by Brown.[11] They did not, however, attempt to improve the design of coils.

III. THEORETICAL

It has already been stated that in an ideal resistance coil for alternating-current work the phase angle should be sensibly zero and the change of resistance with frequency negligible. The most obvious thing to do to make the phase angle small is to wind the coil in such a way that its inductance will be a minimum. However, there are three other factors which need to be considered in a discussion of the phase angle, viz: (1) "Skin effect," (2) capacity in parallel with the resistance, and (3) absorption in the dielectric between the wires. Each gives rise, also, to a change in the effective resistance of the coil. It is the purpose of the following theoretical investigation to show how each of these three factors enters and to give an idea of the magnitude of the effects which they produce.

1. "SKIN EFFECT"

The resistance of a conductor for alternating current is greater than for direct current, because in the former case the current

[6] Verh. d. Deut. Phys. Ges., 12, p. 949; 1910.
[7] Electrician, 61, p. 1000; 1908.
[8] Zs. für Instrk., 29, p. 241; 1909.
[9] Electrician, 66, p. 341; 1910.
[10] Phys. Rev., 26, p. 417; 1908.
[11] Phys. Rev., 29, p. 369; 1909.

density is greater near the outside of the conductor than near its center. On the other hand, this change in current distribution with the frequency decreases the inductance, though the relative change is much less than the change in the resistance. In the case of a circuit containing only resistance and inductance $\tan \theta = pL/R$ where θ is the phase angle, p is 2π times the frequency, L the self-induction and R the resistance. Hence, if the change in R, due to skin effect, can be shown to be negligible, the effect upon the phase angle will also be negligible.

Formulas [12] have been compiled by which the change of resistance with frequency can be computed in the case of a straight cylindrical wire. If now this wire is wound on a spool, or put in any shape such that its total inductance is increased, then its change of resistance will be increased; but if it is so wound that its inductance is decreased, then the change of resistance will be decreased. As the latter form is always used in resistance coils, a maximum value of the change in resistance may be found by computing the case of a straight wire.

For a manganin wire the formula shows that with 3000 cycles the diameter must be as large as 2 mm before the skin effect produces a change of 1 part in 100 000. All of the wires used in this investigation are much smaller than this. The computed resistance change for the largest wire used (0.24 mm) is only 1 part in 10^9. In some of the smaller-valued coils strip manganin of 0.1 mm thickness has been used. Computing by the formula of Bethenod [13] the resistance change in this case is also found to be negligible.

Hence, in none of the coils which we have designed will the skin effect be appreciable. In fact, it may be said that the resistance of most commercial coils having a resistance of 1 ohm or over is not affected by the skin effect up to 3000 cycles.

2. CAPACITY EFFECTS

In a resistance coil it is often necessary to consider not only the resistance and inductance of the conductor, but also the electrostatic capacity between the different parts of the conductor. To

[12] Rosa and Grover, this Bulletin, 8, p. 172; 1911. Reprint No. 169.
[13] Jahrb. d. drath. Telegraphie, 2, p. 397; 1909.

show this effect in a simple case, consider a coil (Fig. 1) having a resistance R and inductance L which is in parallel with a condenser of capacity C. Then the impedance of the upper circuit is $R+ipL$ and of the lower is $-\frac{i}{pC}=\frac{1}{ipC}$. Then if A is the impedance of the two circuits in parallel

$$\frac{1}{A}=\frac{1}{R+ipL}+ipC=\frac{1-p^2CL+ipCR}{R+ipL}$$

or

$$A=\frac{R+ipL}{1-p^2CL+ipCR}=\frac{R+ip(L-CR^2-p^2CL^2)}{(1-p^2CL)^2+p^2C^2R^2} \qquad (1)$$

We may write $A=R'+ipL'$ where R' and L' are, respectively, the effective resistance and effective inductance of the system. Then by separating real and imaginary parts

$$R'=\frac{R}{1-p^2C(2L-CR^2)+p^4C^2L^2} \qquad (2)$$

and

$$L'=\frac{L-CR^2-p^2CL^2}{1-p^2C(2L-CR^2)+p^4C^2L^2} \qquad (3)$$

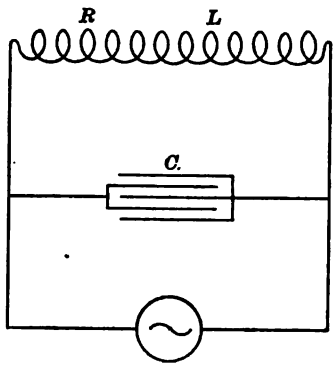

Fig. 1.—*Diagram to Show a Capacity in Parallel with a Coil which has Resistance and Inductance*

The phase angle $\theta=\tan^{-1}\frac{pL'}{R'}$

$$=\tan^{-1}p\frac{(L-CR^2-p^2CL^2)}{R} \qquad (4)$$

If we make a first approximation assuming both L and C small, we have

$$R'=R[1+p^2C(2L-CR^2)] \qquad (5)$$

$$L'=L-CR^2 \qquad (6)$$

$$\tan\theta=\frac{p(L-CR^2)}{R}$$

It should be noted that the phase angle will be zero if $L-CR^2=0$, but that the change in resistance will be zero if $C=0$, or if

$2L-CR^2=0$. Hence, both the phase angle and the change of resistance will be zero only under the condition that $C=0$ and $L=0$.

If we consider the case of two parallel wires, the capacity is distributed, so that the effect is not the same as that given above. The solution in this case requires the integration of the differential equations [14] which express the emf and current at each point of the line at each instant. This solution gives, as a first approximation,

$$R'=R\left[1+p^2C\left(\frac{1}{3}L-\frac{2}{15}CR^2\right)\right] \tag{8}$$

$$L'=L-\frac{1}{3}CR^2 \tag{9}$$

$$\tan\theta=\frac{p\left(L-\frac{1}{3}CR^2\right)}{R} \tag{10}$$

Where R, C, and L are the total resistance, capacity, and inductance of the wires.

Again, it will be seen that an ideal coil is only possible, provided C and L are both zero. However, if C is small and if $L-\frac{1}{3}CR^2=0$, then the phase angle becomes zero, and the change in resistance is negligible. As an example, let us compute the maximum capacity which may be present in a 10 000-ohm coil to to be used on a frequency of 3000 cycles where the allowable uncertainty in the resistance may be 1 part in 100 000, assuming that there is sufficient inductance to make the phase angle zero.

Then $$L-\frac{1}{3}CR^2=0$$

and $$p^2C\left(\frac{1}{3}L-\frac{2}{15}CR^2\right)=\pm 10^{-5}$$

Solving, $C=10^{-10}$ farad, or a ten-thousandth of a microfarad. This is a very small capacity, and great care must be taken if such a coil is to be realized. With a 1000-ohm coil, the allowable capacity is a hundredth of a microfarad, while with a 100-ohm coil a microfarad can be tolerated.

[14] Steinmetz, Alternating-Current Phenomena, 3d ed., p. 167.

If 100, 1000, and 10 000-ohm coils were to be constructed from wire of the same size and wound in the same manner, then the capacity would increase as rapidly as the resistance, so that the change in resistance of the 1000-ohm ocil would be nearly 10 000 times as much as the change in resistance of the 100-ohm coil, while the 10 000-ohm coil would change 100 000 000 times as much as the 100-ohm coil. If they are constructed of wires of different size, such that with a single layer they will each have the same surface, then a 1000-ohm coil will change in resistance 400 times as much as a 100-ohm coil, and a 10 000-ohm coil 160 000 times as much as the 100-ohm coil. In the above it has been assumed that for the 100-ohm coil the phase angle is zero and that the formulas for parallel wires will hold for bifilar coils.

The effect upon phase angle in these cases is also interesting. In the first case, where the wire for the different coils is all of the same size, the capacity, inductance, and resistance all increase directly as the length l, so that $L' = (K - K_1 l^2)$, where K and K_1 are constants. This shows that the term containing capacity increases much more rapidly than the inductance term, so that if a 100-ohm coil has zero phase angle, a similar 1000-ohm coil will have a decided negative phase angle, the real inductance being only 1 per cent of the effective inductance. With the 10 000-ohm coil this effect is greatly increased, so that the real inductance becomes negligible.

With coils similarly wound, but of different sizes of wire, so that they have the same surface, the inductance and capacity increase in the same *ratio* as the resistance increases. Hence, while the increase in the phase angle is less rapid, the relative importance of the inductance and capacity terms is the same.

From what is given above, it will be seen that the effect of capacity upon resistance is appreciable only in the case of high-valued coils, 1000 ohms and over. The effect of capacity upon phase angle is more marked. In most cases it is the predominating factor in coils of 1000 ohms or over, but it also produces a measurable effect in a 10-ohm coil, and perhaps even in a 1-ohm coil.

(a) EFFECT OF TEMPERATURE UPON THE CAPACITY

Since in many coils the phase angle is largely dependent upon the capacity, the temperature coefficient of the phase angle will be nearly that of the capacity. For a poor dielectric, such as is generally found in resistance coils, this temperature coefficient of the capacity is usually quite large. In one bifilar coil which had been cut at the middle point it was measured and found to be 1 per cent per degree. Hence, if sufficient current is used to heat the coil the phase angle will change while the coil is under observation. As the dielectric heats rather slowly when the current begins to pass through a coil, the phase angle will depend upon the length of time which the current has been flowing as well as upon the magnitude of the current.

This drift of the effective inductance has often been observed by us, and before the cause was understood it gave us considerable trouble. The following example will show the magnitude of this for an ordinary coil. A 1000-ohm bifilar coil, suspended in the air, was connected in a circuit through which a current of 50 milliamperes could be sent. Within five minutes after closing the circuit a drift of 5 microhenrys in the effective inductance was observed.

The winding was then cut at its middle point and it was found to have a capacity of 0.0014 μf whose temperature coefficient was 1 per cent per degree. Hence, a rise of 1 degree in the temperature of the dielectric will explain the change in the effective inductance which was observed.

The effect of the temperature coefficient of the capacity upon the high-frequency resistance would be very small. It is so interconnected with other effects that we have never observed it. In well-designed coils the capacity should be so small that its change with temperature will not produce any noticeable effects.

(b) EFFECT OF HUMIDITY UPON THE CAPACITY

It has been shown [15] that the shellac of resistance coils absorbs moisture from the air. On account of the high dielectric constant of water, one would expect that this would affect the effective inductance of the coil as a result of the change in the capacity. This was found to be the case. In one coil the effective induc-

[15] Rosa and Babcock, this Bulletin, 4, p. 121; 1907.

tance was increased algebraically by 20 per cent by baking at 110° C for 15 minutes. Also, coils where the insulation is of uncovered silk show large changes of the capacity with humidity. This trouble may be overcome by coating the coil with paraffin or sealing it in a metal case.

3. EFFECT OF ABSORPTION IN THE DIELECTRIC BETWEEN THE WIRES

In coils for direct current work the only property required of the insulating material between the turns is that it shall not allow an appreciable leakage between the wires. However, where alternating current is used the absorption of a poor dielectric may produce an apparent leakage between the turns of a coil whose direct current insulation resistance is sufficiently high. If we again consider the coil as equivalent to two parallel wires, the effect of this absorption upon the resistance can be shown to be

$$R' = R\left[1 - \frac{1}{3} pCR \tan \theta\right] \qquad (11)$$

where θ is the phase difference of the capacity due to absorption.

The absorption also causes the effective inductance to change with frequency, since the capacity of an absorbing condenser is a function of the frequency. The change in effective inductance is $(C_1 - C_2)\ R^2$, where C_1 and C_2 are the capacities at two frequencies.

The magnitude of these effects can only be determined by experiment. For this purpose the coil which had been opened at the center was again employed. It was a commercial 1000-ohm coil, wound bifilar with silk-covered manganin wire, then shellaced and baked.

The following data were obtained:

Frequency of A. C.	100 ~	1500 ~	2700 ~
Capacity	0.00160 μf	0.00158 μf	0.00152 μf
Phase angle	2°.6	1°.5	1°.2

Having joined the wires at the center again, the following measurements were made:

Frequency	Zero	100 ~	1500 ~	2700 ~
Resistance (Ohms)	1002.97	1002.93	1002.77	1002.55
Effective inductance		0.54 mh	0.54 mh	0.50 mh

From the first set of measurements the change in resistance and inductance can be approximately computed, while the observed values come directly from the second set. These give the following results:

			0 to 100~	0 to 1500~	0 to 2700~
Change in resistance (Ohms)	Computed	$\frac{1}{3}pCR \tan \theta$	1.4×10^{-6}	12×10^{-6}	18×10^{-6}
		$\frac{2}{15}p^2C^2R^2$	0.01×10^{-6}	3×10^{-6}	8×10^{-6}
		Total	1.4×10^{-6}	15×10^{-6}	26×10^{-6}
	Observed		4×10^{-6}	20×10^{-6}	42×10^{-6}

		100~ to 1500~	100~ to 2700~
Change in inductance (Henrys)	Computed $(C_1-C_2)R^2$	2×10^{-6}	8×10^{-6}
	Observed	0	4×10^{-6}

While these results are not all that could be desired, yet they are probably all that can be reasonably expected, considering the difficulties in such measurements.

The phase difference of the capacity is a characteristic of the dielectric. Therefore, the above values are typical of what may be expected with shellaced coils. Hence, from equation (11), we see that the higher-valued coils will have a larger percentage change in resistance, due to absorption. It is interesting to see how much this will be under the conditions which were assumed in computing the effect of the capacity upon the resistance. For the case where the coils are made of wire of the same size, this decrease in resistance will be proportional to the square of the resistance. Where the coils have the same area, the decrease will be proportional to the four-thirds power of the resistance.

The relative change of capacity with frequency is also a characteristic of the dielectric. Hence, in coils where the capacity is the predominating factor in the effective inductance, the percentage change in phase angle with frequency will be independent of the resistance of the coil.

4. CONCLUSIONS FROM THEORETICAL WORK

(1) It is possible to make a resistance coil in which the phase angle will be small and practically the same for all frequencies. This requires for a bifilar coil that $L=\frac{CR^2}{3}$ and that C shall be so small that its change with frequency may be neglected.

(2) The alternating-current resistance of ordinary coils for frequencies up to 3000 cycles, is not affected by the skin effect, but may be decidedly affected by capacity between the turns and by absorption in the dielectric. This can only be overcome by keeping the capacity as low as is consistent with the demands for a small phase angle.

IV. APPLICATIONS

The preceding investigation shows that it is desirable to have both the inductance and capacity small. This means that the coil shall contain as little wire as possible, besides having the necessary wire put in the most advantageous shape. As all of the heat which is lost from a coil in continuous operation passes through the surface, there is no advantage, as far as carrying capacity is concerned, in making a multiple layer coil. With regard to inductance and capacity effects there is a decided disadvantage. Hence, as far as possible all coils should be single-layer coils.

The area of surface required will depend upon the amount of energy to be expended in the coil and upon the allowable rise of temperature. It seemed desirable to have a surface of approximately 50 cm^2, as then an expenditure of 1 watt upon a coil immersed in oil will raise its temperature only about 1 degree when the coil is wound on a poor conductor of heat. If the coil is in air, the rise in temperature will be nearly 10 times as great. Even with a given area, the size of the spool will affect the phase angle as is shown in Fig. 2. However, this effect is small. A spool 2.5 cm in diameter is a convenient size, and all of the results given are for spools of this size.

The material and treatment of the coils is the same as for direct-current coils. They are made of silk-covered manganin wire. After winding they are shellaced and baked at 120°–130° C for several hours. The baking is to anneal the wire as well as to

dry the shellac. They should be protected from the moisture of the air by dipping in molten paraffin after baking or by sealing in air-tight cases. This protection is necessary to prevent changes in the effective inductance as well as changes in the resistance.

The collected results of our experiments are given in Table I. In no case is the time constant greater than 5×10^{-8} second. Hence, with 3000 cycles the maximum phase angle is 3.5′. The change in resistance with frequency is also negligible in all cases. A brief description of each coil is given in the table and a more detailed description follows.

TABLE I

Constants of Resistance Coils for Alternating-Current Measurements

Resistance of coil	Material of spool	Shape of resistance material	Size of resistance material			Method of winding	Effective inductance of sample coil	Time constant
			Width	Thickness	Diameter			
ohms			mm	mm	mm		microhenrys	seconds
1	Brass or porcelain	Strip	3 to 5	0.1		Bifilar	+ 0.05	5×10^{-8}
10	do	do	1	0.1		do	+ .2	2×10^{-8}
10	do	Wire			0.24	Three 30-ohm bifilar sections in parallel	+ .3	3×10^{-8}
100	do	do			.24	Bifilar	− 1.6	1.6×10^{-8}
1000	Porcelain	do			.10	Five 200-ohm bifilar sections in series	− 16	1.6×10^{-8}
5000	do	do			.05	Inductive for 20 turns, then reversed for 20 turns, etc.	+210	2.1×10^{-8}
5000	do	do			.05	Inductive, reversed each turn	+ 30	0.6×10^{-8}
10 000	do	do			.05	do	+100	1×10^{-8}

Each coil is made of manganin wound on spools 2.5 cm in diameter. With the exception of the coils made of strip, all coils are single layer. Coils are shellaced and baked in the usual manner, then paraffined or sealed.

1. HUNDRED-OHM COILS

A large number of measurements extending over a period of several years have shown that a 100-ohm coil usually has a small phase angle. It was natural, therefore, first, to see what was the smallest value of the effective inductance which could be attained with coils of this value, and to see how nearly two coils made in the same manner would agree. An ordinary bifilar winding was used and found to be satisfactory. Two coils made as nearly alike as possible differed in effective inductance by 1 microhenry, one having -1.6 microhenrys and the other -0.6 microhenry. This gives a difference in phase angle between the two coils of 40″ at 3000 cycles, or less than 1″ at 60 cycles. There was no appreciable difference between coils wound on brass and on porcelain.

A computation of the inductance of such a coil by Rosa's formula [16] for a "noninductive" winding gives a value of 1.6 microhenrys. Hence, if the coil has an effective inductance of -1.6 microhenrys, it must have a capacity of 3.2×10^{-4} microfarads. The effect of this capacity upon the resistance is less than 1 part in 10 000 000. If properly treated, the effective inductance at any frequency can be depended upon to remain constant to 0.1 or 0.2 microhenry. The data given in Sec. III–3 show a change in capacity between 100 and 3000 cycles of 5 per cent. Other measurements show that the change is rarely twice this in the case of shellaced coils. If the capacity of a 100-ohm coil were 3×10^{-4} microfarad, as estimated above, this would cause a maximum change in effective inductance of 0.3 microhenry in the range of frequencies mentioned. Measured values are not much different from this.

2. TEN-OHM COILS

With 10-ohm coils, the effect upon the resistance due to the capacity and absorption between the turns of the winding is negligible. Even the CR^2 term in the effective inductance is small in most cases. Hence, in the design of a 10-ohm coil the important point is to keep the inductance as small as possible.

Two methods of effecting this result have been tried. One is to make the coil of strip manganin, and the other is to use three

[16] This Bulletin, 4, p. 338. 1907. Reprint No. 80.

30-ohm coils in parallel. While the first method gave somewhat better results, the difference in general is not of sufficient importance to justify the additional trouble of making a coil of strip manganin.

The 30-ohm coils were wound of the same size wire as was used upon the 100-ohm coil just described. They were wound bifilar in the ordinary manner, the three coils being wound upon the same spool. Whether the spool is of brass or porcelain is of little importance, as practically all the capacity is between adjacent wires. Still, better results could doubtless have been obtained by using wire of smaller diameter, and making five 50-ohm coils in parallel. This was not considered necessary.

The strip coil was made of manganin 0.1 mm in thickness, the width of the strip being 1 mm. Between the strips of the bifilar winding was placed a layer of silk 0.09 mm in thickness. This brings the strips close together, making the inductance small and at the same time making the capacity larger than would be the case with round wires of the same cross-sectional area. If a thinner insulator, such as mica, be used, the effective inductance could be made still less.

One of the chief difficulties in winding a strip coil is in keeping the outer strip exactly over the inner strip. This can be largely overcome by binding the two strips rather loosely together with silk thread before winding them into a coil.

3. ONE-OHM COILS

In these coils all capacity effects are negligible. Hence, the only thing to consider is the reduction of the inductance to a minimum.

There seems to be no satisfactory method except to use the resistance material in the form of a strip. It is generally considered inadvisable to use manganin thinner than 0.1 mm. With this material, winding it with a double thickness as is necessary in a strip coil, the strip must be about 5 mm wide in order to give a surface of 50 cm^2. However, as this coil can be wound on a metal spool, so large a surface is not necessary, and a 3-mm strip is satisfactory for most purposes. Using silk insulation of approx-

imately 0.09 mm thickness the effective inductance was 0.1 microhenry, but with a thin mica insulation this was reduced to 0.05 microhenry.

4. TENTH-OHM COILS

It is shown in Table II that in many resistance boxes each tenth-ohm coil has an inductance comparable with that of a 1-ohm coil. Hence, in construction of resistance boxes the design of coils of this value should not be neglected.

The best method is to use 0.1 mm manganin strip, making the width about 3 mm and the length 10 cm. This is folded in the center and bound together with a thin insulation between. In general, no support will be required. The current-carrying capacity of these is no larger than that of the 1-ohm coils. The inductance, however, is only one-tenth that of the 1-ohm coil, and where these coils are to be used in series with other coils of higher denomination this advantage is more important than that of a large carrying capacity.

5. THOUSAND-OHM COILS

With coils of 1000 ohms the problem is quite different from that met with in the coils just described. As fine wire is used in their construction, the inductance of a bifilar winding becomes relatively small, while the capacity effect increases very decidedly. A method which has often been used is to connect several coils of smaller value in series. In order to obtain data in regard to the correct size of wire, and the proper resistance in each section to make the effective inductance zero, five coils were constructed in which these factors were systematically varied. At the same time two other coils were constructed in order to determine the effect of the size of the spool. The results of these experiments are shown in the curves of Fig. 2. From a study of these curves it was decided that with 0.10-mm wire (a convenient size for a 1000-ohm coil) a 200-ohm section would have a very small effective inductance. This proved to be the case and a coil was constructed of five 200-ohm sections in series.

In this case it is important that the coil should be wound upon an insulating spool, or that each section be wound upon a metal

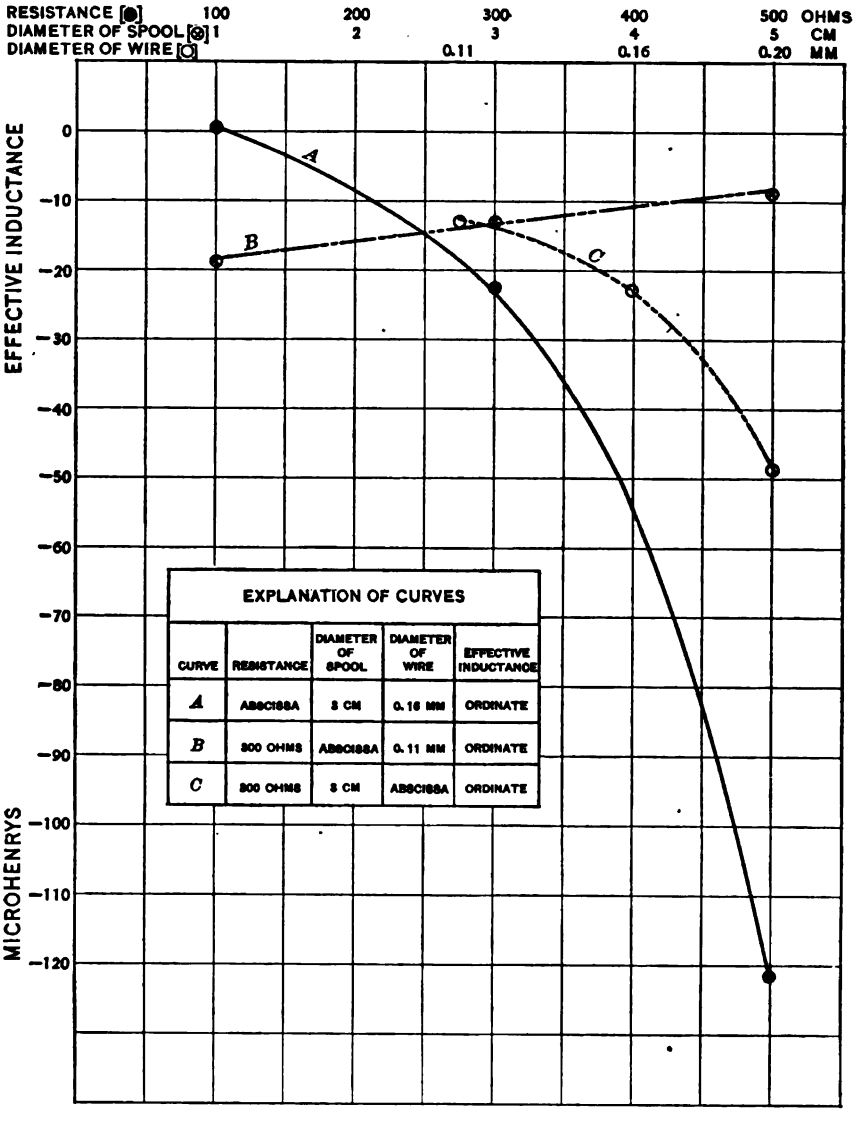

Fig. 2.—*Curves Showing the Variation of the Effective Inductance of a Bifilar Coil when the Resistance, Diameter of Spool, and Diameter of Wire are Varied*

spool, and the spools insulated from one another. The necessity for this method of construction is caused by the fact that, if all the sections are upon one metal spool, the capacity between the sections is greatly increased, so that the effective inductance of the sum of five sections will not be five times that of a single section. The importance of this is shown by the following data:

Five times the inductance of a 200-ohm coil.... −1.5 microhenrys

Inductance of five 200-ohm coils in series
- (a) on porcelain..... −12 microhenrys
- (b) on brass..... −74. microhenrys

From these measurements it appears that the capacity between sections influences the phase angle even when the sections are wound upon an insulating spool. However, the capacity is nearly seven times as large when the sections are wound on a brass spool, as is shown by the fact that the effective inductance has a much larger negative value in the latter case.

Two coils were constructed according to these specifications, one being wound on a porcelain spool, and each section of the other on a brass spool, the latter spools being insulated by fiber washers on a fiber tube. The first had an effective inductance of −16 microhenrys, and the second −9 microhenrys. If we take the larger value of the effective inductance and, assuming it is all due to capacity, compute the effect of this capacity upon the resistance when the frequency is 3000 cycles, we find that the change in resistance is not more than 1 part in 10 000 000. The effect of absorption is larger, but using the most probable phase differences of the capacity, it will not exceed 2 or 3 parts in 1 000 000. This makes it very probable that the change of resistance is negligible. We have also failed to detect any change experimentally.

6. FIVE-THOUSAND-OHM COILS

As the difficulties in designing coils for alternating-current work increases rapidly above 1000 ohms, it seemed desirable first to construct a 5000-ohm coil instead of attempting immediately to construct a 10 000-ohm coil. The capacity here becomes the important factor, and questions of inductance can be largely neglected.

At this point it seemed desirable to give up the bifilar winding, and wind an ordinary inductive winding, reversing the direction of winding every few turns. Two such coils were constructed, one with 20 turns before reversing and the other with 50 turns. These coils gave fairly satisfactory results. The change in resistance from 0 to 1200 cycles was less than 1 part in 100 000 (the limit of the sensitiveness of the bridge). The effective inductance of the coil with 20 turns per section was found to be +210 microhenrys, while the one having 50 turns per section was +330 microhenrys. For most purposes these may be regarded as satisfactory coils.

However, as the inductance predominates, it was evident that a still better coil could be made by reversing the direction of winding oftener. This was done by a method suggested by Mr. Joseph Ludewig, of the instrument shop of the Bureau. A cylinder of biscuit porcelain 2.8 cm in diameter and 15 cm long was obtained. This is porcelain which has received only a preliminary baking and is sufficiently soft to be worked with a steel tool. A narrow slit was cut along one diameter for two-thirds of its length. The cylinder was then baked to increase its strength. This reduced the diameter to about 2.7 cm. The coil was then wound with 0.05-mm manganin wire in the following manner: The wire goes once around the spool, then passes through the slit and one turn is made in the opposite direction, then the wire passed through the slit and one turn made in same direction as the first. The process is repeated until the desired amount of wire has been wound on the spool. This method of winding is shown diagrammatically for two turns in Fig. 3.

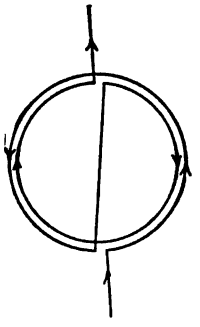

Fig. 3.—*Diagram to Show Method of Winding Resistance Coils to Have Small Inductance and Small Capacity*

It will be noticed that each turn has two adjacent turns in which the current flows in the opposite direction. This is equivalent to reversing every turn and it means that the inductance will be as small as with a bifilar winding, while the capacity effect will also be small since there is only a small potential difference between

adjacent wires. The coil constructed according to this method had an inductance of only +30 microhenrys, which was about the limit of measurement.

This method of winding a coil is rather difficult, but it can be done more rapidly than would seem possible at first sight. It gives a very satisfactory coil.

7. TEN-THOUSAND-OHM COILS

The method of winding just described was so satisfactory in the case of 5000-ohm coils that it was applied to the construction of a 10 000-ohm coil. The same size of wire (0.05 mm) and the same size of spool (2.7 cm) were employed. This made a winding about 7.5 cm long. In order to leave the porcelain tube sufficiently strong, it was not desirable to cut the slit for more than three-fourths of its length. Hence a tube 10 cm long was required, since the portion which is not slit could not be used for the winding.

The results were very satisfactory. The effective inductance was too small to be accurately measured. It is not more than +100 microhenrys.

The chief objection to this coil is the difficulty of winding it. Experience has shown that an apprentice can wind a 10 000-ohm coil in a day. Thus the cost is greater than for ordinary commercial coils, but is not excessive.

8. COMPARISON OF NEW COILS WITH ORDINARY NONINDUCTIVE COILS

The relative merits of the new coils are shown in Table II, in which the values of the effective inductance and of the change in resistance from 0 to 1200 of the new coils and of a representative coil from resistance boxes by German and American manufacturers are given. No attempt has been made to make the table complete, but sufficient results are given to show the improvements which have been effected.

TABLE II

Nominal resistance of coil	Effective inductance in microhenrys at 1200 ~			Change in resistance, 0 to 1200 ~		
	New coil	A	G	New	A	G
0.1	+ 0.005	+ 0.14	+ 0.18			
1	+ 0.05	+ 0.4	+ 0.5			
10	+ 0.3	+ 0.9	+ 1			
100	− 1.6	− 5	− 2			
1000	− 16	−400	− 100	<0.001%	−0.08%	−0.05%
5000	+ 30		− 27 500	<0.001		−0.2
10 000	+100		−100 000	<0.001		−1

A form of resistance proposed by Rowland [17] is quite satisfactory for high resistance alternating current work. It consists of fine wire wound inductively on thin sheets or cards of mica. The effective inductance of resistances of this type is small, the time constant being seldom more than 10^{-6} and often as small as 10^{-7} second. However, such resistances have considerable surface, so that the capacity between two cards which are connected in series may appreciably affect the phase angle. The magnitude of the capacity C can be approximately computed by assuming that the cards are plates of a condenser. The effect of the capacity upon the effective inductance was found by experiment to be approximately $-\frac{1}{3} CR^2$, where R is the resistance of the two cards. While this type of resistance has been extensively used in multipliers for voltmeters and wattmeters, it has not found extensive use in resistance boxes.

A type of resistance which accomplishes the same results as the above is due to Duddell and Mather.[18] It consists of a fine wire woven as the woof in a silk warp. While no recent measurements have been made in this laboratory on such resistances, yet it is evident that satisfactory results may be obtained by this method.

9. METHODS OF CONNECTING COILS IN BOXES

The best method of connecting the coils in a box will depend on the resistance of the coils. For low resistances (up to 100 ohms)

[17] Amer. J. Sci. (4), 8, p. 35; 1899.

[18] Electrician 56, p. 54; 1905.

the connections should be so arranged that the inductance is small. For high resistances (1000 ohms and over) the inductance of the leads is of little importance, but the capacity between the leads, between the leads and the coils, and between the coils themselves becomes of importance.

For low-resistance coils, the method in common use at present suffices for most purposes. To the terminal block is soldered a copper rod which extends below the bottom of the coils when they are fastened in place. To the lower end of this is attached one terminal of each of the two adjacent coils. If the effective inductance of each coil is measured from the terminals on the top of the box, then the inductance of the terminal rods will be included in each measurement, while in the ordinary use of the box there will never be more than two rods in the circuit. To show this effect, the inductance of each of the ten 1-ohm coils of a decade box was measured by connecting to the terminals on the top of the box. The 10 were then measured in series. The sum of the inductances of the 10 coils was 0.57 microhenry larger than the inductance of the 10 in series. This shows the inductance of each rod to be 0.03 microhenry.

This could be slightly reduced by making the rods of larger diameter so as to diminish the self-inductance. Another method is to use two rods from each terminal block and to connect only one coil to a rod. There will then be the mutual inductance between the two rods from one block when two coils are used in series. This will be smaller than the self-inductance, so that the inductance of a group of coils in series will more nearly equal the sum of the inductances of the individual coils. However, it will not often be necessary to use this refinement.

For high-resistance coils, the capacity effects must be closely guarded against, while on the other hand, the inductance of the leads is negligible. With coils connected as described above for low-resistance coils, the capacity between the terminal blocks of the coils forms a shunt on the impedance of the coil. In this case, however, the capacity is always a shunt on the coil and produces the same effect whether the coil is measured from its terminals or is in series with other coils. Although this capacity is small, it

can not always be neglected. In one of the dial decade boxes in this laboratory it amounts to about 5×10^{-6} μf. This would change the effective inductance of a 1000-ohm coil by 5 microhenrys (an amount which is sometimes of importance), and the effective inductance of a 10 000-ohm coil would be changed by 0.5 millihenry. The effective resistance at 3000 cycles is not altered by more than 1 part in 1 000 000 in either case. It is evidently desirable that the terminal blocks of high-resistance coils should be small, and as far apart as is consistent with good operation.

The capacity between the coils themselves also measurably affects the phase angle of the coils when they are joined in series. With two 1000-ohm coils it was found that the effective inductance was decreased algebraically by 4 microhenrys when their distance apart was changed from 20 cm to 2 mm. With two 5000-ohm coils the effective inductance was decreased algebraically 200 microhenrys by bringing close together coils which had been separated by a distance of 10 cm.

Perhaps a more troublesome factor is what is sometimes called the "dead end effect." When only a part of a resistance box is being used, the remainder of the coils are connected to the circuit, and their capacity to the other coils as well as to earth may produce an appreciable effect. To minimize this, the resistance box should be so constructed that each coil is entirely disconnected from the others when not in the circuit. No satisfactory design of a resistance box which accomplishes this is known to the authors.

The position of the terminals of the box is sometimes of importance. If only a small resistance is to be used from the box and it is desired to keep the inductance of the circuit low, a twisted cord should be used for leads. In this case the terminals should be near each other. If, on the other hand, a large resistance is to be used, then it is more important to decrease the capacity between the leads than to decrease their inductance. In this case the leads, and hence the terminals, should be as far apart as possible. In the case of boxes designed for general use, a mean value should be chosen. If the terminals are from 5 to 10 cm apart, they will meet all ordinary requirements.

V. SUMMARY

1. A theoretical discussion is given to show the conditions which must be fulfilled in the construction of resistance coils in order that the phase angle shall be small and the change of resistance with frequency negligible.

2. These principles are applied to the design of coils of different denominations, and specifications are given for the construction of coils of different denominations from 0.1 to 10 000 ohms.

3. There is given the results of measurements on sample coils of each denomination, constructed according to these specifications.

4. A comparison is made between the constants of these coils and those of commercial coils.

5. Suggestions are made as to the method of connecting coils when they are to be used in resistance boxes.

WASHINGTON, September 1, 1911.

THE HYDROLYSIS OF SODIUM OXALATE AND ITS INFLUENCE UPON THE TEST FOR NEUTRALITY

By William Blum

CONTENTS

	Page.
I. Introduction	519
1. Purpose of the investigation	519
2. Sources of uncertainty in Sörensen's method	520
II. Hydrolysis of Sodium Oxalate	521
1. Theoretical considerations	521
(a) Calculation of the theoretical hydrolysis	521
(b) Calculation of standards for camparison	523
(c) Choice of indicator for comparison	524
2. Experimental part	524
(a) Purification and preparation of materials	524
(b) Stability of sodium oxalate solution	527
(c) Effect of sodium oxalate solution on glass	529
(d) Presence of sodium bicarbonate in sodium oxalate	530
(e) Colorimetric comparisons	531
3. Conclusions from comparisons	533
4. Discussion of discrepancies	534
III. Determination of Excess Alkali or Acid in Sodium Oxalate	535
1. Error caused by neglecting hydrolysis	535
2. Calculation of impurities present	535
3. Effect of impurities on standardizing value	536
IV. Summary	537

I. INTRODUCTION

1. PURPOSE OF THE INVESTIGATION

The use of sodium oxalate as a primary standard for acidimetry and oxidimetry was suggested in 1897 by Sörensen,[1] who described

[1] Zs. anal. Chem., 36, pp. 639–643; 1897.

its preparation, testing, and use in subsequent papers.[2] Its general adoption as a standard has been hindered by the difficulty of securing from the manufacturers material of a purity conforming to the specifications prepared by Sörensen.[3] In order to determine the composition of sodium oxalate as purchased, and to secure, if possible, a material of the requisite purity to issue as a standard sample, specimens of sodium oxalate were obtained by purchase in the open market or directly from the maker, from two European and three American manufacturers. In the course of the subsequent tests the methods described by Sörensen were found satisfactory, with the exception of that for the determination of sodium carbonate or sodium acid oxalate, in which discrepancies were found, which led to the following investigation. Sörensen's directions are stated in the following paragraph; and an improved method, based upon this investigation, is described in the summary.

2. SOURCES OF UNCRERTAINTY IN SÖRENSEN'S METHOD

Sörensen's method for testing the neutrality of the sodium oxalate is as follows: "Introduce into a conical Jena flask about 250 cc of water and 10 drops of phenolphthalein solution (0.5 g phenolphthalein dissolved in 50 cc alcohol and 50 cc of water), and evaporate to 180 cc while passing in a current of pure air, free from carbon dioxide. Allow to cool to the ordinary temperature and add 5 g of sodium oxalate. Upon shaking carefully, while maintaining the current of air, the oxalate slowly dissolves. If the solution is red, not more than 4 drops of decinormal acid should be required to render it colorless, while if the solution is colorless, it should acquire a distinct red color upon the addition of not more than 2 drops of decinormal sodium hydroxide."

Preliminary experiments showed that the following points in the above method required investigation:

(*a*) Is pure sodium oxalate neutral toward phenolphthalein, or is the hydrolysis sufficient to produce an alkaline reaction?

(*b*) Does sodium oxalate solution decompose on boiling, or is the increased alkalinity due to its action upon the glass?

[2] Zs. anal. Chem., 42, pp. 333–359 and 512–516; 1903.

[3] Merck "Prüfung der Chemischen Reagenzien auf Reinheit," 1905, and Krauch-Merck "Chemical Reagents," 1907.

(*c*) Is any excess alkali present entirely in the form of Na_2CO_3, or may $NaHCO_3$ actually exist in a material heated to 240° C?

(*d*) Under given conditions of titration, to what form should the excess alkali or acid be calculated, and what errors in the use of sodium oxalate as an acidimetric or oxidimetric standard are caused by the presence of a given excess of acid or alkali?

II. THE HYDROLYSIS OF SODIUM OXALATE

1. THEORETICAL CONSIDERATIONS

Souchay and Lenssen [4] stated that "the aqueous solution does not affect curcuma paper, but blues red litmus paper, especially on boiling." Sörensen [5] considered that the slightly alkaline reaction which most of his preparations showed toward phenolphthalein was probably due to a trace of sodium carbonate, though admitting the possibility of alkalinity due to hydrolysis, in the sense of the equation

$$Na_2C_2O_4 + H_2O = NaOH + NaHC_2O_4.$$

Since the apparent alkalinity of his samples (a maximum of about 0.05 per cent Na_2CO_3) represented a negligible error when used as an acidimetric standard, Sörensen dismissed the subject without further study.

(a) CALCULATION OF THE THEORETICAL HYDROLYSIS

The hydrolysis of sodium oxalate may be represented thus:

$$Na_2C_2O_4 + H_2O \rightleftarrows NaHC_2O_4 + NaOH$$

which may be expressed as follows in terms of ions:

$$Na_2C_2O_4 \rightleftarrows 2Na^+ + C_2O_4^{--}$$

$$C_2O_4^{--} + H_2O \rightleftarrows HC_2O_4^- + OH^-.$$

The hydrolysis may be considered as due entirely to the small value of the ionization constant of the second hydrogen of oxalic

[4] Ann. d. Chem. u. Pharm., 99, p. 33; 1856.

[5] Za. anal. Chem., 42, p. 351; 1903.

acid, which is only about one-thousandth of that of the first hydrogen of this acid.[6] In such a solution, therefore $[HC_2O_4^-]$, is practically equal to $[OH^-]$, while $[C_2O_4^{--}]$ is equal to the product of the concentration of the salt, and its ionization. Thus

$$[HC_2O_4^-]=[OH^-] \qquad K_{A2}=\frac{[H^+]\,[C_2O_4^{--}]}{[HC_2O_4^-]}$$

$$[C_2O_4^{--}]=\gamma C \qquad K_w=[H^+]\,[OH^-]$$

where c = molar concentration of the solution, i. e., the number of gram molecules in one liter of the solution.

γ = ionization of the salt.

K_{A2} = ionization constant for the second H of $H_2C_2O_4$.

K_w = ionization constant for H_2O.

By substitution $[H^+]=\sqrt{\frac{K_{A2}K_w}{\gamma C}}$

To calculate $[H^+]$ for 0.1 M (molar) solution of sodium oxalate, the following values were employed:

$$K_{A2} = 4.5\times 10^{-5} \qquad (7)$$

$$K_{w\,25^\circ} = 1.1\times 10^{-14} \text{ and } K_{w\,18^\circ} = 0.6\times 10^{-14} \qquad (8)$$

$$\gamma = 0.7 \qquad (9)$$

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[6] Chandler, J. Am. Chem. Soc., **30**, pp. 694–713; 1908.

[7] Chandler, J. Am. Chem. Soc., **30**, pp. 694–713; 1908, found by partition experiments 4.1×10^{-5}, and by conductivity, 4.9×10^{-5}. The mean value, 4.5×10^{-5} has been employed for both 18° and 25°, although determined by Chandler at 25°; since the temperature coefficient is probably less than the uncertainty in the value for 25°.

[8] Derived from the following values:

	25° $K\times 10^{14}$	18° $K\times 10^{14}$
Arrhenius, Zs. phys. Chem., **11**, p. 805; 1893	1.2	
Wijs & Van't Hoff, Zs. phys. Chem., **12**, p. 514; 1893	1.4	0.64
Löwenherz, Zs. phys. Chem., **20**, p. 283; 1896	1.4	
Kanolt, J. Am. Chem. Soc., **29**, p. 1402; 1907	0.8	0.46
Hudson, J. Am. Chem. Soc., **31**, p. 1130; 1909	1.0	
Lorenz & Böhi, Zs. phys. Chem., **66**, p. 733; 1909	1.2	0.72
Heydweiller, Ann. phys., **28**, p. 503; 1909	1.0	0.59
(Recalc. from Kohlrausch & Heydweiller, Wied. Ann., **68**, p. 709; 1894.)		
Mean	1.1	0.60

[9] Kohlrausch ("Leitvermögen der Elektrolyte," p. 161; 1898) and Noyes and Johnston (J. Am. Chem. Soc., **31**, p. 987; 1909) both found for 0.1 M $K_2C_2O_4$ at 18°, γ=0.7; which value has been assumed to hold for $Na_2C_2O_4$.

		Round number employed.
from which $[H^+]_{18°}$	$=1.96\times10^{-9}$	2.0×10^{-9}
and $[OH^-]_{18°}$	$=3.06\times10^{-6}$	3.1×10^{-6}
$[H^+]_{25°}$	$=2.66\times10^{-9}$	2.7×10^{-9}
$[OH^-]_{25°}$	$=4.14\times10^{-6}$	4.1×10^{-6}

In the subsequent calculations the round numbers indicated have been employed.

(b) CALCULATION OF STANDARDS FOR COMPARISON

For the purpose of comparison, standards of calculated alkalinity were prepared from mixtures of ammonium chloride and hydroxide, on the following basis. The ionization constant[10] of ammonium hydroxide at $25°=1.8\times10^{-5}$, i. e.,

$$\frac{[NH_4^+]\,[OH^-]}{[NH_4OH]}=1.8\times10^{-5} \text{ and } \frac{[NH_4^+]}{[NH_4OH]}=\frac{1.8\times10^{-5}}{[OH^-]}$$

$$\therefore\ \frac{[NH_4^+]}{[NH_4OH]}=\frac{1.8\times10^{-5}}{4.1\times10^{-6}}=4.4$$

Since, however, 0.1 N NH_4Cl is about 85 per cent ionized[11], it is necessary to use $4.4\div.85=5.2$ parts of 0.1 N NH_4Cl to 1 part 0.1 N NH_4OH to prepare a solution in which $[H^+]_{25°}=2.7\times10^{-9}$.

A second comparison was made with 0.1 N sodium acetate, for which the following constants were employed:

$$K_{Ac}=1.8\times10^{-5} \qquad (12)$$
$$\gamma=0.8$$

from which $[H^+]=\sqrt{\frac{K_{Ac}K_w}{\gamma C}}=1.6\times10^{-9}$

i. e., 0.1 N sodium acetate should be slightly more alkaline than sodium oxalate.

Mixtures of 0.1 N sodium borate and hydrochloric acid, and of glycocoll and sodium hydroxide, were also prepared, according to

[10] Noyes and Sosman, "Electrical Conductivity of Aqueous Solutions," Carn. Inst. Pub., 63, p. 228; 1908.
[11] Calculated from Kohlrausch, "Leitvermögen der Elektrolyte," p. 159; 1898.
[12] Noyes and Sosman, loc. cit.

Sörensen,[13] who calculated $[H^+]_{18°}$ for such solutions from emf measurements with a hydrogen electrode.[14] As his measurements were made at 18°, the observations were confined to that temperature.

(c) CHOICE OF INDICATOR FOR COMPARISON

Preliminary tests showed that phenolphthalein is the only indicator that is sensitive in solutions of exactly this alkalinity. Alizarin, cyanin, and dinitro-hydroquinone all proved unsatisfactory for accurate matching, though the latter was sufficiently sensitive to confirm approximately the results with phenolphthalein. The indicator selected for the final comparison was therefore phenolphthalein, for which the most probable value[15] of K is 1.7×10^{-10}. In a solution in which $[H^+]_{25°} = 2.7 \times 10^{-9}$ this indicator should therefore be transformed into its salt[16] to the extent of 5.9 per cent.

$$\frac{[H^+]\,[I^-]}{[HI]} = 1.7 \times 10^{-10}$$

$$\frac{[I^-]}{[HI]} = \frac{1.7 \times 10^{-10}}{2.7 \times 10^{-9}} = 6.3 \times 10^{-2} = \frac{5.9}{94.1}$$

These calculations are based upon the assumption that the ionization constant of phenolphthalein is the same at 18° and 25°, and that the color of the indicator is not materially affected by the presence of neutral salts.

2. EXPERIMENTAL PART

(a) PURIFICATION AND PREPARATION OF MATERIALS AND SOLUTIONS

Air.—Air was freed from carbon dioxide by passage through two wash bottles containing 30 per cent potassium hydroxide and a

[13] Sörensen, Compt. Rend. Trav. Lab. Carlsberg, 8, pp. 1–168; 1910, and Biochem. Zs., 21, pp. 131–304; 1910.

[14] Several of Sörensen's mixtures were checked electrometrically and colorimetrically by Auerbach and Pick, Arbeit Kais. Gesundheitsamte, 38, pp. 243–274; 1911.

[15] Hildebrand, Zs. Elektrochem., 14, p. 351.
Wegscheider, Zs. Elektrochem., 14, p. 510.
Rosenstein (Mass. Inst. Tech.), Private communication from W. C. Bray.

[16] Noyes, J. Am. Chem. Soc., 32, p. 861; 1910.

U tube with a 10-inch column of soda-lime, followed by absorbent cotton.

Water.—The regular distilled water was redistilled from alkaline permanganate. It was then boiled in a seasoned Jena flask for two hours in a stream of pure air and preserved in a Jena flask provided with a siphon and a soda-lime guard tube. This water had a specific conductivity of 2×10^{-6} reciprocal ohms and evolved no carbon dioxide when a small portion was boiled for one-half hour in a stream of pure air, which after being dried with calcium chloride was passed through a weighed soda-lime tube.

Sodium oxalate.—Pure sodium oxalate was prepared from two commercial samples. One, designated A, contained originally (as determined in the final tests) 0.02 per cent $NaHC_2O_4$; while that marked B contained 0.04 per cent Na_2CO_3, and considerable CaC_2O_4. These materials were recrystallized in platinum, the solutions being electrically heated, and the surrounding atmosphere being kept as free from carbon dioxide as possible. The fine crystals obtained were sucked dry on a platinum cone and dried in an electric oven at 240°. A portion of the product of the third recrystallization of B was dissolved in water and precipitated by the addition of double distilled neutral alcohol, the product being filtered out and dried as above. The following samples of sodium oxalate were thus obtained:

Water crystallization A_1, A_2; B_1, B_2, B_3.

Alcohol precipitation B_4.

Since in subsequent tests these samples were found to produce almost the same colors with phenolphthalein, we may conclude that a single water crystallization eliminated practically all of the impurities above noted; and as a constant and uniform product was obtained by recrystallization of materials originally respectively acid and alkaline, it is reasonably certain that the product represented pure sodium oxalate.

Ammonium chloride.—A normal solution was prepared from J. T. Baker's special ammonium chloride, dried at 110° for two hours. This material was found to contain no organic bases when tested according to Krauch.

Ammonium hydroxide.— Decinormal solution was prepared from redistilled ammonia and standardized against decinormal hydrochloric acid, which in turn was standardized gravimetrically by means of silver chloride. Methyl red was used as the indicator. The solution was preserved in a ceresin-lined bottle and measured in a burette closed with a soda-lime tube.

Sodium hydroxide.— Pure material was prepared from metallic sodium with exclusion of carbon dioxide.[17] The resulting strong solution was titrated against standard hydrochloric acid and diluted to centinormal solution, which was preserved in a ceresin-lined bottle.

Oxalic acid.—A centinormal solution was prepared from J. T. Baker's crystallized oxalic acid.

Sodium acetate.— Merck's crystallized sodium acetate was recrystallized in platinum. The products of the crystallizations were sucked dry on a platinum cone, dried between filter paper, and finally air-dried.

Sodium borate ($Na_2B_4O_7$) solution was prepared from equivalent quantities of twice recrystallized boric acid and pure sodium hydroxide.

Glycocoll (amino-acetic acid) solution was prepared according to Sörensen by dissolving 7.505 g glycocoll (Kahlbaum) and 5.85 g sodium chloride in 1 liter.

Phenolphthalein.—A 1 per cent alcoholic solution of the indicator was employed, i. e., about 0.03 N. In each comparison exactly 0.2 cc of this solution was employed in a total volume of 150 cc, thus giving a solution in which the phenolphthalein is about $\frac{1}{22000}$ N. According to McCoy[18] the saturated aqueous solution of this indicator is about $\frac{1}{11000}$ N. A solution of such concentration (i. e., 0.4 cc in 150 cc) was, however, always turbid, so that the smaller concentration was employed, giving a color of

[17] Findlay, Practical Physical Chemistry, p. 176; 1906.

[18] Am. Chem. J., 31, p. 503; 1904.

convenient intensity for comparison. Alcohol was not present in amount sufficient to affect the color [19] of the indicator.

(b) STABILITY OF SODIUM-OXALATE SOLUTIONS

Before attempting to prepare and test pure sodium oxalate it was necessary to determine the effect of heat upon its solution. Sörensen [20] found that a sodium oxalate solution became more alkaline upon boiling, whether in water that had previously been boiled or not, and equally in a stream of air, oxygen, or nitrogen. He attributed the increased alkalinity to a decomposition in the sense of the equation

$$2Na_2C_2O_4 + H_2O = CO_2 + Na_2CO_3 + 2HCOONa$$

quoting Carles [21] who found that oxalic-acid solution decomposed similarly if boiled in a stream of an inert gas.

That no appreciable decomposition takes place upon boiling sodium-oxalate solution was shown by the following experiments:

Five grams of the salt were boiled with pure water in a stream of pure air, which was passed through calcium chloride and then through a weighed soda-lime tube. The latter was reweighed at the end of two hours, replaced, and the solution acidified and again boiled to determine whether any sodium carbonate was formed in the boiling of the aqueous solution. The amount of carbon dioxide evolved on direct acidification was also determined. The apparatus and method were tested with known small quantities of sodium carbonate.

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[19] McCoy, Am. Chem. J., 31, p. 503; 1904.
Hildebrand, Zs. Elektrochem., 14, p. 351, 1908; and J. Am. Chem. Soc., 30, p. 1914, 1908.
[20] Zs. Anal. Chem., 42, p. 352; 1903.
[21] Compt. rend., 71, p. 226; 1870.

TABLE I

Evolution of Carbon Dioxide on Boiling Solutions Containing 5 Grams of Sodium Oxalate

Expt.	Sample	Solution	Time	Weight of CO_2
			minutes	mg
1	A	Acidified	15	0.3
2a	"	Aqueous	15	0.0
b	"	"	+120	0.2
c	"	Acidified	+ 15	0.2
3a	B	"	15	1.3
b	"	"	30	1.4
4a	"	Aqueous	15	0.6
b	"	"	+120	0.3
c	"	Acidified	+ 15	0.5

Since in experiments 1 and 2b practically the same amount of CO_2 was evolved on 15 minutes' treatment with acid as upon 2 hours boiling with water, we may conclude that no appreciable decomposition takes place in 2 hours' boiling, though the solutions become markedly alkaline. The evolution of 0.6 mg CO_2 in 4a is believed to be due to the presence of $NaHCO_3$ (or occluded CO_2) in this sample. See 2 (d), p. 530.

Further tests showed that there was no increase in alkalinity upon boiling the sodium-oxalate solution for two hours in a platinum dish or in a quartz flask in a current of pure air. (Cf. Table II, 1b, c, d; p. 529.) The increased alkalinity noted by Sörensen[9] must therefore have been due to the attack of the glass (Jena glass) employed by him. Since it was impracticable to employ entirely quartz flasks, tests were carried out to determine the relative effect of sodium oxalate solutions upon different kinds of glass. Flasks of several kinds of glass were prepared of approximately the same shape and size, with tubes ground in for the inlet and exit of the pure air (except the quartz flask). Two gram portions of sodium oxalate were boiled with 125 cc pure water and 0.2 cc phenolphthalein solution in a stream of pure air. At the end of the test they were titrated with 0.01 *N* oxalic acid till the solutions were colorless. The results were as follows:

TABLE II

(c) Effect of Sodium Oxalate Solutions upon Glass

Increased Alkalinity, in terms of 0.01 N Oxalic Acid

Expt.	Glass	Sodium oxalate	Time	Volume of 0.01 N $H_2C_2O_4$
			minutes	cc
1a	Quartz	A	120	0.0
b	"	F	"	[a] 0.5
c	"	"	60	0.5
d	"	"	10	0.4
2a	Durax	A	120	0.5
b	"	"	"	0.5
c	"	F	"	0.7
3a	Jena Geräte	A	"	4.9
b	"	"	"	4.6
c	"	F	"	5.1
4a	German soft	A	"	6.8
b	"	"	"	6.7
c	"	F	"	6.5
5a	Resistance	A	"	8.7
b	"	"	"	6.6
c	"	F	"	8.5
6a	Jena Verbund	"	"	4.1

[a] All the 2-hour results for F have been corrected for this "original" alkalinity.

From Table II it is evident that the Durax glass [22] is not appreciably attacked by neutral or faintly alkaline sodium oxalate solutions during two hours' boiling—i. e., much longer than is required to expel any CO_2 from the solution. For the subsequent experiments vessels of quartz, platinum, or Durax glass were therefore employed. It may be noted that in none of the flasks did the solution acquire more than a very faint pink color at the end of 15 minutes, though the Jena Geräte glass was most rapidly attacked, and is least suited for this purpose. As the successive tests of each glass were made on the same vessels with approximately equal results, there is no marked evidence of "seasoning" of the glass by several hours' boiling.

[22] Durax glass, made by Schott & Gen., is furnished only in the form of tubing, which formerly was distinguished by a longitudinal blue line, which has recently been changed to a green line. Verbund glass, formerly made with no distinguishing mark, now has a blue line. This information was secured directly from the maker, and is of importance in view of the confusion of terms in some of the recent catalogues of glassware.

(d) PRESENCE OF SODIUM BICARBONATE IN SODIUM OXALATE

Sörensen dismissed the question of the existence of $NaHCO_3$ by stating that it could not possibly be present in material dried at 240°, since it decomposes at or below 100°. In the course of the testing, however, it was found that several of the samples when dissolved in pure water containing phenolphthalein were colorless, or nearly so, but after a few minutes' boiling became strongly pink and remained pink on cooling, indicating the presence of appreciable excess alkali, which would have been overlooked if tested according to Sörensen. Such samples, e. g., B, evolved carbon dioxide upon boiling the aqueous solution, and upon boiling with acid evolved a total amount of carbon dioxide greater than would correspond to the titrated alkalinity if calculated as Na_2CO_3. From Table III it may be seen that in every case the gravimetric and volumetric results agreed more closely if calculated to $NaHCO_3$ than to Na_2CO_3, and that in all but E the carbon dioxide is apparently even in excess of that calculated for $NaHCO_3$.

TABLE III

Determination of Excess Alkali in Sodium Oxalate

Expt.	Sample	Gravimetric determination of CO_2, calculated as—		Volumetric determination of alkali, calculated as—	
		Na_2CO_3	$NaHCO_3$	Na_2CO_3	$NaHCO_3$
		Per cent	*Per cent*	*Per cent*	*Per cent*
1	B	0.065	0.051	0.036	0.057
2	C	0.291	0.230	0.107	0.171
3	D	0.308	0.244	0.120	0.192
4	E	1.31	1.05	0.70	1.11

It is evident, therefore, that if such samples were tested with the object of making corrections for accurate work, appreciable errors might arise if the alkalinity were calculated to Na_2CO_3. For samples containing less impurity than B the differences would be entirely negligible.

With our present means of analysis it is impossible to state whether such "excess" carbon dioxide (above Na_2CO_3) is present

as $NaHCO_3$ or in some occluded form. That it is not readily given off at 240° was shown in E, which lost only 0.14 per cent in two hours' heating at 240°, though it contained about 0.40 per cent of such excess "carbonic acid." When we consider that as usually prepared (by neutralization of oxalic acid with sodium carbonate) the salt separates from a solution charged with carbon dioxide, it is not impossible that the latter should be occluded in it, in a form in which it is not readily expelled at high temperatures; just as water has been shown to exist in materials heated to high temperatures.[23]

(e) COLORIMETRIC COMPARISONS

All the solutions for comparison were prepared in 300 cc flasks of Durax glass with a long narrow neck to facilitate exclusion of carbon dioxide. Before being used these were boiled for two hours with pure water, allowed to soak in water for 36 hours, and were filled with 0.1 *M* sodium oxalate (or solutions of approximately the same alkalinity) for several days before the final tests were made.

Decimolar solutions of sodium oxalate were prepared by dissolving 2 grams of the salt in 150 cc of pure water, to which was added 0.2 cc phenolphthalein. The solutions were boiled for 10 minutes in a stream of pure air, which was continued during their cooling to room temperature. The flasks were then closed with rubber stoppers and placed in a thermostat adjusted to the desired temperature (±0.2°).

Ammonium chloride—ammonium hydroxide solution of the calculated alkalinity ($[H^+]_{25°} = 2.7 \times 10^{-9}$) was prepared by mixing 15.0 cc *N* ammonium chloride, 28.8 cc 0.1 *N* ammonium hydroxide, and 0.2 cc phenolphthalein, and diluting to 150 cc, the resultant solution being decinormal with respect to the salt—i. e., ammonium chloride. In the determinations of the value of the dissociation constant for phenolphthalein, given above (p. 524), Hildebrand[24] and Rosenstein[25] employed solutions not over 0.05 N, with respect to

[23] Richards, J. Am. Chem. Soc., 33, p. 888; 1911.
[24] Zs. Elektrochem., 14, p. 351; 1908.
[25] Private communication from W. C. Bray.

ammonium chloride, in order to reduce the neutral salt effect.[26] In these tests it was hoped that by making all the solutions 0.1 *N* the neutral salt effect would be largely eliminated. (From the standpoint of hydrolysis, 0.1 *M* sodium oxalate may be considered as 0.1 *N*.)

From Sörensen's chart, mixtures of 7.06 cc of 0.1 *N* $Na_2B_4O_7$ and 2.94 cc 0.1 *N* HCl, or of 9.34 cc glycocoll solution and .66 cc 0.1 *N* NaOH, produce solutions in which $[H^+]_{18°} = 2.0 \times 10^{-9}$ (i. e., $10^{-8.7}$). Mixtures in these proportions were prepared with a total volume of 150 cc.

Phenolphthalein solutions of calculated transformation were prepared[27] by adding an accurately measured amount (0.2 cc) of phenolphthalein to 150 cc of water containing 10 cc of 0.1 *N* sodium hydroxide, whereby the indicator was completely transformed into its red salt. Measured portions of this solution were then diluted to 150 cc to produce the standards representing desired percentage transformation. In each solution it is of course necessary to have 0.2 cc of the original phenolphthalein solution in the same total volume.

The following comparisons were made by direct optical matching of the solutions in the Durax flasks, which were all of the same shape and size. Attempts to use a colorimeter proved unsatisfactory, owing to the very light colors of the solutions and to very rapid fading of the sodium oxalate and acetate solutions upon exposure to air and even slight absorption of carbon dioxide. The results are expressed in percentage transformation of phenolphthalein, the standards for which did not vary appreciably in color between 18° and 25°.

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[26] Michaelis & Rona, Zs. Elektrochem., **14**, pp. 251–253, 1908, and Biochem. Zs., **22**, pp. 61–67; 1909. Bohdan v. Szyszkowski, Zs. Physik. Chem., **58**, p. 420; 1907. Sörensen and Palitzsch, Biochem. Zs., **24**, p. 387; 1910.

[27] Noyes, J. Am. Chem. Soc., **32**, p. 826; 1910.

TABLE IV

Colorimetric Comparisons

Expt.	Solution	Composition or Ratio	COLOR: Per cent Transformation of Phenolphthalein			
			25°		18°	
			Calc.	Obs.	Calc.	Obs.
1	$Na_2C_2O_4$-B_1	0.1 *M*	5.9	3.5	7.8	4.0
2	" B_2	0.1 *M*	5.9	3.5	7.8	3.5
3	" B_3	0.1 *M*	5.9	4.0	7.8	4.0
4	" B_4	0.1 *M*	5.9	4.0	7.8	4.0
5	" A_1	0.1 *M*	5.9	4.0	7.8	4.0
6	" A_2	0.1 *M*	5.9	4.0	7.8	4.0
7	" B_1	0.2 *M*	7.8	6.0	10.2	6.0
8	$NaC_2H_3O_2$	0.1 *N*	9.6	3.0	12.7	3.5
9	"	0.1 *N*	9.6	3.0	12.7	3.5
10	NH_4Cl NH_4OH	5.2 1.0	5.9	6.5	9.8	8.5
11	NH_4Cl NH_4OH	6.7 1.0	...	...	7.8	5.0
12	$Na_2B_4O_7$ HCl	7.06 2.94	...	...	7.8	5.5
13	$Na_2B_4O_7$ HCl	6.84 3.16	...	...	6.8	4.0
14	Glycocoll NaOH	9.34 0.66	...	...	7.8	6.5
15	Glycocoll NaOH	9.52 1.48	...	...	5.9	4.0

3. CONCLUSIONS FROM COMPARISONS

From Table IV the following conclusions may be reached:

1. The color produced by phenolphthalein in sodium oxalate solutions is not appreciably affected by changes in temperature between 18° and 25°.

2. As noted by Hildebrand[28] the colors produced in given mixtures of ammonium chloride and ammonium hydroxide are markedly affected by slight temperature changes, making such solutions unsuitable for practical comparison standards.

[28] Zs. Elektrochem., 14, p. 351; 1908.

3. The colors produced in sodium acetate solutions [29] are far lighter than those calculated from the constants employed.

4. The color of the sodium oxalate solution is matched closely by solutions 13 and 15, in which, according to Sörensen's chart, $[H^+]^{18°}$ is, respectively, 2.35×10^{-9} (or $10^{-8.63}$) and 2.7×10^{-9} (or $10^{-8.57}$). On the basis of Sörensen's emf measurements $[H^+]_{18°}$ for 0.1 *M* $Na_2C_2O_3$ may therefore be considered as equal to 2.5×10^{-9} (or $10^{-8.6}$); and $[OH^-]_{18°}$ as equal to 2.4×10^{-6} (or $10^{-5.62}$). In other words, the salt is hydrolyzed at 18° to the extent of 0.0024 per cent.

5. In all the solutions except No. 10 the colors with phenolphthalein are markedly less than those calculated from the ionization constant 1.7×10^{-10}.

4. DISCUSSION OF DISCREPANCIES

Extended discussion of the causes of the discrepancies between the various calculated and observed colors would be of little interest, owing to the uncertainties in the values of the constants employed, especially that of phenolphthalein.[30] The value accepted depends practically upon the ammonium chloride-hydroxide mixtures used by Hildebrand and Rosenstein, since the individual values of Wegscheider are far from concordant. If the ionization constant of phenolphthalein were calculated from experiments 12 to 15, on the basis of Sörensen's emf measurements, the value $K = 1.1 \times 10^{-10}$ would be obtained, which agrees more closely with McCoy's value of 0.8×10^{-10}. Salm,[31] however, obtained the value $K = 8.0 \times 10^{-10}$ by measurements with the hydrogen electrode. The discrepancies in the present work are in accord with the results of Hildebrand, who found unexplainable irregularities in the value of K with less than 8 per cent transformation. These discrepancies may be due to the fact that phenolphthalein acts as a dibasic acid, as shown by Acree [32] and Wegscheider.[33]

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[29] Salessky, Zs. Elektrochem., **10**, p. 204, 1904, found that *N* $NaC_2H_3O_2$ solution was far less alkaline toward phenolphthalein than was indicated by calculation.

[30] Noyes, J. Am. Chem. Soc., **32**, p. 859; 1910.

[31] Zs. Elektrochem., **10**, p. 344; 1904.

[32] Am. Chem. J., **39**, p. 528; 1908.

[33] Zs. Elektrochem., **18**, p. 510; 1908.

In view of these uncertainties the reaction of pure sodium oxalate may best be defined empirically as equal to 4 per cent phenolphthalein transformation in 0.1 *M* solution, and 6 per cent in 0.2 *M* solution, standards which are readily reproducible and are free from any assumptions as to the value of the constants.

III. DETERMINATION OF EXCESS ALKALI OR ACID IN SODIUM OXALATE

1. ERROR CAUSED BY NEGLECTING HYDROLYSIS

Having shown that pure sodium oxalate (0.1 *M*) produces a pink color equivalent to 4 per cent phenolphthalein transformation, accurate results in testing its neutrality will be obtained by titrating to such a standard color rather than to colorless. In order to determine the magnitude of such a correction the following tests were made: Two gram samples of sodium oxalate dissolved as in the previous tests were titrated with 0.01 *N* oxalic acid or 0.01 *N* NaOH (*a*) to the standard color and (*b*) to colorless. The difference in the volume of 0.01 *N* acid or alkali required was always 0.3 cc, equivalent to 0.016 per cent Na_2CO_3, 0.025 per cent $NaHCO_3$, or 0.017 per cent $NaHC_2O_4$, on the assumption that in the cold the end point is reached when $NaHCO_3$ is formed. The error involved in titrating 4 g in the same volume to colorless instead of the normal color for 0.2 *M* solution (6 per cent phenolphthalein) is 0.6 cc, i. e., the percentage error is about the same. In general, therefore, any errors through titration to colorless are likely to be negligible and far less than those caused by the presence of excess carbon dioxide in the samples or the attack of the glass during the boiling for its expulsion.

2. CALCULATIONS OF IMPURITIES PRESENT

The question as to the form in which the excess alkali exists in a given sample can not be determined without a detailed study of each sample. From the experience with numerous samples most accurate results will probably be obtained by calculating the alkalinity to $NaHCO_3$. In the case of samples containing less than 0.10 per cent alkali the differences will be negligible, i. e., not over

0.04 per cent. Samples containing much more than 0.10 per cent of alkali are unsuitable for standardizing, even with a correction. Excess acidity may be calculated as $NaHC_2O_4$.

In the above calculations we have assumed that 1 cc 0.01 *N* acid[34] is equivalent to 0.00106 g Na_2CO_3 or 0.00168 g $NaHCO_3$, while 1 cc 0.01 *N* NaOH is equivalent to 0.00112 g $NaHC_2O_4$, in accordance with the following equations:

Upon boiling the original solution

$$2NaHCO_3 = Na_2CO_3 + H_2O + CO_2.$$

and upon titration in the cold

$$2Na_2CO_3 + H_2C_2O_4 = Na_2C_2O_4 + 2NaHCO_3,$$

while

$$NaHC_2O_4 + NaOH = Na_2C_2O_4 + H_2O.$$

3. EFFECT OF IMPURITIES ON STANDARDIZING VALUE

The errors caused in the use of sodium oxalate by a given amount of such impurities are as follows: When the sodium oxalate is used as an acidimetric standard, Na_2CO_3 causes a positive error to the extent of only 21 per cent of its amount, i. e., the material is apparently stronger than 100 per cent. $NaHCO_3$ causes a negative error of 25 per cent of its amount, while $NaHC_2O_4$ produces a negative error of 67 per cent of its amount. That is, the same quantity of alkali is present (after ignition) in 134 parts $Na_2C_2O_4$, 106 parts Na_2CO_3, 168 parts of $NaHCO_3$, or 224 parts of $NaHC_2O_4$. If sodium oxalate is to be used as an oxidimetric standard, either Na_2CO_3 or $NaHCO_3$ is an inert impurity, i. e., its effect is proportional to its amount. $NaHC_2O_4$ causes a positive error of 16 per cent of its amount, since 112 parts of $NaHC_2O_4$ have the same reducing power as 134 parts of $Na_2C_2O_4$. The same relations may be expressed as follows: In order to have an error in titration of not over 0.10 per cent, the following amounts of impurities may be present: In alkalimetry, 0.48 per cent Na_2CO_3, 0.40 per cent $NaHCO_3$, or 0.15 per cent $NaHC_2O_4$. In oxidimetry,

[34] It is immaterial whether oxalic acid or some stronger acid be employed, since the latter, in small amount, would immediately liberate its equivalent of oxalic acid.

0.10 per cent Na_2CO_3 or $NaHCO_3$, or 0.63 per cent $NaHC_2O_4$. For use as a general standard, however, the lowest of each of these values is the maximum permissible for the given degree of accuracy.

IV. SUMMARY

1. The solution of pure sodium oxalate is alkaline.

2. Decimolar sodium oxalate solution produces a color with phenolphthalein equivalent to 4 per cent transformation of the indicator and fifth molar a color equal to 6 per cent transformation. The most probable value of $[H]_{18°}$ in these two solutions is, respectively, 2.5×10^{-9} and 2.0×10^{-9}.

3. The use of calculated standards of (*a*) phenolphthalein, (*b*) ammonium chloride and hydroxide, (*c*) sodium acetate, (*d*) borax and hydrochloric acid, or (*e*) glycocoll and sodium hydroxide, for determining the normal alkalinity of sodium oxalate is inaccurate, due either to uncertainty in the values of the constants, to undetermined influence of the salts, or to abnormal dissociation and hydrolysis phenomena.

4. The value of the ionization constant K for phenolphthalein is probably less than 1.7×10^{-10} for solutions in which it is transformed to the extent of less than 8 per cent, i. e., the only solutions adapted to direct optical comparison.

5. Sodium oxalate solution does not decompose appreciably on boiling.

6. Sodium oxalate solutions readily attack glass, "Durax" glass being the least affected of the kinds tested.

7. Commercial samples of sodium oxalate, dried at 240°, may contain $NaHCO_3$ or even occluded CO_2.

8. The following method for testing the neutrality of sodium oxalate is recommended to replace that given by Sörensen: Evaporate 200 cc of water in a quartz or Durax glass flask to 150 cc in a current of pure air, free from carbon dioxide. Add exactly 0.2 cc phenolphthalein solution (1 per cent solution in alcohol), and 4 g of sodium oxalate. Continue to boil for 10 minutes, then cool to room temperature while maintaining the air current. If pure, the solution should have a pink color equivalent to 6 per cent phenolphthalein transformation. (Such a standard may be pre-

pared by adding 0.2 cc phenolphthalein to 150 cc of water containing 10 cc of 0.1 *N* sodium hydroxide and diluting 9 cc of this solution to 150 cc.) Each cc of 0.01 *N* acid or alkali required to titrate to the standard color indicates the presence, respectively, of approximately 0.04 per cent $NaHCO_3$ or 0.03 per cent $NaHC_2O_4$. One or two water recrystallizations in platinum will usually be sufficient for the preparation of the pure salt from materials containing moderate excess of alkali or acid.

The author desires to express his thanks for the information furnished by J. H. Hildebrand, J. Johnston, W. C. Bray, and W. F. Hillebrand during the course of this investigation.

WASHINGTON, November 9, 1911.

WAVE-LENGTHS OF NEON

By Irwin G. Priest

Introduction.—The 10 wave-lengths of neon presented in the accompanying table have been determined by the method of reflection fringes previously described in this Bulletin.[1] Only minor changes of method and apparatus, mostly for convenience, have been made in this research. The lamps were of the high-voltage "vacuum" type and operated on a transformer circuit.

The probable errors of the final results are shown in Columns II and III of the table. They are all less than one part in ten million. The average probable error of the result of a single determination (usually involving 10 measurements of the order of interference) is about one part in eight or nine million. The average residual of the determinations relative to neon 5852 as standard is about 0.001 A, or one part in six million.

Basis and Reliability of the Results.—The ultimate *reference standard* to which these values are referred is the wave-length of cadmium red taken as **6438.4696** I. A.[2] The value **5852.4862** I. A. for the bright yellow neon line is derived from our previous determination[3] of this wave-length relative to cadmium red by

[1] This Bulletin, 6, p. 573; 1911 (Reprint No. 142).
[2] Trans. Inter. Solar Union, 2, 20. Astrop. Jour., 32, 215; 1910.
[3] This Bulletin, 6, 599; 1911.

TABLE I[4]

General Summary of Results

I	II	III	IV	V	VI	VII
Wave-Lengths in International Ångström Units[a]	Probable errors[b] relative to		Number of measurements involved[c]	Average of accepted minus average of rejected determinations	Difference between present and former determinations[d]	
	Neon yellow, 5852	Cadmium red, 6438			Baly minus Priest	Watson minus Priest
5852.4862	.....	±0.0003	1 (5, 5); 6 (10, 10)	None rejected[e]	+0.16	+0.13
5881.8958	±0.0004	5	3 (10, 10)	−0.0004	.14	.16
5944.8344	2	3	(f)	+0.0012	.08	.19
6074.3383	3	4	5 (10, 10); 1 (9, 10); 1 (10, 9)	None rejected	.18	.17
6096.1608	4	4	4 (10, 10); 1 (9, 10)	None rejected	.21	.20
6143.0600	6	6	2 (10, 10)	+0.0006	.22	.25
6266.4948	4	5	1 (10, 9); 1 (9, 10); 2 (10, 10)	−0.0003	.16	.19
6304.7929	4	5	4 (10, 10)	−0.0008	.20	.18
6382.9882	4	5	1 (9, 10); 1 (8, 10); 2 (10, 10)	+0.0024	.16	.15
6402.2392	2	3	(g)	−0.0014	.16	.19

[4] Particular parts of the table are explained by the following notes:

[a] The definition of the International Ångström Unit is

$$1\ \text{I. A.} = \frac{\text{wave-length of cadmium red, in dry air at } 15°,\ 760 \text{ mm } (g = 980.67)}{6438.4696}$$

(Trans. Int. Solar Union, **2**, p. 20; 1907. Astrop. J., **32**, p. 216; 1910).

[b] The probable errors in Column II are obtained by combining the probable errors of the independent determinations. The first probable error in Column III is obtained from the data given in column Eλ, Table II, page 599, vol. 6 of this Bulletin. The other errors in Column III were obtained from corresponding ones in Column II by combining each of these latter with the first one in Column III. The reason for this will be found in a consideration of the remarks under "Reference standards" in the body of this paper.

[c] In Column IV the boldface figures indicate the number of determinations containing the number of measurements indicated by the accompanying light-faced figures. These light-faced figures, in the order given, indicate, respectively, the number of measurements of the order of interference made in Forms II and III, as illustrated in this Bulletin, **6**, 596.

[d] The values of Baly (Trans. Roy. Soc. A 202, p. 183; 1904. Also B. A. Report, 1905, pp. 105–153), as well as those of Watson (Proc. Roy. Soc. **81**, p. 181; 1908), are from grating determinations by the method of interpolation, relative to Kayser's old iron standards. Now, "the differences" between the standards of the old system and those of the new international system "vary irregularly between 0.15 and 0.22 A." (Kayser, Astroph. J., 32, 217.) On this account and since Baly estimated his probable error as ±0.03 A, and Watson only hopes that his error is not greater than the same amount (Proc. Roy. Soc. **81**, p. 183), it appears that my values agree with those of Baly and Watson to within the accuracy of their measurements.

[e] Eight determinations in which this wave-length was used as λ_r relative to 6438.4696 were rejected for quite adequate reasons. The mean of them is 5852.4848.

[f] 4(10, 10) relative to neon yellow, 5852.4862. **2**(10, 10) relative to cadmium red, 6438.4696. 16 determinations as λ_r (see this Bulletin, **6**, 575), relative to 5852.4862.

[g] 1(7, 8); **2**(5, 5) relative to cadmium red. 47 determinations as λ_r relative to 5852.4862.

merely reducing that result to the present definition of the International Ångström Unit. All the other wave-lengths have been determined relative to this yellow neon line as an intermediate standard, assigning it the above value. Also the wave-lengths 5944 and 6402 have been determined *directly* relative to cadmium red and the results combined with those relative to neon yellow to give the values in the table.

Error of Method.—It should be noted that the method assumes an equal degree of homogeneity in the standard and the unknown. This condition was not sufficiently emphasized in my previous paper.[5] Occasion is therefore taken here to point out that the method is not applicable except to sufficiently narrow lines. We have in actual practice,

$$\left(\lambda_x - \frac{W_x}{2}\right) = \frac{p'_s}{p'_x}\left(\lambda - \frac{W_s}{2}\right)$$

instead of simply

$$\lambda_x = \frac{p_s}{p_x}\lambda_s$$

to determine λ_x, where λ_s and λ_x are the standard and the unknown wave lengths; p'_s and p'_x are the corresponding *measured* orders of interference; and W_s and W_x are the widths of the lines. From the first equation above we have

$$\lambda_x - \frac{W_x}{2} + \frac{p'_s}{p'_x} \cdot \frac{W_s}{2} = \frac{p'_s}{p'_x}\lambda_s$$

Now, our method actually gives as the computed result the value of the first member of this equation as it is defined by the second member.

Evidently the error of the method is

$$\frac{p'_s}{p'_x} \cdot \frac{W_s}{2} - \frac{W_x}{2}$$

The value of $\frac{p'_s}{p'_x}$ in these determinations varies between 0.91 and 1.10. Examination of these radiations by the echelon, as well as

[5] See, however, this Bulletin, 6, 602; 1911.

by the interferometer, shows that W for cadmium red is very nearly equal to W for the neon lines. Moreover, Dr. Nutting concludes from the echelon examination that the value of W in neither case can exceed 0.002 A. This being true, it is evident that the value of the above expression for error of method must be considerably less than 0.001 A, and so is negligible in the present case.[6] At the same time I take this opportunity of emphasizing the fact that *an error will be introduced if the above conditions are not sufficiently fulfilled.*[7] The value of the method for many purposes of interferometry is not invalidated by this consideration.

Discarded Results.—As is almost inevitable in any investigation, some determinations have been rejected on account of evident or of suspected errors. However, in the case of each wave-length the mean of *all* rejected determinations has been found. As is shown in Column V of the table, these means differ from the accepted values in every case but one by a negligible amount.

Assumed Values.—The previous results of Baly and Watson were assumed correct only to within about 1. A. This gives a large factor of safety, as will be seen by referring to the accuracy of their values. (See Table, Columns VI and VII and footnote [d].)

Confirmatory Results.—Determinations of wave-lengths 5944 and 6402 relative directly to cadmium red and indirectly through neon yellow give results agreeing to within a few ten-thousandths of an Ångström Unit, thus confirming the first determined value of the neon yellow relative to cadmium red. Using the present method and my value for neon yellow as a standard, I have also confirmed to within less than 0.001 A, the mean of Lord Rayleigh's results[8] for helium 5016. However, this result is 0.004 A less than that of Eversheim.[9] It may be that my result in this case

[6] The foregoing discussion is not presented as a complete and rigorous general treatment, but is considered adequate for the present particular case.

[7] Such an error has been found in attempting to determine the yellow helium wave length. It was found that the *measured* wave length (not corrected for error of method) decreased with increasing current in the lamp. At the same time we have found for the green helium (5015) a value agreeing with that found by Rayleigh, who used the method of diameters.

[8] Reduced to International Ångströms. For original data see Phil. Mag. (6), 11, p. 685; 1906 and (6), 15, p. 548; 1908.

[9] Zs. wiss. Photo, 8, p. 148; 1909.

is low, owing to the width of the helium line. (See "Error of Method," in this paper.) In any case my value for neon yellow appears to be confirmed to within the accuracy of the mean of Rayleigh's and Eversheim's results for the helium line. I have not yet obtained satisfactory results for other helium wave-lengths. (See footnote [7].) They may be taken up again at some future time.

Since completing the work reported here I have determined the wave-length of mercury 5769 relative to our value of neon yellow by an adaptation of Fabry and Perot's method of coincidences. The mean of eight determinations[10] agrees with Fabry and Perot's value[11] to within 0.0015 A. If one of these determinations be suppressed because of its wide discrepancy from the others, the mean of the remaining seven is only 0.0002 A greater than Fabry and Perot's value. This constitutes an indirect confirmation of the correctness of our value for neon yellow independent of our method.

As pointed out in footnote [d] to the table, our results are in agreement with those of Baly and Watson to within the accuracy of their values.

Accuracy.—The reader is now in possession of the data requisite to form his own estimate of the accuracy of these results. I have recorded the results to eight significant figures, because I believe that at least the neon ratios inter se are accurate to a few units in the eighth place, and these ratios themselves are of some value. The only doubt that has arisen in my mind as to the accuracy relative to cadmium red is that discussed under "Error of Method," above. Relying on the conclusion reached there and the further confirmatory evidence of the determination relative to mercury 5769, I see no present reason to doubt that the values relative to cadmium red are accurate to about one one-thousandth of an Ångström unit. However, this investigation appears to be the pioneer attempt to determine the neon wave-lengths to this

[10] Before perfecting the method of observing coincidences a number of discordant results were obtained. The eight determinations here mentioned were obtained all in succession after making improvements. Since then no others have been made.

[11] Reduced to International Ångströms. A. de Chim. et de Phy. (7), 16, p. 320.

degree of accuracy, and like all pioneer efforts must be regarded as subject to revision. It is not expected that these values will command implicit confidence until confirmed by the results of another observer using another method; and it is hoped that such results will not long be lacking.

The interest and criticism of Drs. Stratton and Nutting, as well as the assistance of Mr. Pidgeon in observing and computing, and the more than ordinary service of Mr. Sperling in not only preparing but also in improving the design of the cadmium lamps, have all been of service in this investigation.

It has not been deemed advisable to encumber this paper with a mass of details. However, such information is recorded and preserved at this laboratory and will gladly be furnished to anyone desiring to examine the work more critically.

WASHINGTON, November 23, 1911.

ON THE DEDUCTION OF WIEN'S DISPLACEMENT LAW

By E. Buckingham

1. Although Wien's displacement law may be regarded as quite well established by experiment, its great importance seems to justify attempts to improve or simplify the reasoning by which it may be deduced a priori as a consequence of the general principles of thermodynamics and the electromagnetic theory of radiation. Any such deduction must, in substance, contain the following four elements:

(*a*) The treatment, by Doppler's principle, of the change of wave length produced when diffuse radiation is compressed or expanded within a perfectly reflecting shell, i. e., adiabatically.

(*b*) The evaluation, by means of the principle of the conservation of energy, of the change of the volume density of the radiant energy which occurs during the adiabatic change of volume and accompanies the change of wave length. This step involves the use of the value of the pressure of diffuse radiation on a bounding surface, deduced from the electromagnetic theory and confirmed by experiment.

(*c*) The demonstration, by means of the second law of thermodynamics, that black radiation remains black when its density and temperature are changed by adiabatic change of volume.

(*d*) The use of the Stefan-Boltzmann law to correllate the results obtained by the steps (*a*), (*b*), and (*c*) so that the displacement law shall appear as a necessary consequence of those results.

These parts of the deduction need not be kept entirely separate nor do they necessarily occur in the order given above, which is that followed in this paper, but they must be present in some form or other. In the deductions I have read, the treatment of the change of wave length seems somewhat difficult or obscure and I have attempted to simplify this part of the subject and make it easier to grasp. The remainder of the demonstration contains little that is at all novel, but is given for the sake of presenting a connected whole, comprehensible to those who are not already familiar with the subject. The treatment is elementary and relates only to radiation in vacuo.

2. Let ds be an infinitesimal plane element of surface at a point within a field of radiation. A certain amount of radiant energy passes through ds from the negative to the positive side, in unit time, in various directions. Let us consider only those directions comprised within a cone of the infinitesimal solid angle dw described about the positive normal to ds. The amount of energy of wave lengths between λ and $\lambda+d\lambda$, which passes through ds from the negative to the positive side in one second, in directions close enough to the normal to lie within the cone, may be expressed by

$$R_\lambda d\lambda \cdot ds \cdot dw$$

The quantity $R_\lambda d\lambda$ may be called the strength of the radiation and R_λ the "radiant vector" at the given point, in the given direction of the positive normal to ds, and for the wave length λ. If the value of R_λ is given for all values of λ, for all directions, and at every point within a given region, the radiation within that region is, for our purposes, completely specified, since questions of phase and state of polarization will not enter into our reasoning.

By speaking of "the energy of wave lengths between λ and $\lambda+d\lambda$" we make a somewhat violent though familiar assumption, namely, that no matter what may be the nature of the pulses which constitute radiation, since our spectral apparatus enables us to analyze radiation into series of wave trains of assignable period, the radiation *before analysis* may be treated as the sum of these wave trains coexisting independently. However obvious the

truth of this assumption may appear from a purely mathematical standpoint, it is well to recognize that *physically* it is to be justified by the agreement with experiment of conclusions drawn from it. It is in fact thus justified and we shall make a rather full use of this principle.

3. Let us consider a closed evacuated shell, the walls of which reflect perfectly, but at least somewhat irregularly. Let a beam of approximately monochromatic radiation of wave lengths between λ and $\lambda + d\lambda$ be admitted to the inclosure through a hole, which is then closed by a cover similar in its properties to the rest of the walls. After a short time the radiation within the shell becomes perfectly diffuse, for the directed quality of the original beam is soon obliterated by the successive irregular reflexions, so that thereafter the value of R_λ is the same at all points and in all directions. These reflexions, however, do not change the period of the radiation, since there is no absorption and reemission, but only pure reflexion. The volume density of the energy is now

$$\rho_\lambda d\lambda = 4\pi \frac{R_\lambda d\lambda}{c} \tag{1}$$

where c is the velocity of light; and the whole amount of energy within the shell is $v\rho_\lambda d\lambda$, if v is the volume of the shell.

Let M be a small plane piece of the shell wall of area s, and let M be given a normal velocity βc outward, β being infinitesimal. This motion will disturb the perfect diffuseness of R_λ by an amount which we shall show to be negligible, but at present we shall assume that R_λ remains diffuse.

The reflexion from M also causes a change of period which we must now proceed to evaluate. If a wave train of period T strikes M at an angle of incidence φ, it is easily seen that the period T' of arrival of the waves at a given point of M is

$$T' = \frac{T}{1 - \beta \cos \varphi}$$

The period at a point of the moving surface is therefore greater than at a fixed point in space in the ratio

$$r_a = 1 + \beta \cos \varphi \tag{2}$$

terms of higher orders in β being negligible. A disturbance starting with the period T' at a point of the moving surface and propagated at an angle ψ with the normal has, upon arrival at a fixed point in space, the period

$$T'' = T'(1 + \beta \cos \psi)$$

so that the effect of departure is to increase the period in the ratio

$$r_d = 1 + \beta \cos \psi \tag{3}$$

Our problem is to find the total effect on the original period T, of all the arrivals and departures, at all possible angles φ and ψ from 0 to $\pi/2$, during a long time t; and to do this we must find the number of times that each of these effects is produced on every wave; i. e., every element of the radiation existing within the shell.

4. We base our reasoning on the consideration that in a very long time—though not in a short one—every element of the energy within the shell must undergo reflexion from M at the angles (φ, ψ) just as many times as every other element. The number of times that any particular effect of reflexion at M is produced on each element of the energy is therefore the ratio of the total amount of energy thus affected in the time t to the total amount present within the shell. The changes of period in the ratios r_a and r_d, caused by arrival and departure, occur alternately; and in finding the total effect of a number of successive arrivals and departures we have evidently to evaluate a product of the form

$$r_a r_d \cdot r'_a r'_d \cdot r''_a r''_d, \text{ etc.}$$

But since multiplication is commutative, we shall get the same result if we pursue the more convenient method of treating all the arrivals by themselves, then all the departures by themselves, and finally multiplying the two resulting ratios together to get the combined effect of both arrivals and departures.

We start, then, with the arrivals. In any time t, the amount of energy which strikes M at angles between φ and $\varphi + d\varphi$ is

$$t \cdot R_\lambda d\lambda \text{ s} \cos \varphi \cdot 2\pi \sin \varphi \, d\varphi$$

The total amount present within the shell to be affected is, by equation (1),

$$v \cdot \frac{4\pi R_\lambda d\lambda}{c}.$$

Hence the number of times n that each element of the energy arrives at M at angles between φ and $\varphi + d\varphi$ within the long time t is

$$n = \frac{t \cdot R_\lambda d\lambda \; s \; \cos\varphi \cdot 2\pi \sin\varphi \; d\varphi}{v \cdot \frac{4\pi R_\lambda d\lambda}{c}} = \frac{cts}{2v} \cos\varphi \sin\varphi \, d\varphi \tag{4}$$

By equation (2) the effect of all these n arrivals is to increase the period in the ratio

$$r_a^n = (1 + \beta \cos\varphi)^n = 1 + n\beta \cos\varphi \tag{5}$$

the remaining terms being of higher orders in β. Inserting the value of n from (4) we have

$$r_a^n = 1 + \frac{\beta cts}{2v} \cos^2\varphi \sin\varphi \, d\varphi$$

and since βcts is the infinitesimal increase of volume, Δv, which occurs within the time t as a result of the motion of M, this last equation may be written in the form

$$r_a^n = 1 + \frac{1}{2} \frac{\Delta v}{v} \cos^2\varphi \sin\varphi \, d\varphi \tag{6}$$

So far we have considered only the directions between φ and $\varphi + d\varphi$; but meanwhile the given element of energy has also been arriving a large number of times at every other possible angle between 0 and $\pi/2$, and equation (6), with the appropriate value of φ, is applicable to every such angle. The total effect of all the arrivals at all possible angles will therefore be to change the period in the ratio given by the product of all the expressions of the form (6) for all values of φ; and dropping terms of higher order in β, the value of this product will be

$$1+\frac{1}{2}\frac{\Delta v}{v}\int_0^{\frac{\pi}{2}}\cos^2\varphi\,\sin\varphi\,d\varphi=1+\frac{1}{6}\frac{\Delta v}{v} \tag{7}$$

If we go on to treat the effects of the departures, the reasoning will be found to be the same all the way through, with the mere substitution of ψ for φ, and the total effect will be found as before to be to increase the period in the ratio $1+\frac{1}{6}\frac{\Delta v}{v}$. Hence the combined result of the two sets of effects, which have in reality been occurring alternately, is to increase the period of every element of the radiation in the ratio

$$\frac{T+\Delta T}{T}=\left(1+\frac{1}{6}\frac{\Delta v}{v}\right)^2=1+\frac{1}{3}\frac{\Delta v}{v} \tag{8}$$

Replacing the period by the wave length we therefore have

$$\frac{\Delta\lambda}{\lambda}=\frac{1}{3}\frac{\Delta v}{v} \tag{9}$$

Since this result is independent of the original value of the wave-length λ, it is valid for any value. Hence $\lambda+d\lambda$, and therefore $d\lambda$, the interval within which the wave-lengths are included, is changed in the same ratio as λ.

The result is also valid for any element of the surface of the shell which is small enough to be treated as plane, and for motion either in or out, so that equation (9) may be integrated into the form

$$\lambda=\text{const.}\times\sqrt[3]{v} \tag{10}$$

The meaning of equation (10), reduced to its simplest terms, is as follows: If the shell changes its volume while retaining its shape, the dimensions of the waves change in the same ratio as those of the shell. The whole system of waves and shell remains geometrically similar to itself, the number of waves present being unchanged.

5. In the foregoing reasoning we have treated the radiation as perfectly diffuse, for the cancellation of R_λ from the numerator

and denominator of equation (4) and the corresponding equation for the case of departure, involved the assumption that R_λ was the same at all points and in all directions. This assumption is not exact, for upon reflection from a moving mirror the angle of departure ψ' differs from the angle ψ at which the ray would leave a fixed mirror by a quantity of the order of magnitude of the ratio of the velocity of the mirror to the velocity of light. The result of this is that at any point within our shell the radiant vector in directions away from the moving piece M has not exactly the mean value R_λ, but a value $R_\lambda' = R_\lambda(1+\epsilon)$, where ϵ is an infinitesimal of the same order as β. This departure from perfect diffuseness is not cumulative, but remains of the same order of magnitude whatever the lapse of time, for the disturbing effect of reflection from M is continually being damped out by the diffusing effect of the irregular reflections from the stationary walls.

If we now review our reasoning, we find that R_λ appears only in the expressions for the total energy within the shell and for the total energy which strikes or leaves M at a given angle within the time t. If R_λ represents the average value which satisfies the equation

$$\int \rho_\lambda d\lambda \cdot dv = 4\pi \frac{R_\lambda d\lambda}{c} \cdot v$$

the numerator of equation (4) ought to contain not R_λ but $R_\lambda(1+\eta)$, where η is an infinitesimal of the order βq, q being the ratio of the area s to the whole area of the shell walls. The error in n caused by the omission of this factor $(1+\eta)$ is infinitesimal and negligible. The changes caused in equations (5) to (10) by using the exact value instead of the mean value of R_λ would all be of a lower order of magnitude than the terms which have been retained. Hence the error incurred by our treating the radiation as completely diffuse is infinitesimal, and the result expressed in equations (9) and (10) remains valid.

One further point may be worth notice. If "the long time t" appears to the reader to be possibly not long enough to give all the finite amount of energy within the shell an equal opportunity of being reflected from M in every one of the infinite number of conceivable ways, there is no objection to his making it longer, in

other words, infinite. If this is done, we may still make Δv or $ct\beta s$ infinitesimal, as we want it to be, by making βs of the order $\frac{1}{t^2}$; and this may be accomplished either by making $\beta = \frac{\text{const}}{t^2}$ with s finite, or by making both β and s of the order $\frac{1}{t}$. There is nothing to prevent our adopting either course. The use of the more concrete expression a "long" time instead of an "infinite" time did not, therefore, involve an error in the reasoning, while it obviated the necessity of interrupting the argument at an inconvenient point.

6. We have next to consider the change of the energy of the radiation which accompanies its change of period, and we assume that diffuse radiation exerts a pressure equal to one-third of its density on the walls of a containing envelope. This proposition may be deduced in several ways from Maxwell's theory of the electromagnetic field, which has been confirmed in so many respects that we need not regard it as doubtful but accept its consequences without further discussion.

During an expansion Δv, the work given out is then expressed by $\frac{1}{3}\rho_\lambda d\lambda \cdot \Delta v$; and since the expansion of radiation within a perfect reflector is adiabatic, this work is equal to the simultaneous decrease of the energy within the shell, and we have

$$\frac{1}{3}\rho_\lambda d\lambda \cdot \Delta v = -\Delta(\rho_\lambda d\lambda \cdot v)$$

whence upon developing, rearranging, and dividing by $\rho_\lambda d\lambda \cdot v$, we have

$$\frac{4}{3}\frac{\Delta v}{v} + \frac{\Delta d\lambda}{d\lambda} + \frac{\Delta \rho_\lambda}{\rho_\lambda} = 0$$

But since $d\lambda$ changes in the same ratio as λ, $\frac{\Delta d\lambda}{d\lambda} = \frac{\Delta \lambda}{\lambda}$, and if we eliminate v by equation (9) we have

$$5\frac{\Delta \lambda}{\lambda} + \frac{\Delta \rho_\lambda}{\rho_\lambda} = 0 \tag{11}$$

or

$$\rho_\lambda \lambda^5 = \text{const.} \tag{12}$$

The meaning of this result is that if approximately monochromatic diffuse radiation of strength $R_\lambda d\lambda$ and density $\rho_\lambda d\lambda$ is compressed or expanded adiabatically and infinitely slowly, the quantity ρ_λ changes as the inverse 5th power of the wave length, λ itself being subject to equation (10). If the strip A (Fig. 1), of width $d\lambda$ and height ρ_λ, represents the original energy density, the strip B which represents the energy density after compression, has its height increased in the ratio of the 5th power of the ratio of decrease of the width or of mean wave length. Its area is therefore proportional to the inverse 4th power of the wave length.

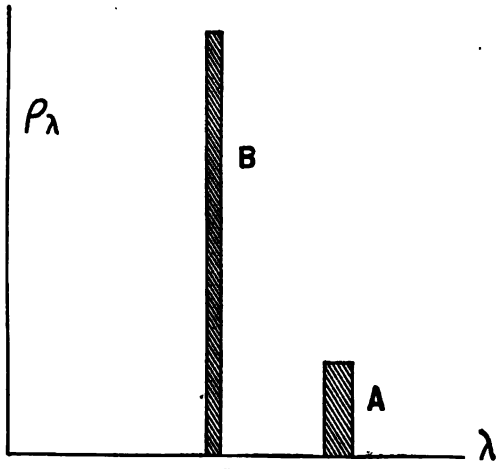

Fig. 1

This change of density and the accompanying change of $R_\lambda d\lambda$ do not interfere with the validity of the reasoning by which we found the value of n in equation (4). For since the motion is infinitely slow, the values of $R_\lambda d\lambda$ which have been canceled from the numerator and denominator may be treated as always equal, within an infinitesimal amount, in spite of the fact that they are not constant in time.

The width $d\lambda$ may be as small as we please, so that the change of ρ_λ may be assigned definitely to the particular wave length λ. There is no occasion for the formation of any new waves; the change in energy is a change in the amplitude of waves already present, which occurs in connection with the change of period

upon reflection from the moving surface. In a *short* time, reflection from the moving mirror would introduce inhomogeneity into the radiation, which would not have time to be all equally affected by reflection at M; but in a long time, the radiation again becomes homogeneous to the same degree as at first, and equation (12) is satisfied for each wave length.

7. Hitherto we have considered only a small interval of wave lengths, but suppose that the strip in question is merely a part of a continuous distribution of diffuse radiation which may be represented by a curve such as is shown in figure 2. If we take full advantage of the principle of the independence of the separate

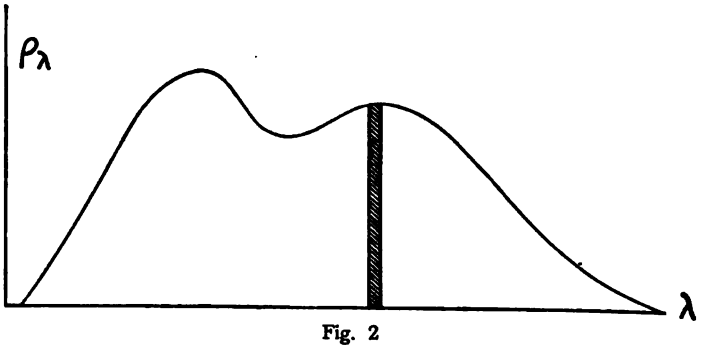

Fig. 2

elements composing the whole spectrum, we must admit that the reasoning given above for an isolated strip is applicable without change to every wave length of the spectrum, no matter what may be the form of the energy curve $\rho_\lambda = f(\lambda)$. We then have the proposition that when any completely diffuse radiation is compressed infinitely slowly within a perfectly reflecting inclosure, the energy curve is changed in such a way that the abscissa of every point is multiplied by some fraction f while the ordinate is multiplied by f^{-5}, so that the area under the curve, or the integral density of the energy present, is multiplied by f^{-4}. This is true for any continuous or discontinuous spectral distribution and not merely for black radiation.

8. The remainder of the deduction contains nothing new, but may be given for the sake of completeness. Let the shell be

filled with black radiation of temperature θ and density ρ, by first covering a hole in the shell with an ordinary body of temperature θ, and then, after equilibrium has been established, closing the hole with a cover which has the same reflecting properties as the rest of the shell wall. Let the volume of the shell, which may, if we prefer, have the form of a cylinder closed by a piston, be decreased a finite amount by an infinitely slow compression. The fact that this requires an infinite time need not concern us. The density of the energy is increased on account both of the work done and of the decrease of volume of the energy already present. The spectral distribution also changes in the manner already given.

At the end of the compression we introduce into the shell a particle of ordinary matter so small as to be of negligible thermal capacity. If the spectral distribution after compression was not that of black radiation of the same integral density, absorption and reemission by the particle cause a redistribution and a "blackening" of the radiation without change of density. This establishment of stable equilibrium by redistribution of the energy among the different periods is spontaneous and therefore irreversible.

We now expand to the original volume. The radiation remains black on account of the presence of the particle, and the work given out is the same as that put in during the compression, because the pressure depends only on the total density of the energy present and not on its spectral distribution. At the end of the cycle, which may be completed by removing the particle, we have therefore reestablished the original state exactly. No heat has been added to or taken from any outside body, the work done has been regained, and no changes remain. The cycle is therefore reversible and can not have included any irreversible element. Hence the introduction of the particle after compression did not cause any change in the spectral distribution of the energy which must therefore already have been that of radiation from a black body. Hence we conclude that during infinitely slow adiabatic change of density, radiation which was initially black remains black.

9. We may now apply equation (12) to an adiabatic change of the volume and density of black radiation. The integral density changes from that needed for equilibrium with an absorbing shell at the absolute temperature θ_1 to that needed for some other temperature θ_2. The abscissa λ_1 of any point on the original energy curve is changed to a new value λ_2, such that

$$\frac{\lambda_2}{\lambda_1}=x=\sqrt[3]{\frac{v_2}{v_1}} \tag{13}$$

At the same time the ordinate of the point changes so that

$$\frac{(\rho_\lambda)_2}{(\rho_\lambda)_1}=\frac{1}{x^5} \tag{14}$$

and the area under the curve, which represents the integral density ρ, changes so that

$$\frac{\rho_2}{\rho_1}=\frac{1}{x^4} \tag{15}$$

But we know by the Stefan-Boltzmann law that for diffuse black radiation at θ_1 and θ_2

$$\frac{\rho_2}{\rho_1}=\left(\frac{\theta_2}{\theta_1}\right)^4 \tag{16}$$

whence it follows that

$$x=\frac{\theta_1}{\theta_2} \tag{17}$$

From equations (13) and (17) we thus obtain the relation

$$\lambda_1\theta_1=\lambda_2\theta_2 \tag{18}$$

and the displacement law contained in equations (12) and (18) may be stated as follows: Given the spectral energy curve of black radiation at any temperature θ_1, to construct the curve for any other temperature θ_2, multiply the abscissa of each point by $\frac{\theta_1}{\theta_2}$ and the coordinate by $\left(\frac{\theta_2}{\theta_1}\right)^5$. "Corresponding" points on the two curves have the same value of $\lambda\theta$.

From this we may easily deduce the more familiar special forms of the displacement law

$$\lambda_{max}\theta = \text{const},$$

$$\rho_{max}\lambda^5{}_{max} = \text{const},$$

and

$$\rho_{max} = \text{const} \times \theta^5,$$

as well as the fact that if the displacement and Stefan-Boltzmann laws are to be satisfied, the complete equation must have the general form

$$\rho_{\lambda,\theta} = C\lambda^{-5}F(\lambda\theta)$$

The only novelties in the above deduction, if there are any at all, occur in sections 4 to 7; but it seems to me that greater clearness has been attained without any real sacrifice of rigor and with only very elementary reasoning.

WASHINGTON, February 28, 1912.

THE FOUR-TERMINAL CONDUCTOR AND THE THOMSON BRIDGE

By Frank Wenner

CONTENTS

	Page
I. Introduction	559
II. The Four-terminal Conductor:	
1. Definition of resistance	560
2. Definition of inductance	562
3. The reciprocal theorem	563
4. Relation between resistances	569
5. The design of standards of resistance	571
6. Standards for use with alternating current	577
III. The Thomson Bridge:	
1. Theory with linear conductors	580
2. The correction terms	581
3. Adjustments making correction terms small	584
4. An illustrative problem	586
5. Sensitivity with D'Arsonval galvanometer	589
6. The multiple bridge	595
7. Theory with nonlinear conductors	596
8. Theory using alternating current	598
IV. Summary	609

I. INTRODUCTION

The increased demand for the accurate measurement of power has necessitated the construction of low-resistance standards capable of carrying large currents. In such standards the surfaces of the contacts through which the current enters and leaves must be large, otherwise the heating will be excessive. Using different current leads or making the connections under different conditions necessarily introduces some variations in the current distribution in the terminals. On account of the relative proportions of the standard these changes are often not limited to the

terminals, but extend to parts of the conductor between the potential terminals. As a result the resistance often depends, to some extent, on the manner in which the current leads are attached. Yet the standard, to be reliable, must have a definite value which must be determined in terms of the values of other standards.

Where the resistances are to be used in alternating-current measurements, it is also necessary to know the inductance, or to know that at the frequency used the phase angle between the current and the drop in potential is so small that it may be considered zero.

Various methods have been proposed or used in the comparison of the resistances of four-terminal conductors. One of these, known as the Thomson bridge method, has been in use a half century. In many cases where it is being used, moreover, sufficient precautions are not taken to get the accuracy easily attainable. However, this method has not come into use in proportion to its merits.

In considering the four-terminal conductor it is the purpose of this paper to point out the conditions which must necessarily be fulfilled in order that the resistance be definite or both the resistance and inductance be definite and to discuss some of the points to be observed in the design of low-resistance standards which are to carry large currents, especially if the current is alternating. In connection with the Thomson bridge we shall consider first the theory where the four-terminal conductors are linear, then discuss some of the ways in which the measurements have been made in the past, describe the way in which they are carried out at the Bureau of Standards, and show that the method being used is equally applicable where the four-terminal conductors are not linear, and finally consider adjustments which should be made when using alternating current.

II. THE FOUR-TERMINAL CONDUCTOR

1. DEFINITION OF RESISTANCE

The generalized four-terminal conductor is a mass of conducting material of any size or shape and has four limited portions of the

surface arbitrarily selected and adapted for making electrical connection to other conductors. In what follows we shall refer to these four limited portions of the surface (see Fig. 1) as terminals 1, 2, 3, and 4; that is, we shall limit the meaning of the word terminal to that part of the surfaces of the conductors adapted for making electrical connections. If a current enters and leaves the conductor through any two of the terminals, there is in general a difference in potential between the other two. The ratio of this difference in potential to the current may be considered one of the resistances of the conductor.

We shall define the ratio of the drop in potential from 1 to 2 to the current entering at 3 and leaving at 4 as the resistance 1-2/3-4, likewise the ratio of the drop in potential from 3 to 1 to the current entering at 4 and leaving at 2 as the resistance 3-1/4-2, etc.; in all, 24 resistances. We also have the resistances 3-2/3-4, 2-3/2-3, 1-2/4-2, etc. In these cases, however, only two or three of the terminals are used, so these resistances will not be considered in this paper, except in so far as may be necessary for a complete understanding of the case where the four terminals are used. These various resistances and the relations between them have been considered fully in a recent paper by Searle.[1]

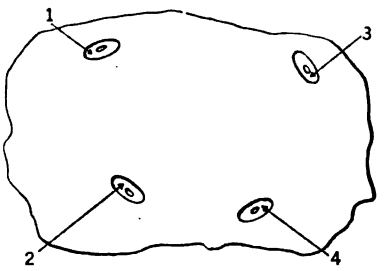

Fig. 1

With the current entering at 1 and leaving at 4, 2 may be either at a higher or at a lower potential than 3. The resistance 2-3/1-4 may therefor be either positive or negative. As the notion of sign is foreign to our usual conception of resistance, we may, if we wish, consider it as associated with the drop in potential and with the direction of the current.

It will readily be seen that certain of the resistances are equal or have the same magnitude. For example

$$\begin{aligned} 3\text{-}1/4\text{-}2 &= -1\text{-}3/4\text{-}2 = 1\text{-}3/2\text{-}4 = -3\text{-}1/2\text{-}4 \\ 1\text{-}2/3\text{-}4 &= -2\text{-}1/3\text{-}4 = 2\text{-}1/4\text{-}3 = -1\text{-}2/4\text{-}3, \text{ etc.} \end{aligned} \tag{1}$$

[1] Electrician, 66, p. 1002; 1911.

Therefore, instead of 24 different values there can not be more than 6. We shall see later that there are other relations between these six values, so that only two are really independent.

Obviously, if any one of these resistances is to have a definite value, the ratio of the drop in potential between two of the terminals to the current entering and leaving by the other two must be independent of the part of each of the four terminals used in making electrical connections to the current and potential leads; that is, all parts of the terminals to which the potential leads are connected must be at the same potential and this potential must not be affected by the current density in the other two terminals so long as the total current is constant. From relations which will be shown later it follows that the only condition necessary for all the resistances to have a definite value is that there be no difference in potential between the parts of any one of the terminals, no matter what current may be entering and leaving the conductor through any two of the remaining terminals.

2. DEFINITION OF INDUCTANCE

If the current is changing, there is, in general, an additional difference in potential between the potential terminals. The ratio of this additional difference in potential to the rate of change of the current is the inductance. However, in using the four-terminal conductor, leads must be connected to each of the four terminals and there is no way by which to distinguish between the emf developed (by mutual inductance) in the two potential leads which do not carry a current and the emf which is developed between the terminals to which they are connected. There is, therefore, no reason for separating the emf developed between the terminals from the total emf in the circuit composed of the two leads and the part of the conductor between the two terminals to which they are connected. We may, therefore, define the inductance of a four-terminal conductor as the ratio of the emf induced in a circuit composed of the conductor and one pair of leads to the rate of change of the current in a circuit composed of the conductor and the other pair of leads. It is to be understood

that each of these circuits is closed or practically closed. Obviously, the number of inductances is the same as the number of resistances and can be designated in the same way.

If any one of these inductances is to have a definite value, the terminals must either be brought out in such a way that the mutual inductance between the two pairs of leads may be made negligible or the parts of the leads which contribute to the mutual inductance between the two circuits must always be arranged in the same definite relative positions.

3. THE RECIPROCAL THEOREM

Before proceeding to the consideration of the less obvious relations between the resistances and inductances obtained by using the four terminals in different combinations, it will be necessary to call attention to a general theorem. This theorem may be stated as follows: In any conductor or system of conductors having four terminals 1, 2, 3, and 4 selected in any way, the drop in potential from 1 to 2 caused by a current entering at 3 and leaving at 4 is equal to the drop in potential from 3 to 4 caused by an equal current entering at 1 and leaving at 2.

For a network of linear conductors this theorem was given by Kirchhoff[2] in the same paper in which he gave the laws in regard to the sum of the currents to a point and the emf and current in a colsed circuit.

For an isotropic and homogeneous nonlinear conductor this theorem was developed theoretically by Helmholtz.[3] He also considered the case of a conductor having two parts of different conductivities and tested the relation experimentally on a carbon cylinder 3.5 inches long and 2 inches in diameter. In this work electrical connections were made by means of four small quantities of mercury held in place by paper rings, and the currents compared by reading the deflections of a galvanometer. Rosen[4]

[2] Pogg. Ann., 72; 1847. Kirchhoff, Ges. Abh., p. 32. See also Maxwell Elec. and Mag., 1, p. 371, and Gray Abs. Meas., 1, p. 160.

[3] Pogg. Ann., 89, p. 211; 1853. Helmholtz, Wiss. Abh., 1, p. 496.

[4] Öfvers. af k. Vetensk. Akadem. Vorhandl., p. 197; 1887.

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nals to the current [illegible] the theorem to the case where the current is independent of t[illegible] necessary to consider the whole matter electrical cor[illegible] for the simple case (Case I, below) where the con-all parts of [illegible] and the current constant the proof which will must h[illegible] isotropic and developed independently, may be considered affec[illegible] though the same as that given by Rosen, Heaviside, and as [illegible] In considering this matter a vector notation will Bromwich. be used in which all vectors are designated by bold-faced type and scalar products indicated by a dot (·) between the vectors.

According to the theorem of Gauss, sometimes referred as the divergence theorem [5]

$$\int \boldsymbol{F}_n \delta s = \int \nabla \boldsymbol{F}\, \delta v \tag{2}$$

where $\boldsymbol{F}$ is a finite, single valued, and continuous vector function of space, δv an element of volume, δs an element of surface and $\boldsymbol{F}_n$ is the component of $\boldsymbol{F}$ normal to the surface. Here the surface integral is to be extended over a closed surface and the volume integral extended through the volume inclosed by the surface.

Consider a four-terminal conductor so designed as to have definite resistances [6] and let the surfaces over which we are to integrate (see Fig. 2) include all of the conductor but coincide with the terminals 1, 2, 3, and 4.

Let ψ be the potential at any point above that at 3 caused by a current I entering at 1 and leaving at 4,

φ be the potential at any point above that at 4 caused by a current C entering at 2 and leaving at 3,

i be the density at any point of the current I,

c be the density at any point of the current C.

---

[5] For the derivation of this equation and equations similar to those which follow, see Abraham und Föppl Theorie der Electrizitite, 1, p. 54.

[6] For explanation of definite resistance, see p. 562.

Now if we substitute φi for F [7] we have

$$\int(\varphi i)_n \delta s = \int \nabla(\varphi i)\delta v \tag{3}$$

If we substitute ψc for F we have

$$\int(\psi c)_n \delta s = \int \nabla(\psi c)\delta v. \tag{4}$$

Carrying out the indicated differentiation of the right-hand members of these equations and subtracting gives

$$\int(\psi c)_n \delta s - \int(\varphi i)_n \delta s = \int(\psi \nabla c - \varphi \nabla i)\delta v + \int(c\nabla\psi - i\nabla\varphi)\delta v. \tag{5}$$

In regard to the surface integrals, the current density c is zero everywhere except on 2 and 3 while ψ is zero on 3 and constant on 2. The integration of c_n over the surface of the terminal 2 gives the total current C or the value of the first integral of equation (5) is $-\psi_2 C$. Likewise the value of the second integral is $\varphi_1 I$ or

$$\int(\psi c)_n \delta s - \int(\phi i)_n \delta s = \varphi_1 I - \psi_2 C. \tag{6}$$

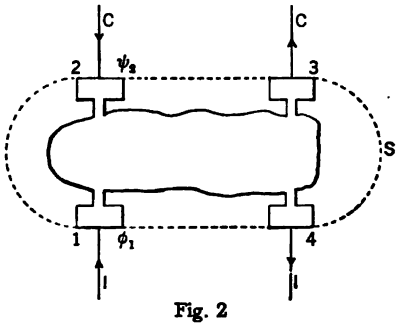

Fig. 2

The value of the right-hand member of equation (5) will depend upon a number of conditions or set of conditions, of which we will consider four special cases. In each case we shall consider that the resistance is definite, and that the conductor is free from internal sources of current.

Case I: Conductor isotropic and current constant.

If ρ represents the volume resistivity,

$$\begin{aligned} &-\nabla\varphi = c\rho \text{ and } -\nabla\psi = i\rho \\ &\therefore\ c\nabla\psi - i\nabla\varphi = 0 \end{aligned} \tag{7}$$

for every point in the conductor. The conductor being free from sources of current, ∇i and ∇c are equal to zero for every point

[7] For the conditions which the functions F, ϕ, and ϕ must fulfill, see Peirce, Newtonian Potential Function, p. 93.

within the conductor. Therefore the entire second member of the equation is zero and we have

$$\varphi_1 I - \psi_2 C = 0 \tag{8}$$

or the resistance 1-4/2-3 equals the resistance 2-3/1-4.

Case II: Conductor nonisotropic or crystalline, current constant.

The first volume integral is zero for the same reason as in Case I.

In each individual crystal or nonisotropic element of volume there is, in general, an angle between the potential gradient and the current density. There are, however, in most if not all such cases three directions in which if a potential gradient is impressed the current will be in the same direction. These directions may be considered the electrical axes of the element of the conductor and the current densities $\boldsymbol{i}$ and $\boldsymbol{c}$ may be resolved into components $\boldsymbol{u}i_u$, $\boldsymbol{v}i_v$, $\boldsymbol{w}i_w$, $\boldsymbol{u}c_u$, $\boldsymbol{v}c_v$, and $\boldsymbol{w}c_w$ parallel to these axes. Here $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are unit vectors parallel to the axes and i_u, i_v, i_w, c_u, c_v, c_w, the magnitudes of the components of the currents. If, then, we let ρ_u, ρ_v, and ρ_w be the resistivity in directions parallel to the axes, we have

$$\begin{aligned} \boldsymbol{c}\nabla\psi - \boldsymbol{i}\nabla\varphi = {} & \boldsymbol{u}\cdot\boldsymbol{v}(i_u c_v - i_v c_u)(\rho_v - \rho_u) \\ & + \boldsymbol{u}\cdot\boldsymbol{w}(i_u c_w - i_w c_v)(\rho_w - \rho_u) + \boldsymbol{v}\cdot\boldsymbol{w}(i_v c_w - i_w c_v)(\rho_w - \rho_v). \end{aligned} \tag{9}$$

If either (1) the axes are mutually perpendicular, (2) the resistivity is the same for current in the direction of all of the three axes, or (3) the resistivity is the same for current in the direction of two axes which are themselves perpendicular to the third axis, then

$$\boldsymbol{c}\Delta\psi - \boldsymbol{i}\Delta\varphi = 0 \tag{10}$$

for every point, the same as in an isotropic conductor, and the resistance 1-4/2-3 is equal to the resistance 2-3/1-4.

Here it is shown that certain conditions are necessary in order that the theorem may be true. It is known, however, from experiment that most, if not all, conductors fullfil at least one of these conditions. In the papers referred to above it is pointed out that in order for the theorem to be true it is necessary that if a current density i_x in the direction of the x axis requires a com-

ponent of the potential gradient e_y along the y axis then a current density i_y along the y axis must require a component of the potential gradient e_x along the x axis, and e_x/i_y must be equal to e_y/i_x. Also similar relations must exist between the other components. This relation between the current density and the components of the potential gradient in the element of volume is almost identically the relation with which we are concerned for the four-terminal conductor as a whole.

Case III: Conductor isotropic, current changing.

Here
$$c\Delta\psi = -c\cdot i\rho - c\cdot m_i\frac{\delta I}{\delta t}$$
and
$$i\Delta\varphi = -i\cdot c\rho - i\cdot m_c\frac{\delta C}{\delta t} \tag{11}$$

where m_i is the mutual inductance of the total circuit in which the current I flows upon a unit length of the conductor taken in a direction parallel to the induced emf. As the direction of the induced emf has no relation to the direction of either of the current densities i or c, m_i must be considered as a vector quantity. Likewise the m_c is the mutual inductance of the total circuit in which the current C flows upon a unit length of the conductor taken in a direction parallel to the emf induced by changes in the current C. Therefore

$$\int(c\Delta\psi - i\Delta\varphi)\delta v = \frac{\delta C}{\delta t}\int i\cdot m_c\delta v - \frac{\delta I}{\delta t}\int c\cdot m_i\delta v. \tag{12}$$

For the first integral of the second member we may take as the element of volume the part of the conductor included between two equipotential surfaces in one of the tubes of current constituting a part of the current I. If $\delta i'$ is the current in the tube, δa the area of the tube at the point in question, and δl the distance between equipotential surfaces then

$$i\delta a = \delta i' \quad \text{also } \delta v = \delta a\delta l$$

or
$$\frac{\delta C}{\delta t}\int i\cdot m_c\delta v = \frac{\delta C}{\delta t}\int\int m_c\cdot\delta i'\delta l = IM_c\frac{\delta C}{\delta t} \tag{13}$$

likewise
$$\frac{\delta I}{\delta t}\int c\cdot m_i \delta v = CM_i\frac{\delta I}{\delta t} \tag{14}$$

where M_i is the mutual inductance of the total circuit in which the current I flows upon the part of the circuit between terminals in which the current C flows. Also M_c is the mutual inductance of the total circuit in which the current C flows upon the part between the terminals of the circuit in which the current I flows.

From this it follows that

$$\varphi_1 I - \psi_2 C = IM_c\frac{\delta C}{\delta t} - CM_i\frac{\delta I}{\delta t} \tag{15}$$

If the connectors and leads are so arranged that the leads contribute nothing to the mutual inductance M between the two circuits in which the current I and C flow, and if the current distribution is the same with changing as with constant current, then $M_c = M_i = M$. Or, if the parts of the leads near the conductor contribute to the mutual inductance between the two circuits, the inclosing surface may be enlarged so as to include these parts of the leads, in which case $M_c = M_i = M$. If, further, both I and C are alternating currents of the same frequency and wave form, then $M_c = M_i = M$ even though the current distribution is affected by the frequency. Therefore,

since
$$I = I'\epsilon^{\iota pt} \text{ and } \frac{\delta I}{\delta t} = \iota p I'\epsilon^{\iota pt}$$

and since
$$C = C'\epsilon^{\iota pt} \text{ and } \frac{\delta C}{\delta t} = \iota p C'\epsilon^{\iota pt}$$

where I' and C' are proportional to the maximum values of the currents, $p = 2\pi$ times the frequency, $\iota = \sqrt{-1}$ and ϵ is the base of natural logarithm; we have

$$M\left(I\frac{\delta C}{\delta t} - C\frac{\delta I}{\delta t}\right) = 0 \tag{16}$$

and the impedence 1-4/2-3 is equal to the impedence 2-3/1-4. Therefore, to alternating currents of the same frequency, the resistances 1-4/2-3 and 2-3/1-4 are equal.

Case IV: Conductor nonisotropic, current changing.

Since we have here only a combination of Cases II and III, it follows that, with the limitations of both of these cases, the theorem is applicable here.

In showing that the theorem is applicable to such conductors and under such conditions as we may find in resistance measurements it has been necessary to assume a conductor free from sources of current (such as thermoelectromotive forces, the Hall, and similar effects) that there are at least three axes in each element of volume along which the current flows parallel to the potential gradient, and either that these three axes are mutually perpendicular, or the conductivities to current along each of the three axes are equal, or the conductivities to current along each of two axes are equal and these two axes are both perpendicular to the third. In order for the theorem to be applicable in case the current is alternating it has been shown that the total mutual inductance between the two circuits must be taken into consideration.

The effect of electrostatic capacity has not been considered, since, in the case of resistance standards designed to carry large currents, it is known to have no appreciable effect. Furthermore, in the case of resistance coils the effect of capacity is known to be the opposite to the effect of a self-inductance.

We shall make use of this theorem not only in the consideration of the relations between the different resistances and inductances of the four-terminal conductor but more particularly in showing the relations between the resistances and inductances of the Thomson bridge.

4. RELATION BETWEEN RESISTANCES

Making use of the reciprocal theorem we have the following relations between the six resistances considered above

$$2\text{-}3/1\text{-}4 = 1\text{-}4/2\text{-}3,\ 1\text{-}2/4\text{-}3 = 4\text{-}3/1\text{-}2, \text{ and } 1\text{-}3/2\text{-}4 = 2\text{-}4/1\text{-}3 \qquad (17)$$

This shows that in case no two of the four leads are connected to the same terminal and considering only positive values there are three and only three different resistances.

In order to show the further relation between the three values we shall consider the ratio of the difference in potential between the different terminals to the current. If, for example, the current enters at 1 and leaves at 4, the differences in potential between 1 and 4, 2 and 4, 3 and 4, 1 and 2, 1 and 3, and 2 and 3, divided by the current are the resistances 1-4/1-4, 2-4/1-4, 3-4/1-4, 1-2/1-4, 1-3/1-4, and 2-3/1-4.

As the difference in potential between 2 and 3 is the difference in potential between 2 and 4 minus the difference in potential between 3 and 4, we have the following relation between the resistances:

$$2\text{-}3/1\text{-}4 = 2\text{-}4/1\text{-}4 - 3\text{-}4/1\text{-}4 \tag{18}$$

Likewise

$$1\text{-}3/2\text{-}4 = 1\text{-}4/2\text{-}4 - 3\text{-}4/2\text{-}4 \tag{19}$$

and

$$2\text{-}1/3\text{-}4 = 2\text{-}4/3\text{-}4 - 1\text{-}4/3\text{-}4 \tag{20}$$

Adding (19) and (20) gives

$$1\text{-}3/2\text{-}4 + 2\text{-}1/3\text{-}4 = 1\text{-}4/2\text{-}4 - 1\text{-}4/3\text{-}4 = 2\text{-}4/1\text{-}4 - 3\text{-}4/1\text{-}4 \tag{21}$$

since $3\text{-}4/2\text{-}4 = 2\text{-}4/3\text{-}4$, $1\text{-}4/2\text{-}4 = 2\text{-}4/1\text{-}4$, and $1\text{-}4/3\text{-}4 = 3\text{-}4/1\text{-}4$. Therefore, from (18) and (21) it follows that [8]

$$2\text{-}3/1\text{-}4 = 1\text{-}3/2\text{-}4 + 2\text{-}1/3\text{-}4 \tag{22}$$

Four terminal conductors which are to be used as resistance standards, ammeter shunts, etc., usually have two of the terminals arranged for connection to the current leads and two for connection to the potential leads. For convenience we may designate the potential terminals as 2 and 3, and the current terminals as 1 and 4, 2 being the potential terminal at the higher potential when the current enters at 1. The resistance with which we are most concerned, then, is 2-3/1-4 (or 1-4/2-3), and in what follows we shall refer to it as the direct resistance or simply as the resistance. The resistance 2-1/3-4 or 3-4/2-1 we shall refer to as the cross resistance, and the resistance 2-4/1-3 or 1-3/2-4 we shall refer to as the diagonal resistance. From equation (22) it follows that the resistance is equal to the cross resistance plus the diagonal resist-

[8] Searle: Electrician, 66, p. 1029; 1911. Wenner: Phys. Rev., 32, p. 616; 1911.

ance. Where the cross resistance is negligibly small in comparison with the direct resistance the diagonal and direct resistance are equal. In this case, if the conductor is connected into any system of conductors, the same effects are obtained as would be obtained with a conductor having definite branch points joined by a single-wire conductor. Such a conductor we shall refer to as linear to distinguish it from the more general case.

In most well-designed resistance standards the ratio of the cross to the direct resistance is small. For the smaller-size standards of the Reichsanstalt type constructed by Otto Wolff and having values of 0.001 ohm and above, this ratio is generally less than 10^{-6}. Such standards may therefore be considered as linear. Even where the value is 0.0001 ohm the ratio is only a few hundredths per cent.

In standards of low resistance, if the potential connectors are located at or near the side of the current connectors, the ratio is often of the order of 1 per cent and may readily be much larger.

While the inductance 2-3/1-4 equals the inductance 1-4/2-3, etc., we can not have the same simple relations between the direct, cross, and diagonal inductances as between the corresponding resistances, since in changing from one to the other the relative position of the leads and consequently their mutual inductance must be changed.

5. THE DESIGN OF STANDARDS OF RESISTANCE

The main feature to be considered here is a construction such as will make the resistances definite without making the mass unnecessarily large. The condition necessary can only be realized approximately, yet we shall see that we can make the direct resistance at least as definite as it can be measured. A matter of secondary importance is to make the cross resistance small in comparison with the direct resistance.

With terminals of a given size, shape, and relative location, the condition for a definite resistance is more nearly fulfilled the higher the conductivity of the material used in making the connectors, and with infinite conductivity the resistance would all be definite. For other reasons copper is generally used for

the connectors of the lower resistances. As little could be gained by the substituting of any other material for copper, even if the matter of cost were not to be considered, this suggestion can lead to few if any improvements in the design of such conductors.

In the more usual cases the condition is more nearly realized the smaller the terminals are made. With very small terminals the potential is necessarily the same all over each terminal when there is no current through it. Reducing the section of the connectors even for a very short length has the same effect as reducing the size of the terminals. The area of the potential terminals or section of the potential connectors may be made very small, since they are seldom required to carry much current. On the other hand, the area of the current terminals can not be made small nor the connectors of small section even for a short length on account of the large current which will sometimes be used. Nevertheless, many resistance standards, ammeter shunts, etc., could be materially improved by a judicious reduction of the section of the current connectors, even though doing so makes it necessary to dissipate a larger amount of power in the conductor. Ordinarily the potential connectors have a sufficiently small section so that little if anything would be gained by a further reduction of their section.

The effect of reducing the area of the terminals or the minimum section of the connectors is to make the current distribution throughout the major part of the conductor more nearly independent of the current distribution on the terminals, through which the current enters and leaves. Thus the same effect may be obtained by increasing the length [9] of the connectors. But this not only increases the amount of power which must be dissipated in the conductor for a given current, but also increases the mass and cost, especially in the case of conductors intended to carry very large currents.

Experiments made by the author show that for a terminal of a definite size and shape the current distribution is more nearly independent of the part of the terminal used if the connector is

[9] The effect of the relative dimensions of the connectors on the definiteness of the resistance is discussed fully by Searle: Electrician, 66, p. 1032, and 67, p. 12; 1911.

made of materials having widely different conductivities, using the material of lower conductivity between portions of the material of higher conductivity. The material of the lower conductivity, then, has very much the same effect as would be obtained by making the connector longer and using the material of the higher conductivity throughout. While in many cases improvements might be made either by reducing the area of the terminals or the section near the middle of the connectors, by increasing the length of the connectors, or by making the connectors of materials of different conductivities, yet for conductors which are to carry large currents better results will be obtained by working along somewhat different lines.

A matter which has considerable effect upon the definiteness of the resistances is the relative locations of the different terminals. This will readily be understood by reference to Fig. 3, which shows two heavy conductors used as terminal blocks connected by three nearly equal wire resistances. Near the junction of the wires and terminal blocks are brought out six terminals a, b, c, a', b', and c', the whole constituting a conductor with large current terminals and three pairs of potential terminals. The different potential connectors being of small section, the potential must necessarily be constant over the terminals when there is no current through them. Current entering and leaving through the current terminals may not divide equally between the three wire conductors, so the potential difference between a and a', b and b', and c and c' is not necessarily the same. If a and a' are used as the potential terminals and the major part of the current enters the conductor in the vicinity of A and leaves in the vicinity of A', then more than one-third of the current will flow through the conductor (1) and consequently the resistance will be more than one-third the mean value. If, on the other hand, the current enters mainly in the vicinity of C and leaves mainly in the vicinity of C', then the resistance will be less than one-third the mean value.

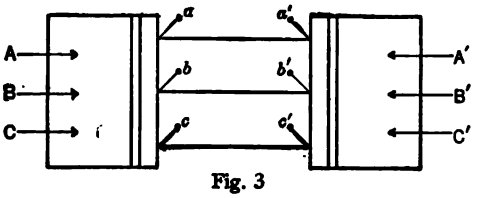

Fig. 3

Again, if b and b' are used as potential terminals, the resistance is very nearly the same whether the current enters in the vicinity of A or C and leaves in the vicinity of A' or C'. There is, therefore, a decided advantage in having the connectors to the potential terminal brought out symmetrically with respect to the parts of the current terminal.

It will be observed, however, that if the current enters and leaves in the vicinity of B and B' the resistance will be higher than if it enters and leaves in the vicinity of A or C and A' or C'. Obviously, if we can prevent the current from entering and leaving in the vicinity of B and B', the resistance of the conductor will be more nearly definite. This may be accomplished by cutting away the part of the terminals including the area around B and B'. The same result may be accomplished without an appreciable reduction of the area of the terminals by cutting away a portion of the connector, as shown in Fig. 4. In this case the current entering and leaving at B and B' necessarily distributes itself much as it would with half entering at A and leaving at A' and the other half entering at C and leaving at C'. That is, the proportional part of the current carried by the center conductor is about the same as it would be with the current entering at A or C and leaving at B or B'. With this arrangement the resistance is much more nearly definite than with the arrangement shown in Fig. 3, especially if a and a' or c and c' are to be used as potential terminals.

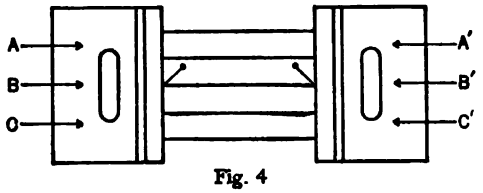

Fig. 4

It will easily be seen that the addition of two nearly equal linear conductors symmetrically located, as shown in Fig. 4, does not make the resistance of the system any more indefinite. Likewise, if instead of any number of pairs of equal linear conductors symmetrically located we have a single uniformly conducting sheet, the resistance is slightly more definite.

Where the resistance material is in the form of sheets a number of which are set in parallel the parts of the current terminals

should be arranged symmetrically with respect to the individual sheets and also in respect to the whole number of sheets. Also the potential connector should be brought out from a point about which the rest of the conductor, including the current terminals, is symmetrical in two planes at right angles to each other.

With potential terminals on the side the indefiniteness of the resistance of the conductors designed to carry large currents has been found to amount to as much as one-half per cent and might easily amount to more. In limiting the part of each current terminal to two not very small areas symmetrically located and using symmetrically located potential terminals the indefiniteness of the resistance of these same conductors became so small that it could scarcely be detected. The advantage of locating the terminals and connectors symmetrically is that the resistance is made much more definite without increasing either the mass or the resistance of the current connectors.

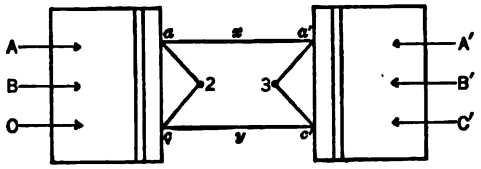

Fig. 5

Another way of attacking the problem is the use of branched potential connectors. The advantage of such an arrangement will readily be understood by reference to Fig. 5, which shows two heavy conductors connected by two wire conductors of practically equal resistance. The ends of the wire conductors are connected, as shown, by approximately equal resistances to the terminals 2 and 3 and the whole constitutes a four-terminal conductor. With this arrangement the difference in potential between the potential terminals 2 and 3 is very approximately the mean of the difference in potential between the points a and a' and c and c', regardless of the way the current divides between the two wires. Here, unless the resistance from A to B and from A' to B' is unnecessarily large, the part of the current carried by each wire changes by only a small amount (a few per cent at most) on using different parts of the current terminals. The indefiniteness of the resistance then must be exceedingly small, since even

a large change in the current distribution produces only a small change in the resistance.

If the resistance from a through 2 to c and from a' through 3 to c' is large in comparison with the resistance of the wires x and y, four more conductors of nearly equal resistances may be added without materially affecting the definiteness of the resistance, if two are placed near and symmetrically with respect to x and the other two are placed near and symmetrically with respect to y. This is necessarily the case, since any change in the current distribution, such as the increase in the part carried by x, will be accompanied by proportional increase of current in the two conductors near x and a decrease of the current in the two conductors near y in such a way that the fractional part of the total current carried by x as compared with that carried by y will be just about the same as it was before the change.

For conductors of the usual type where either wires or sheets are used the number of branches which the potential terminals may have is not limited to two but may be almost any number desired. If the resistance material is in the form of a uniform sheet, there may be two or more branches from each potential terminal connected near where the sheet joins the terminal block and spaced uniformly along the junction. Where there are a number of sheets in parallel a very definite value for the direct resistance would be obtained by having two or more branches from each terminal to each sheet. This, however, would make the construction very complicated and is an arrangement which need not be considered, since the use of good conducting material for the connectors, a fairly long connector, a symmetrical arrangement of the terminals, and branched potential connectors each supplement the other in making the resistance more definite. If in the design all these ideas are kept in mind, the resistance of conductors, even if they are to carry excessively large currents, can be made sufficiently definite for the most precise measurements without making the current connectors excessively heavy nor their resistance of the order of that of the four-terminal conductor. This can be accomplished, too, without carrying any one of these ideas to an extreme.

6. STANDARDS FOR USE WITH ALTERNATING CURRENT

If the conductor is to be used as a standard of resistance in alternating current measurements, the more important requirements in addition to those for use in direct-current measurements are that the resistance shall be the same or nearly the same for alternating as for direct current and that direct inductance be small and definite. If the standard is to carry a large current, these additional requirements make the design a much larger problem, since the usual arrangement of a number of sheets in parallel can not be used and without such an arrangement it is difficult to get a sufficient surface through which to dissipate the heat developed in the resistance by the current. The first requirement is more nearly fulfilled if the material is made in the form of a tube [10] having a thin wall and a considerable diameter. In this case, then, if the frequency is low, the current distribution and consequently the resistance will be nearly the same with alternating as with direct current. Further, if the current leads lie in the axis of the tube, the increase in resistance if sufficient to be considered, may be calculated to a fair accuracy.[11] To get the necessary surface to keep the rise in temperature from becoming excessive requires a tube of large diameter and of considerable length. In general, a change in current distribution results in a change in the inductance. However, symmetrical location of the terminals or connectors and the use of branched potential connectors each have about the same effect in making the inductance definite as they have in making the resistance definite. The inductance may be made very small by making the potential connectors of considerable length and so locating [12] them with respect to the circuit in which the current flows that the flux cutting across them, for any change in the current, is the same or nearly the same as that cutting across the resistance material between the points to which they are joined. If, then, the potential connectors extend some little distance from the main part of the

[10] Patterson and Rayner: Collected Researches Nat. Phys. Lab., 6, p. 90; 1910.
[11] Russell: Phil. Mag., 17, p. 524; 1909.
[12] Orlich: Zs. Instk., 29, p. 148; 1909. Campbell: Electrician, 61, p. 1000; 1909.

conductor and the terminals are close together, the potential leads can easily be arranged so that they contribute nothing to the mutual inductance between the current and potential circuits. In this case the inductance will be very small and definite and is as likely to have a negative as a positive sign.

The way in which these ideas may be embodied in the construction of resistance standards is illustrated by Fig. 6, which shows the design of a resistance standard for use with alternating current. The resistance is to be 0.001 ohm and the current capacity, with a forced circulation of oil to assist in dissipating the heat developed, is to be 1000 amperes. The main part of the conductor is to be of manganin in the form of a tube 15 cm in diameter, about 25 cm long, and .25 mm thick. The current connectors and the lugs for the current leads are to be of the best grade of cast copper. The potential connectors are each to have six or eight branches, and each branch is to have a resistance of 0.02 ohm. The branches of each potential connector are to be attached to the tube at points uniformly spaced along the junction between the connector and the tube. The potential connectors are to extend from the inside of the tube through an insulating bushing to the potential terminals, only one of which is shown in the figure. The other potential terminal and that part of the connectors on the outside of the tube are to be placed near the one shown, so that the flux linking this part of the potential circuit will be very small. This arrangement should make the inductance entirely negligible for measurements with low-frequency alternating currents. The large amount of heat developed is to be dissipated by a vigorous circulation of oil over both the inside and outside surface of the tube. The circulation through the inside is made possible by having four large holes through both of the current connections.

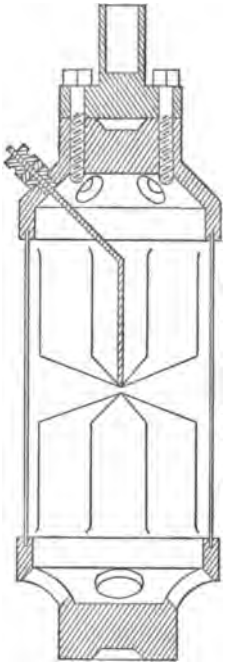

Fig. 6

The oil bath, not shown in the figure, is to be circular and of slightly larger diameter than the standard. If it is desired to cool the oil, a coil of copper tubing may be placed inside the bath and water circulated through it. The bottom of the oil tank is to be made of cast copper, which can now be obtained very pure, and amalgamated on a part of the upper surface. Also the lower current terminal is to be amalgamated and electrical connection between the two made by means of mercury. The lower current lead is attached to the upper side of the tank and extends downward along the axis of the tube. The lower current connector is to be drilled and tapped like the upper one, so that for smaller currents the standard may be used without the oil bath.

This design was developed mainly with the idea of getting a standard suitable for use with the Thomson bridge in the determination of the inductance and resistance of other standards to be used in alternating-current measurements and of such form that both its inductance and change in resistance with frequency can be calculated. It is not presumed that such a standard will be found satisfactory for general use, mainly on account of the strong magnetic field produced even at a considerable distance from the axis of symmetry.

In considering the design of conductors to be used as standards of resistance we have given but little consideration to the cross resistance and none to the cross inductance. However, most of the suggestions made above for increasing the definiteness of the resistance also increase the definiteness of the cross resistance and some of them tend to make it smaller.

Bringing the potential connectors out in such a way so as to make the direct inductance very small results in making the cross inductance of considerable magnitude. But we are very much concerned in keeping the one small and care very little about the value of the other. In some of the measurements, in which a part of the current enters or leaves through the potential terminals, the cross inductance and the cross resistance have an effect, which, if not taken into account, may lead to errors. This matter will be considered further in connection with the theory of the Thomson bridge. In most measurements, however, neither the cross induc-

tance nor the cross resistance can have any effect upon the results obtained.

III. THE THOMSON BRIDGE

1. THEORY WITH LINEAR CONDUCTORS

In 1862 Sir William Thomson described a method for the comparison of low resistances and pointed out many of its advantages.[13] The arrangement of conductors which he used has since come to be known as the Thomson bridge, though it is often referred to as the Kelvin double bridge. The method has been modified from time

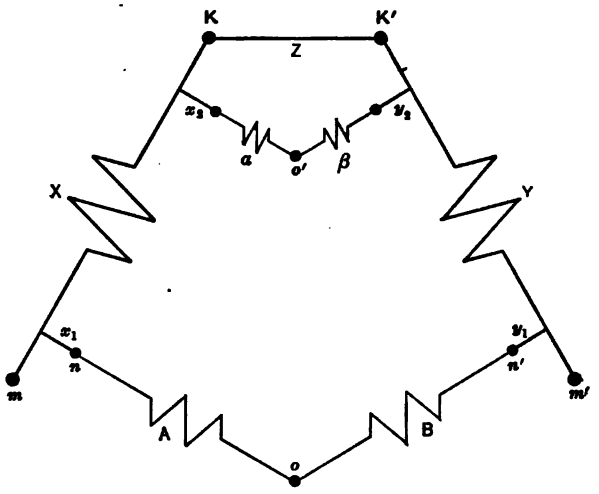

Fig. 7

to time to adapt it to different classes of work, to get a higher precision, or to reduce the number of separate measurements necessary for determining the relation between the resistance and the standard with which it is compared.

The arrangement of conductors for a Thomson bridge is shown in Fig. 7. Here the low resistances under comparison are designated by X and Y, the resistances for the main ratio by A and B, and the auxiliary ratio by α and β. Further, the connecting resistances from the points where the current and potential terminals divide to the terminals of the ratio sets are designated by x_1, x_2, y_1, and y_2. These letters are used both to designate the conductors and to

[13] Phil. Mag., 24, p. 149; 1862.

represent the values of these resistances. C represents the value of the resistance between X and Y from branch point to branch point, while Z designates the low resistance in parallel with the auxiliary ratio. If, then, the conductors are all linear and the bridge is balanced, we have the following relation between the resistances:

$$X = Y\frac{A+x_1}{B+y_1} + \frac{A+x_1}{B+y_1}C\frac{B+y_2}{\alpha+\beta+x_2+y_2} - C\frac{\alpha+x_2}{\alpha+\beta+x_2+y_2}. \quad (23)$$

This equation can easily be established from relations which exist in the simple bridge if we observe that the resistance C is divided in two parts having the same ratio as $\alpha+x_2$ to $\beta+y_2$ and that the first is in the same arm as X and the second in the same arm as Y. Before considering the matter further we will put this equation in a more convenient form. This may be done by substituting

$$\frac{A}{B}(1+a) \text{ for } \frac{A+x_1}{B+y_1}, \quad \frac{A}{B}(1+b) \text{ for } \frac{\alpha+x_2}{\beta+y_2}$$

and D for $\frac{C}{Y}\,\frac{\beta+y_2}{\alpha+\beta+x_2+y_2}$ which gives

$$X = Y\frac{A}{B}[1+a+D\,(a-b)]. \quad (24)$$

Equations (23) and (24) are the same except as to form and in what follows we shall refer to either indiscriminately as the fundamental equation of the Thomson bridge. It will be seen that here a is the difference, in proportional parts, between the main ratio, including the connecting resistances to the branch points of the conductors, and the ratio A/B; that $a-b$ is the difference, in proportional parts, between the main and auxiliary ratios; and D is a quantity usually small in comparison with unity.

2. THE CORRECTION TERMS

In the article referred to above it is pointed out that if the main and auxiliary ratios $(A+x_1)/(B+y_1)$ and $(\alpha+x_2)/(\beta+y_2)$ are made nearly equal and if the resistance C is small in comparison with the resistances X and Y, the sum of the second and third terms, or $D\,(a-b)$, is small and in many cases entirely negligible.

Reeves [14] has shown how the bridge may be balanced and at the same time the main and auxiliary ratios be adjusted to equality. His method consists in (a) adjusting the main ratio to give a zero current in the galvanometer; (b) removing the connector Z thus making a simple bridge and adjusting the ratio

$$(X+x_2+\alpha)/(Y+y_2+\beta) \text{ equal to } (A+x_1)/(B+y_1);$$

(c) restoring the connector Z and again establishing a balance by adjusting the main ratio. While both the second and third adjustments disturb those made before, yet the result is that the ratios are more nearly equal than before the adjustments were made and consequently the correction term is smaller. Repeating the adjustments in the same order makes the correction term still smaller. Thus, by successive approximations the three ratios

$$X/Y,\ (A+x_1)/(B+y_1), \text{ and } (\alpha+x_2)/(\beta+y_2)$$

are made more and more nearly equal until there is no change in the balance on removing the connector, or the correction term D $(a-b)$ is so small that no error is introduced on considering it zero.

This method was used by Dr. F. A. Wolff in the office of weights and measures in the Coast and Geodetic Survey prior to 1901 and has since been in use in the Bureau of Standards [15] and in the national laboratories of Germany [16] and England.[17] Unless the resistance C is small in comparison with $X+Y$ it is necessary to repeat the adjustment several times before a sufficiently close approximation is reached to neglect the correction term. In the case of very low resistances the number of successive approximations required is large and the process tedious. The time required for making the adjustments is materially reduced if the dials and switches for adjusting the main and auxiliary ratios are mechanically connected so that they can be easily changed by the same amount. With this arrangement the value of the correction term

[14] Proc. Phys. Soc. London, **14**, p. 166; 1896. See also Fleming, Hd. Book, **1**, p. 276; 1901; and N. F. Smith, Phys. Rev., **28**, p. 113; 1909.

[15] Lloyd: Proc. Eng. Soc. W. Pa., **19**, p. 403; 1903.

[16] Jaeger, Lindeck, and Diesselhorst: Zs. Instk., **23**, p. Z5; 1903.

[17] Smith: Electrician, **57**, p. 1011; 1906.

is independent of the setting of the dials. With additional variable resistances in the auxiliary ratio the connector Z may be removed and the two ratios set to an approximate equality, making the correction term negligible. Then with the connector replaced the balance may be made keeping the two ratios equal. Another alternative is to make the final adjustment by a variation of the resistance X or Y. In some cases small changes in X or Y may be made by shunting them with a comparatively high resistance, the value of which need not be known to a high accuracy. In other cases the adjustment is made by sliding one of the potential connectors along the conductor. These different ways of making the necessary adjustments of the resistance are in more or less common use. In general, however, the use of variable ratios, the dial switches of which are mechanically connected, is found to be more convenient and reliable.

The resistances x_1 and y_1 while small in comparison with A and B are often large enough to introduce appreciable errors unless the proper corrections are made. It is necessary, therefore, to make some auxiliary measurements for determining the term a which is the difference in proportional parts between $(A+x_1/B+y_1)$ and A/B. Where the main ratio can be varied in known small steps without changing the auxiliary ratio it is sufficient, with the connector Z removed, to observe the change in the main ratio necessary to reestablish the balance on shifting [18] the battery connections from m and m' to n and n'. If this change in proportional parts be denoted by w, it is readily shown that to the required accuracy,

$$a=\frac{\alpha}{\alpha+A}w. \tag{25}$$

Where the dials of the two ratio sets are mechanically connected, this difference may be determined from the galvanometer deflections. This requires a determination of the sensitivity in addition to the change in deflection on shifting the connections from m and m' to n and n'. From the sensitivity and the deflection the magnitude of a is readily calculated. However, a may be either positive or negative and it often requires more time to

[18] Lloyd: Proc. Eng. Soc. W. Pa., 19, p. 403; 1903. Smith: Electrician, 57, p. 1009; 1906.

determine the sign than the magnitude. All things considered, the time and attention necessary for determining this correction a is often as much as or more than that required for determining the correction term $D(a-b)$ or making the adjustments which make it negligibly small.

There is, therefore, in many cases as much as or more reason for carrying out a simple adjustment which makes the correction term, a, negligibly small as for carrying out an adjustment which makes the correction term, $D(a-b)$, negligibly small. What is really wanted is an adjustment which not only makes b very nearly equal to a but makes both very nearly equal to zero.

Jaeger and Diesselhorst[19] recognized that it would be well to make such an adjustment and devised a means for carrying it out. The special feature of their method is the transfer of the connector Z (see Fig. 7) from the terminals k and k' to n and n' and then balancing the bridge by an adjustment of x_1 or y_1. This gives

$$x_1/y_1 = X/Y \text{ approximately,}$$

and this makes a very small if x_1 and y_1 are small in comparison with A and B, since in the final adjustment $(A+x_1)/B+y_1$ is made equal to X/Y. While the method is good, the apparatus which they used was not well adapted to the purpose, so that in making the measurements it was necessary to make a number of successive approximations.

3. ADJUSTMENTS MAKING CORRECTION TERMS SMALL

When the main balance is established by an adjustment of the ratios, there is a decided advantage in having the dial switches of the main and auxiliary ratios mechanically connected. If, then, both a and b are to be made negligibly small, some means must be provided for an independent adjustment of the ratios x_1/y_1 and x_2/y_2. This can be accomplished by the use of variable low resistances forming a part of the resistances x_1 or y_1 and x_2 or y_2.

A variable low[20] resistance developed jointly by J. H. Dellinger and the author is shown in Fig. 8 in section. Here the letters have

[19] Wiss. Abh. d. P. T. R., 4, p. 119; 1904.
[20] Phy. Rev., 32, p. 614; 1911. 33, p. 215; 1911.

the following significance: a, a hard rubber tube; b, mercury; c and c_1, copper terminals amalgamated where they come in contact with the mercury; d, an amalgamated copper rod; e, a spring clamp for holding the copper rod in position. Electrical connection is made through the terminal blocks c and c_1, and the resistance is varied by changing the position of the copper rod. If the tube is made 12 cm long and has a bore of 3-mm diameter, the range is about 0.01 ohm, and the adjustment can easily be made to 0.00005 ohm. If a larger bore is used, closer adjustment may be obtained, though, of course, the range is reduced. Where a larger range is desired, it can be obtained by reducing the bore. Experience has shown, however, that it is not desirable to reduce the bore to less than 1 mm diameter. This gives a range, for a 12-cm tube, of about 0.1 ohm. In a number of measurements, especially with alternating currents, it is not necessary to know the resistances accurately, yet it is necessary to establish accurately some particular relation between them. In most of such cases these variable resistances make the adjustment a very simple matter. With one of these variable resistances on one side of each ratio and a small resistance on the other side, a and b may be made as small as we please. Obviously, if we desire to make such adjustments, we must begin by making a small rather than making b equal to a as has generally been done in measurements of high precision. If we begin by making b equal to a, then any adjustment of a is impracticable, since it disturbs the equality of a and b. It is in consequence of this that it has been customary to make some auxiliary measurement in order to determine the correction a.

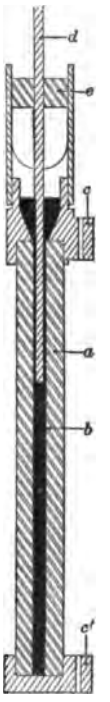

Fig. 8

The procedure which is now being followed in most of the precision resistance measurements at the Bureau of Standards, and which results in making both a and D $(a-b)$ negligibly small, is as follows: Starting with the ratio A/B, α/β and X/Y approximately equal

(1) With n and n' as points of connection, the bridge is balanced by an adjustment of x_1 or y_1.

(2) With m and m' as points of connection and the connector Z between X and Y removed, the bridge is balanced by an adjustment of x_2 or y_2.

(3) With m and m' as points of connection and the connection Z between X and Y restored, the bridge is balanced by a proportional adjustment of both the main and auxiliary ratios, or in some cases by an adjustment of Y.

Where the connecting resistances are small in comparison with the ratio resistances and starting with the three ratios X/Y, A/B, and α/β approximately equal, the first operation makes a negligibly small, the second makes b equal to a, and the third fulfills the conditions of the fundamental equation. The three adjustments then give

$$X = YA/B \tag{26}$$

which is the equation for the corresponding simple bridge.

Where the connecting resistances are large, the three adjustments are not entirely independent, since any change in making the second or third disturbs the first. It may, therefore, be necessary in some cases to repeat the adjustments in the same order, using those already made as first approximations.

4. AN ILLUSTRATIVE PROBLEM

To illustrate the way in which the adjustments work out we shall consider the particular case of the comparison of 0.001-ohm standards, using the bridge regularly used in the comparison of standards designed to carry large currents.

In comparison of standards of the same denomination the ratio sets used with this bridge each have a total resistance of 200 ohms and a ratio of 0.998 approximately. The higher side of each ratio is provided with a shunt whose resistance can be varied by means of four dials from a small value up to 100 000 ohms. By changing the settings of the dials the ratios may be varied from 0.999 up to 1.001, or higher if necessary. The dials on the two shunts operate together, so that the two ratios always have the same setting. As used, one end of each ratio set is connected to the potential terminals of one of the low resistances through a wire having a resistance of about 0.005 ohm, while the other end is connected

through a variable mercury resistance to the potential terminals of the other low resistance. The wires form the major part of the resistances y_1 and y_2 and the variable mercury resistances the major part of the resistances x_1 and x_2, shown in Fig. 7.

In the comparison of 0.001-ohm standards the following values chosen as an illustration are not only possible but are as probable as any others we might choose:

$$Y = 0.001 \text{ ohm}$$
$$X = 0.001001 \text{ ohm}$$
$$y_1 = 0.005 \text{ ohm}$$
$$y_2 = 0.005 \text{ ohm}$$
$$A = 100 \text{ ohms}$$
$$\alpha = 100 \text{ ohms}$$
$$C = 0.001 \text{ ohm}$$

It follows, then, that for the three ratios X/Y, $A+x_1/B+y_1$, and $\alpha+x_2/\beta+y_2$ to be equal, or $a=b=0$ we must have

$$B = 99.9$$
$$\beta = 99.9$$
$$x_2 = 0.005005$$
$$x_1 = 0.005005$$

But on beginning the adjustments we do not know the exact ratio of X to Y nor of x_2 to y_2, nor what should be the ratio of A to B. As the resistances to be compared are generally adjusted to within 0.1 per cent of their nominal value, A and B have values to begin with which may differ by not more than 0.2 per cent from the values they should have and will have after the final adjustment. Also, x_2 has a value which may differ by 0.005 ohm from the value it will have after adjustment.

Let us assume, then, that

$$B = 100.1$$
$$\alpha = 100.1$$
$$\text{and } x_2 = 0.001$$

It should be observed that the values assumed are such that the lack of adjustment of the ratios and x_2 cause errors in the same direction in the adjustment of x_1.

1. With these values for B, β and x_2 and the values given above for the other resistances, with the connector Z in place and n and n' as

branch points adjusting x_1 so as to balance the bridge necessitates making x_1 = 0.004 950 ohm. This differs from the value which it should have as given above by 0.000 05 ohm, which in comparison with 100 ohms is 5 parts in 10 000 000. In this case, then, the first adjustment leaves $a = -0.000\ 000\ 5$, which is negligible in most if not quite all resistance measurements.

2. With the connector Z removed and using m and m' as branch points, adjusting x_2 so as to balance the bridge makes the ratios $(A+x_1)/(B+y_1)$ and $(\alpha+x_2)/(\beta+y_2)$ equal, to the precision with which the balance is established. In this adjustment b is easily made equal to a to well within the limits required for the most precise measurements. However, there is a possibilty that this equality may be disturbed in adjusting the main and auxiliary ratios.

3. With the connector Z in place and using m and m' as branch points an equal adjustment of A and α so as to balance the bridge still leaves $(A+x_1)/(B+y_1)$ and $(\alpha+x_2)/(\beta+y_2)$ at least very nearly equal since x_1 and y_1, and x_2 and y_2 are very nearly equal and small in comparison with A, α, B, and β. As this last adjustment can not ordinarily be made to a precision better than 1 part in 1 000 000 the three adjustments give

$$X = YA/B \tag{27}$$

to as high a precision as can be obtained in the final balance of the bridge.

For convenience in carrying out the adjustments it is sometimes desirable to have two keys and two rheostats in the battery circuit. Key No. 1 closes the circuit through the bridge and the rheostat having the higher resistance, while key No. 2 closes the circuit through the bridge and the rheostat having the lower resistance. Key No. 2 is used in making the adjustment (2) while key No. 1 is used in making the adjustment (3). For making the adjustment (1) key No. 1 is generally used, though in most cases No. 2 could be used without a change in the setting of the rheostat. As some of the measurements are actually carried out a single key is used for making the three adjustments and no change is made in the setting of the rheostat.

Had the ratio coils been 10 instead of 100 ohms, the first adjustment would have left a about ten times as large and for the most precise measurements would have necessitated repeating the adjustments in the same order. In conductivity measurements of samples in the form of rods or wires the ratio of the resistances of the conductors may not be known even approximately. In such cases it is better to make an approximate adjustment using m and m' as branch points before taking up the adjustments in order. Even then if the adjustment 3 requires much of a change in the ratio A/B the adjustments should be repeated. If, on the other hand, the change necessary in this ratio is small, we know that we need make no further adjustments.

Where the adjustments are made in this way the accuracy is as high or higher than where any of the connecting resistances are determined and the corrections are applied, while the time and attention necessary for making the adjustments are generally considerably less.

5. SENSITIVITY WITH D'ARSONVAL GALVANOMETER

In most of the work at the Bureau of Standards with the Thomson and Wheatstone bridges, D'Arsonval galvanometers of high sensitivity are used. These galvanometers are so designed and constructed that the motion of the moving system is critically damped with a comparatively low resistance in the external circuit, in some cases as low as 35 ohms. Their operation is so much more satisfactory when critically damped that adjustment to this condition is regularly made.

It is easily seen that if the bridge is out of balance by a small amount d, and the galvanometer is not connected, the difference in potential between the points to which it is to be connected is

$$d\sqrt{PX}\,Y/(X+Y).$$

Here P is the power dissipated in the conductor X, and d is the equivalent of $\Delta X/X$, where ΔX is the amount by which if X were changed the bridge would be balanced. If in adjusting to a balance proportional changes are made in A and α or in B and β, then $d = -\Delta A/A$ or $\Delta B/B$.

(a) If the resistance of the bridge between the terminals to which the galvanometer is to be connected is less than the external critical resistance of the galvanometer, then some additional resistance must be connected in series to make up the deficit. Under these conditions the deflection of the galvanometer is

$$S_e d\sqrt{PX}\,Y/(X+Y)$$

or the sensitivity S, the ratio of the deflection to the lack of balance of the bridge, is given by the equation

$$S=S_e\sqrt{PX}\,Y/(X+Y) \tag{28}$$

when S_e is the voltage sensitivity of the galvanometer under the critically damped condition. This equation and the others which will be given were derived from a consideration of the relations which exist in the Thomson bridge. However, they give the sensitivity not only of the double bridge but also of the simple bridge and the multiple bridge.

(b) If the resistance of the bridge is larger than the external critical resistance of the galvanometer, then some resistance must be placed in parallel with the galvanometer, otherwise the motion of the moving system is underdamped. As the resistance is larger than in the previous case, the current for a given emf is less. Also a part of the current passes through the shunt instead of through the galvanometer. As a result the deflection is less than in the former case by the factor r_c/r_b, where r_c is the external critical resistance of the galvanometer and r_b is the resistance of the bridge. Therefore

$$S'=S_e\sqrt{PX}\,Y/(X+Y)r_c/r_b \tag{29}$$

where S' is the sensitivity in case it is necessary to shunt the galvanometer.

(c) If changes in temperature are to be measured by changes in the resistance, X and if α is the proportional change in X per degree$_c$, the sensitivity S_t of the combination of resistance thermometer, bridge, and galvanometer is given by the equation

$$S_t=\alpha S_e\sqrt{PX}\,Y/(X+Y)$$

or

$$S_t=\alpha S_e\sqrt{PX}\,Y/(X+Y)r_c/r_b \tag{30}$$

depending on whether r_b is less or more than r_c. These equations (28, 29, and 30) show that if the resistance of the bridge is less than the external critical resistance of the galvanometer the sensitivity is directly proportional to the voltage sensitivity of the galvanometer, the square root of the power dissipated in the conductor times the square root of its resistance, and the ratio $Y/(X+Y)$. If the resistance of the bridge is more than the external critical resistance of the galvanometer, the sensitivity is reduced by the factor r_c/r_b.

It will thus be seen that the voltage sensitivity and the external critical resistance of the galvanometer are matters of first importance in fixing the sensitivity of the system. There are also other constants of the galvanometer which have some effect upon the precision which may be obtained. The more important of these are the period of the moving system and the resolving power of the optical system. We know, for example, that under ordinary conditions if the galvanometer has a period of 20 seconds we are not able to obtain as high a precision as with a galvanometer having the same sensitivity but a period of only 3 seconds. On the other hand, we can not state that the precision is inversely proportional to the period or depends in any definite way upon the period. Likewise, we can not state definitely the way in which other constants affect the precision. However, the voltage sensitivity and the external critical resistances are by far the more important constants, and their effect upon both the sensitivity of the system and the precision which may be obtained is definite and known. In galvanometers having a fairly large restoring moment or "stiff" suspensions, the effect of particular changes in the design such as increase in the diameter of the wire used in winding the coil, an increase in the field strength, etc., can be calculated from the theory. On the other hand, in D'Arsonval galvanometers of the highest sensitivity the moment of restoration is so small and the intensity of the field is so large that the magnetic impurities in the coil, even where every precaution has been taken to exclude them, have a marked influence on the behavior of the instrument. On account of these impurities one of two instruments, constructed as near alike as possible, may easily have two or three times the sen-

sibility of the other. Consequently an instrument maker in carrying out a particular design can not produce a galvanometer having just the constants which, according to the theory, we should have expected. Instead of trying to get a galvanometer having particular constants we should select the best galvanometer available for the work and then design the rest of the apparatus so the system will have its maximum sensitivity for the particular galvanometer. If, on the other hand, the bridge is designed first, generally no galvanometer can be obtained which will work to its best advantage with the particular bridge.

To obtain the highest sensitivity the power dissipated in X should be as large as possible, without the uncertainties in its resistance, on account of the heating by the test current, becoming equal to the precision sought. That the test current may not be limited by the heating in some other part of the bridge the various resistances should be so designed that their proportional changes, on account of heating by the part of the current which they carry, is less than the proportional change in X. Further, the resistance of the ratio coils should be large, so that the effects of the connecting resistances will be small and the various adjustments easily made. On the other hand, the resistance between the branch points to which the galvanometer is connected should preferably be less than the external critical resistance of the galvanometer, so it need not be shunted, thus reducing the sensitivity of the system. Ordinarily, the resistances of the ratio coils should be so chosen that the galvanometer may be critically damped by putting only a few ohms in series.

The conductor X should be so designed that the proportional change in resistance will be small for a given current. Also the resistance X should be comparatively large. Generally, however, the conductor is already constructed so we can neither change the load which it will carry nor the value of its resistance. In most cases, too, either the ratio X/Y is fixed by other considerations or the test current which may be used is limited on account of the heating which it may produce in Y. It will thus be seen that of the various factors which affect the sensitivity we are generally at liberty to change only a few. In comparing resistance standards of low value there is seldom any difficulty about the heating

of the ratio coils by the part of the test current which they carry. Therefore nearly everything which may be done toward getting a high sensitivity is done when the best galvanometer available, considering mainly its voltage sensitivity and external critical resistance, is selected and the resistance of the bridge between the galvanometer connectors is made only slightly less than the external critical resistance of the galvanometer.

However, the sensitivity of the system is not the only factor which has an effect upon the precision which may be obtained. Reference has already been made to the period of the galvanometer and the resolving power of its optical system. Of the other things which should be mentioned the thermoelectric effects are probably of the most importance. Under ordinary conditions the galvanometer circuit may be expected to have a thermoelectromotive force which is almost always changing and may easily amount to a few microvolts. Since to get the precision desired often requires the detection of a hundredth of a microvolt, it is necessary to work with a "false zero;" that is, the bridge is considered balanced when with the galvanometer circuit closed there is no change in the deflection following the closing or opening of the battery circuit. What is better than this is the reversal of the test current, since for the same lack of balance the change in deflection is twice as large. If the thermoelectromotive forces are changing rapidly, they limit the precision of the balance even when made in this way. Trouble on this account can be considerably reduced by keeping as much of the galvanometer circuit as possible in a well stirred oil bath. The rest of the circuit should be made up as far as possible from a homogeneous conductor. All loose connections should be avoided, since the surfaces in contact are usually oxidized and so have a high thermal resistance. If, then, there is a flow of heat across the connection, there is a high temperature gradient in the oxide and so the two junctions between the oxide and metal may be at different temperatures, in which case there is in general a thermoelectromotive force.

Where the balance is made using a false zero or a closed galvanometer circuit the unbalanced inductances in the bridge are often troublesome and may limit the precision of the balance.

It will thus be seen that the expressions derived for the sensitivity do not contain all the factors having an effect on the precision which may be obtained. It is believed, however, that with the apparatus used and with the conditions under which the measurements are carried out these expressions contain the more important factors and all the factors whose effects can be definitely stated. Investigators working with different apparatus and under different conditions have found the precision attainable limited in different ways. In deriving expressions for the sensitivity [21] they have therefore included different factors. For example, at one time Daniell cells were used as the source of the test current and as their resistance limited the value of the current it is not uncommon to find expressions for the sensitivity which contain a term representing the resistance of the battery. Thomson,[22] in 1862, called attention to the fact that the heating of the conductor by the test current is one of the important factors in limiting the precision which may be attained in the measurement. However, a number of papers appeared later in which there is no reference to the heating and it was not until 1895 that Schuster [23] again called attention to it and derived expressions for the sensitivity based on the current which the conductor will carry with the permissible rise in temperature.

In work with resistance thermometers the compensation for the changes in the resistance of the leads is not always easily accomplished to the precision desired. For this reason some resistance thermometers are being made with current and potential terminals and have the branch points so located that under the conditions of use all parts of the conductors between them are to be at the same temperature. The resistance between the branch points is generally several ohms and the resistance of the connectors from a few hundredths to a tenth ohm. On account of the comparatively high resistance, the high temperature

[21] The following is a partial list of the papers in which the sensitivity is discussed:

Schwendler: Phil. Mag., **31**, p. 364; 1866. Heaviside: Phil. Mag., **45**, p. 114; 1873. T. Gray: Phil. Mag., **12**, p. 283; 1881. Rayleigh: Proc. Roy. Soc., **49**, p. 203; 1891. Schuster: Phil. Mag., **39**, p. 175; 1895. Gray: Abs. Meas., **1**, p. 331. Flemming: H. Book, **1**, p. 333. Jaeger: Za. f. Instk., **26**, p. 69; 1906. Callender: Proc. Phys. Soc., **22**, p. 220; 1910.

[22] Phil. Mag., **24**, p. 149; 1862.

[23] Phil. Mag., **39**, p. 175; 1895.

coefficient of the resistance material, and its small surface, the test current is limited to a very small value. Even 1000-ohm coils of the ordinary construction will carry more current without an appreciable change in resistance. Therefore, when such a conductor is connected into a Thomson bridge in the usual way, there is no reason why the battery and galvanometer connections may not be interchanged. With the usual connections the ratio coils make up a large part of the resistance between the galvanometer connectors. As a result the resistance of the ratio coils must be kept comparatively low or some sacrifice made in the sensitivity. On the other hand, with the connections reversed the resistance of the bridge depends almost entirely on the resistance $X+Y$. With this arrangement, therefore, the advantage of using ratio coils of high resistance may generally be obtained without a sacrifice in the sensitivity.

6. THE MULTIPLE BRIDGE

As the Thomson bridge is ordinarily used the galvanometer is connected to the points between the ratio coils, while in the case we have just been considering the battery was connected to these same points. Having used an auxiliary ratio set with good results first in the galvanometer circuit and then in the battery circuit, the next step naturally would be the use of auxiliary ratios in both the galvanometer and battery circuits. The author has mentioned this matter to different persons who have used the Thomson bridge in precision measurements and found that each one had considered it.

The connections for a bridge of this kind with three sets of auxiliary ratios are shown in Fig. 9. If adjustments are made so that there is no current through the galvanometer with the connectors Z, Z_1, and Z_2 in place and with them removed alternately it will easily be seen that

$$X/Y=A/B \tag{31}$$

Since in this case at least one set of auxiliary ratios must carry the test current, either the ratio set must be of low resistance or the test current must be small. The advantage of the use of an auxiliary ratio is largely lost unless its resistance is large in com-

parison with the resistance of the conductor with which it may be considered in parallel (the resistance Z referred to above, p. 581). The multiple bridge may therefore be considered as limited to

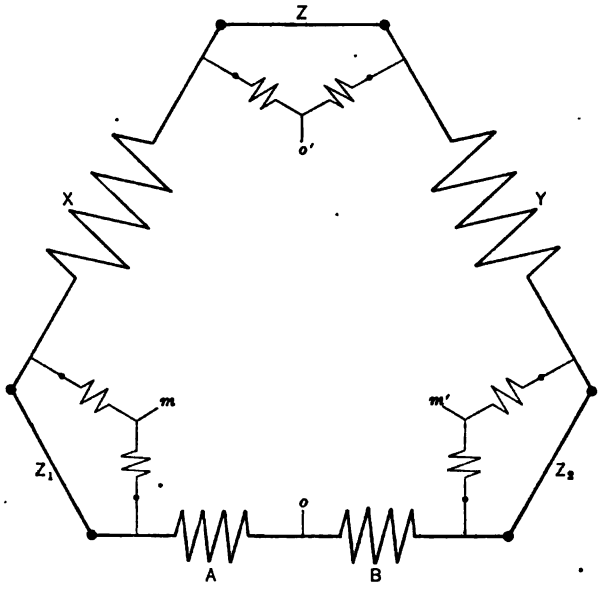

Fig. 9

that class of measurements in which a small test current is used, as in the case of resistance thermometers.

7. THEORY WITH NONLINEAR CONDUCTORS

Equation (23), on which the preceding discussion of the Thomson bridge is based, was derived for the case in which the four-terminal conductors are linear.[24] We can not, therefore, assume that it is applicable in the more general case. In fact, if we will consider the simplest particular cases where the four-terminal conductors are not linear, we will see that this equation does not give the exact relations necessary for a zero current through the galvanometer. If, however, the ratio coils have a large resistance in comparison with the resistance of the four-terminal conductors, and if the cross resistances of the latter are small in comparison with their direct resistances, the errors introduced by using the

[24] For explanation of the difference between linear and nonlinear conductors, see p. 571.

equation will generally be so small that they need not be considered. In the limiting case where the ratio coils have a very high resistance, since they carry no appreciable part of the current, it is easily seen that the ordinary equation may be considered to give the exact relations between the different resistances as it does in the case of linear conductors. Searle[25] has recently published equations giving the general relation between the resistances necessary for the zero current in the galvanometer. He also points out the fact that in the particular case in which the

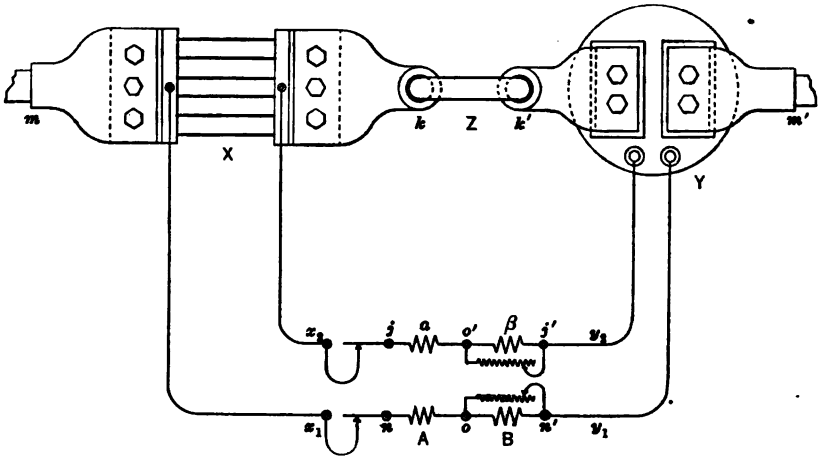

Fig. 10

bridge is balanced, both with the connector Z in place and with it removed, the term or terms containing the cross resistance disappear.

Without going further into the theory we wish now to show that, whether the four-terminal conductors are linear or nonlinear, if the adjustments are carried out as outlined above (p. 585), the same simple relations exist between the resistances. To show this we shall consider the adjustments to have been carried to the point where with either m and m' or n and n' as battery connections (see Fig. 10) and with the connector Z either in place or removed the current through the galvanometer, connected to o and o', may be considered zero. It follows, then, from the recip-

[25] Electrician, 67, p. 56; 1911.

rocal theorem referred to above, that if o and o' are used as battery connections m and m' and n and n' may be considered to be at the same potential, whether the connector Z is in place or not. With the battery connected at o and o' the conditions necessary for the terminals m and m' and n and n' to be at the some potential are that

$$A/B = (x_1 + X + x_2 + \alpha)/(y_1 + Y + y_2 + \beta) \tag{32}$$

and

$$(A + x_1)/(B + y_1) = (X + x_2 + \alpha)/(Y + y_2 + \beta). \tag{33}$$

Here X and Y are the four-terminal resistances $k\text{-}m/q\text{-}n$ and $k'\text{-}m'/q'\text{-}n'$, while x_1, x_2, y_1, and y_2 are the three-terminal resistances $n\text{-}m/n\text{-}k$, $j\text{-}k/j\text{-}m$, $n'\text{-}m'/n'\text{-}k'$, and $j'\text{-}k'/j'\text{-}m'$.

The fact that removing the connector Z produces no change in the difference in potential between m and m' or n and n' shows that it carries no appreciable part of the current, so the terminals k and k' may be considered to be at the same potential. The condition necessary for k and k' to be at the same potential is

$$(A + x_1 + X)/(B + y_1 + Y) = (x_2 + \alpha)/(y_2 + \beta) \tag{34}$$

From equations (39), (40), and (41) it follows that

$$X = YA/B \tag{35}$$

which is the same simple relation as was obtained when we were considering linear conductors. Where the four-terminal conductors are nonlinear the Thomson bridge constitutes a system which can hardly be said to be simple. It seems rather remarkable, therefore, that by a few simple adjustments we can obtain a relation between the resistances which is sufficiently accurate for use in the most precise measurements.

8. THEORY USING ALTERNATING CURRENT

The Thomson bridge has been used with alternating current by Sharp and Crawford[26] in the comparison of inductances in heavy current resistances and by Barnett[27] in the measurement of inductances and capacities. The equations used by these authors can,

[26] Trans. Amer. Inst. E. E., 29, p. 1540; 1910.
[27] Phys. Rev., 34, p. 74; 1912.

however, give accurate results only in special cases or in cases in which the Thomson bridge has little, if any, advantage over the ordinary bridge.

In discussing the comparison of the resistances of nonlinear conductors by means of the Thomson bridge, we found it desirable to consider only the case where certain auxiliary adjustments had been made. When we come to the consideration of the relations between the various resistances, inductances, and the frequency, when alternating current is used, auxiliary adjustments become of much more importance. A general relation between the various quantities affecting the balance of the bridge would necessarily be very complicated; for in addition to the two cross resistances (which are responsible for much of the complication when direct current is used), we should have to consider two cross inductances and various self and mutually induced electromotive forces caused by three components of the current, each of which differs in phase from the others. We shall, therefore, not consider the general problem but limit our discussion to the special case in which, in addition to the auxiliary resistance adjustments outlined above (p. 585) corresponding adjustments of the inductances are made and the parts of the bridge are so arranged that certain of the mutual inductances have a negligible effect upon the conditions of the balance.

Let us assume that adjustments have been carried out so that, with alternating current supplied either through the leads m and m' (see Fig. 11) or n and n', and with the connector Z either in place or removed, there is no current through a galvanometer connected to o and o'. We then have three independent balances of the bridge with alternating current which correspond to the three independent balances with direct current which we have considered above. Here the galvanometer leads must be brought out in such a way that the alternating test current can induce no emf in them.

With this adjustment, it follows from the reciprocal theorem (considered above, p. 563), that if the current supply is led in through the leads connected to o and o', a galvanometer connected alternately to m and m' and to n and n' will indicate a

zero current, if the galvanometer leads are located the same as the current leads were in making the adjustments. Now, as before, the connector Z may be removed without disturbing the balance. This shows that with this connection it carries no appreciable part of the current.

If, then, a galvanometer were connected to the terminals k and

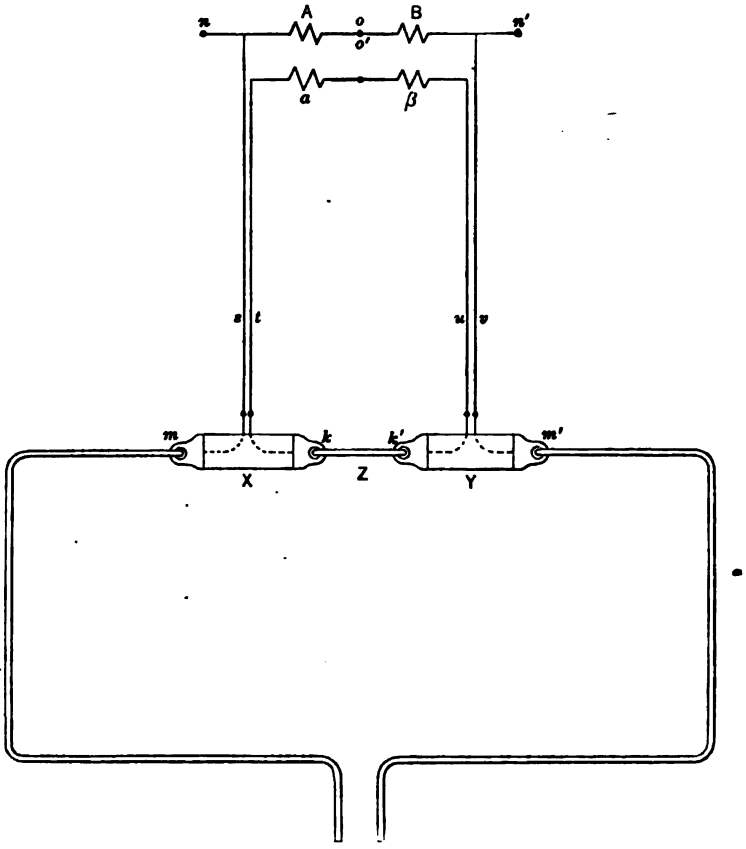

Fig. 11

k', it would indicate a zero current, providing the leads were brought out in such a way as to have the same mutual inductance with respect to the rest of the system as the connector Z, when in place.

The relations between the various quantities are therefore the same as in a simple bridge which is balanced when either of three

pairs of potential connections are used. In this case we may consider the bridge as made up of two conductors in parallel, each of which has five terminals, o, n, m, k, and o', and o, n', m', k', and o'. If no appreciable error is introduced on neglecting the mutual inductance between these two conductors—that is, between the two halves of the bridge—we have the following relations between the impedances:

$$\frac{o\text{-}n/o\text{-}o' + \iota p(\overline{o\text{-}n/o\text{-}o'})}{o\text{-}n'/o\text{-}o' + \iota p(\overline{o\text{-}n'/o\text{-}o'})} = \frac{n\text{-}m/o\text{-}o' + \iota p(\overline{n\text{-}m/o\text{-}o'})}{n'\text{-}m'/o\text{-}o' + \iota p(\overline{n'\text{-}m'/o\text{-}o'})}$$
$$= \frac{m\text{-}k/o\text{-}o' + \iota p(\overline{m\text{-}k/o\text{-}o'})}{m'\text{-}k'/o\text{-}o' + \iota p(\overline{m'\text{-}k'/o\text{-}o'})} = \frac{k\text{-}o'/o\text{-}o' + \iota p(\overline{k\text{-}o'/o\text{-}o'})}{k'\text{-}o'/o\text{-}o' + \iota p(\overline{k'\text{-}o'/o\text{-}o'})} \tag{36}$$

Here $o\text{-}n/o\text{-}o'$, $n\text{-}m/o\text{-}o'$, $m\text{-}k/o\text{-}o'$, and $k\text{-}o'/o\text{-}o'$ are the resistances and $\overline{o\text{-}n/o\text{-}o'}$, $\overline{n\text{-}m/o\text{-}o}$, $\overline{m\text{-}k/o\text{-}o'}$, and $\overline{k\text{-}o'/o\text{-}o'}$ are the inductances of the four-terminal conductors, ι is the square root of minus one and p is 2π times the frequency of the alternating current. The same system is used to designate the resistances and inductances of the right-hand side of the bridge. Since we are considering o and o' as current terminals common to two conductors in parallel, the symbols do not have quite the same significance as that given above. (See p. 561.)

If we may consider the mutual inductance between A and the other parts of the system, including the lead n as negligible, then $\overline{o\text{-}n/o\text{-}o'}$ is the inductance L_A. Under the same conditions $\overline{o\text{-}n'/o\text{-}o'}$ is the inductance L_B. If we may consider the mutual inductance between X, with its leads m and Z, and the rest of the system beyond the potential terminals as zero, then $\overline{m\text{-}k/o\text{-}o'}$ is the inductance L_X, if the lead m and the connector Z are in the normal position of the current leads to the conductor X.

As both the resistance and inductance of X will in most cases be very small, it is important that the mutual inductance between X, with its current leads and the rest of the system, should be very small, otherwise considering it zero may introduce an error. How important it is that this mutual inductance be made very small will be understood when we consider that the resistance of the conductor X may be 0.0001 ohm or less and the current through

it may be 1000 amperes or more. With an alternating current of this magnitude the mutual inductance does not need to be large for the induced emf to be appreciable in comparison with 0.1 volt, the difference in potential between the potential terminals. With the arrangement shown in Fig. 11, even when the ratio sets are at a considerable distance from the four terminal conductors, the emf induced in the ratio coil A is very appreciable in comparison with the drop in potential in the low resistances. Unless this emf is balanced by an equal emf, induced either in the auxiliary ratio arm α or in the other main ratio arm, an error will be introduced. The magnitude of this error may be of the order of the difference of the induced emf in A and α divided by the voltage across the potential terminals of X, though generally it will be considerably less.

If care is taken to place A and α at a considerable distance from the conductors which carry the large current and in such positions that they have very nearly the same mutual inductance with respect to that part of the system which carries the large current, and if the potential leads are placed near each other or are twisted together, the mutual inductance between X, with its current leads and that part of the left side of the bridge beyond the potential terminals, may be considered zero.

Under the conditions similar to those just considered $\overline{m'\text{-}k'/o\text{-}o'}$ is the inductance L_Y. As a matter of convenience we shall call $\overline{n\text{-}m/o\text{-}o'}$, $\overline{k\text{-}o'/o\text{-}o'}$, $\overline{n'\text{-}m'/o\text{-}o'}$, and $\overline{k'\text{-}o'/o\text{-}o'}$ the inductances l_s, l_t, l_u, and l_v. Making these substitutions in equation (36) we have

$$\frac{A+\iota pL_A}{B+\iota pL_B}=\frac{s+\iota pl_s}{v+\iota pl_v} \tag{37}$$

$$\frac{A+\iota pL_A}{B+\iota pL_B}=\frac{X+\iota pL_X}{Y+\iota pL_Y} \tag{38}$$

$$\frac{A+\iota pL_A}{B+\iota pL_B}=\frac{t+\alpha+\iota p(l_t+l_\alpha)}{u+\beta+\iota p(l_u+l_\beta)} \tag{39}$$

If the bridge is balanced under the conditions given above, all of these equations are satisfied. It is, however, the second which gives the relations between the quantities which we wish to com-

pare and with which we are most concerned. This equation may be put in the following form

$$\begin{aligned} AY - p^2L_AL_B + ip(AL_Y + YL_A) = \\ BX - p^2L_BL_X + ip(BL_X + XL_B) \end{aligned} \tag{40}$$

or separating the real and imaginary parts we have

$$AY - BX = p^2(L_AL_Y - L_BL_X) \tag{41}$$

and

$$AL_Y + YL_A = BL_X + XL_B. \tag{42}$$

These equations show that where the time constants are small the balance of the bridge is practically independent of the frequency. Inspection will show that if A and B or A and X or B and Y can be considered to have equal time constants, the balance of the bridge is independent of the frequency except in so far as the resistances and inductances may themselves be functions of the frequency. Low-resistance standards suitable for use in alternating current measurements, such as we are concerned with here, necessarily have small time constants. If, then, the ratio coils A and B have small time constants and the frequency is not high, it will be convenient to write equation (41) in the following slightly different form

$$X = \frac{A}{B}Y\left[1 + \frac{p^2B}{AY}(L_BL_X - L_AL_Y)\right] \tag{43}$$

Where $\frac{p^2B}{AY}(L_BL_X - L_AL_Y)$ is small and may be looked upon as a correction term. We, therefore, have approximately

$$X = \frac{A}{B}Y \tag{44}$$

and

$$\frac{L_X}{X} = \frac{L_Y}{Y} + \frac{L_A}{A} - \frac{L_B}{B} \tag{45}$$

To show that no errors need be introduced by the use of these approximate equations, we may consider the following example

$$p = 2\pi \times 60$$

$A = 10$ ohms	$L_A = 4$ microhenrys
$B = 10$ ohms	$L_B = -3$ microhenrys
$Y = 0.001$ ohm	$L_Y = 0.005$ microhenry
$X = ?$ ohm	$L_X = ?$ microhenry

From equation (45) we have

$$\frac{L_X}{X} = (5 + 0.04 + 0.03) \times 10^{-6} \text{ seconds}$$

and since we know that X must be very nearly 0.001 ohm $L_X =$ 0.00507 microhenry.

This value substituted in (43) shows that the correction term is less than 10^{-6}, a quantity which is negligible even in very precise resistance measurements. We may, therefore, consider X exactly 0.001 ohm, in which case L_X may be considered exactly 0.00507 microhenries. Since ratio coils can be obtained having time constants of only one-tenth the values used in the example and since we shall seldom be concerned with low resistance standards having so large a time constant the approximate equations (44) and (45) can in most cases be used, instead of the exact equations (41) and (42).

We have seen that when the bridge is balanced under each of the three conditions given above and when we can neglect the effect of certain of the mutual inductances, we have fairly simple relations between the resistances, inductances, and frequency, and if the time constants of all the conductors are small, the relations are very simple.

To bring about these relations it is necessary that at least three of the eight resistances and three of the eight inductances be variable. That is, to satisfy equations (37), (38), and (39) at least one resistance and one inductance in each must be variable either continuously or in small steps, over a range corresponding to the range in values of the resistance and inductance of the standards to be compared while using a particular auxiliary standard Y and particular ratio coils A and B. If the time con-

stants of the main ratio coils are small and nearly equal, we may consider equation (37) and (39) satisfied when $\frac{A+s}{B+v}$ and $\frac{\alpha+t}{\beta+u}$ can be considered as equal to A/B, $\frac{pl_s}{A+s}$ can be considered equal to $\frac{pl_v}{B+v}$, and $\frac{p(l_t+l_\alpha)}{t+\alpha}$ can be considered equal to $\frac{(pl_\beta+l_u)}{\beta+u}$. These relations between the resistances being the same as those which we have considered above when using direct current, similar apparatus may be employed in making the adjustments.

As we have assumed the time constants of the ratio coils to be small and nearly equal, we can not change either as a part of the adjustments necessary in establishing the triple balance of the bridge. This makes it necessary to provide some means for adjusting the time constant of conductor X or Y as well as of s or v, and t (or α) or u (or β). A convenient way of providing for an adjustment equivalent to an adjustment of the time constant of the conductor Y is to place a movable coil in one of the potential leads, as shown in Fig. 12, in such a way as to introduce a variable mutual inductance M between the potential lead and one of the current leads. When this mutual inductance is taken into consideration we have L_Y+M in place of L_Y in all the above equations. The relation between the time constants of the four terminal conductors and the main ratio coils is then

$$\frac{L_X}{X}=\frac{L_Y+M}{Y}+\frac{L_A}{A}-\frac{L_B}{B}. \tag{46}$$

A variable mutual inductance suitable for use in the comparison of low resistance standards may consist of a coil of 8 to 10 turns, 8 to 10 centimeters in diameter, and one of the current leads with a suitable support for holding them in the desired relative positions. The mutual inductance may be varied either by rotating the coil or by changing its distance from the current lead. With the former the inductance may be varied continuously from the maximum positive value to a corresponding negative value. It is also the more convenient. In some cases the inductance may be determined by calculation from the dimensions or in general by comparison with other inductances, either self or mutual. The

calibration need not be made to a high accuracy since the time constants to be compared are so small that we shall seldom be concerned with more than two significant figures.

The coil for the mutual inductance M necessarily changes the inductance l_v. Therefore, to make $\frac{l_s}{s}=\frac{l_v}{v}$ it is necessary to put a coil of corresponding self inductance in the conductor s. This coil must be so placed that the mutual inductance between it and the conductor which carries the main part of the test current is

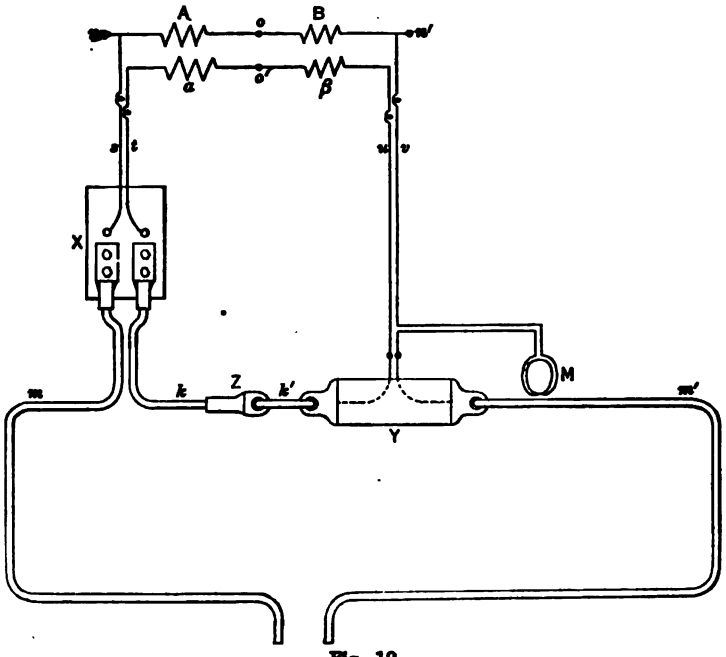

Fig. 12

negligibly small. If this coil is made in two sections, the changes in its inductance necessary for balancing the bridge may be made by changing their relative position. If the auxiliary ratio coils have very nearly equal time constants, the little adjustment necessary can be made by making two or three small loops in one or the other of the potential leads—that is, one or the other of the conductors t and u—and placing them closer together or farther apart, as occasion may demand. These loops must be so placed that the

mutual inductance between them and the conductor carrying the main part of the test current is negligibly small. In the figure the potential leads are shown slightly separated for the sake of clearness. In use they are twisted together to eliminate as far as possible the effects of mutual inductance between them and other parts of the system. The adjustments of the bridge may be carried out in different ways, though in general it is better to use direct current and make resistance adjustments as outlined on page 585 before making any attempt at adjusting the inductances. Then, with alternating current, inductance adjustments may be carried out as follows: (a) With the normal connections adjust M so that the galvanometer indicates a zero current; (b) with the currents supplied through leads connected to n and n' adjust the inductances in s or v, to give a zero current in the galvanometer; (c) with the current supplied through the regular current leads and with the connector Z removed, adjust the inductances in t or u so as to give a zero current through the galvanometer; (d) with the normal connections, again adjust M so as to give a zero current in the galvanometer. If this requires much change in M, the series of adjustments should be repeated, using those already made as first approximations.

It will be observed that except for a preliminary adjustment, the method of making the inductance adjustments is the same as that given above for making the resistance adjustments.

If the resistances X and Y change appreciably on changing from direct to alternating current, a balance can not be established without a further adjustment of the resistances. This adjustment can be made by resetting the main and auxiliary resistance ratios and the change required is the same as the change in the ratio of X to Y on changing from direct to alternating current of the frequency used.

Where alternating current only is used in making the adjustments, we obtain the ratio of the resistances of the four-terminal conductors at only one frequency, unless, of course, a second adjustment is made with a different frequency. Also, unless some special device is used we can not tell, except by trial, what changes should be made. Consequently, more time is required for making

the comparisons than where we begin by making the resistance adjustments using direct current. Here we have assumed the use of a vibration galvanometer. At the usual commercial frequencies other detectors are lacking in sensitivity or have other defects which make them unsuitable for use in this work. Even the vibration galvanometer used must have certain characteristics not common to all instruments of that type. First, the moving system must be practically nonmagnetic; otherwise it will be set in vibration by the "stray" magnetic field, which necessarily has the frequency to which the moving system is tuned. Second, the vibration galvanometer must be so designed and constructed as to have a high voltage sensitivity with an external resistance equal to the resistance of the bridge between the points o and o'. This matter requires care, since, on account of the back emf,[28] we can not divide the current sensitivity by the resistance of the galvanometer plus the resistance of the bridge and assume that this gives the voltage sensitivity under the conditions of use.

In bridge measurements, with direct current, substitution methods have certain well-known advantages and are in common use where a number of standards of the same denomination are to be compared. When alternating current is used the advantages of substitution methods are much more pronounced. This is true whether the resistances are high, in which case capacities between the defferent parts of the system including the room in which the apparatus is located must be considered, or whether the resistances are low, in which case the mutual inductance between that part of the system which carries the large current and all other parts becomes a matter of real importance. Since, in addition to other advantages, the substitution method eliminates the effect of certain of the mutual inductances, it should be used, in most cases at any rate, where the standards to be compared are of the same denomination. A general discussion of the substitution method, or of the establishment and use of known ratios in measurements with the Thomson bridge or of devices which may be employed to simplify the calculations are matters which can not be considered in this paper.

[28] Wenner, this Bulletin, 6, p. 143; 1910.

IV. SUMMARY

1. The conditions which must necessarily be fulfilled in order that the resistance of a four-terminal conductor be definite are pointed out.

2. The additional conditions which must be fulfilled in order that the inductances as well as the resistances be definite are pointed out.

3. The theorem regarding the reciprocal relation obtained on interchanging the current and potential connections is discussed and applied in showing the relations between the resistances of a four-terminal conductor. It is also applied in showing the relations between the resistances and the relations between the inductances in the Thomson bridge.

4. It is shown that on using the four terminals in different combinations, three and only three values for the resistance are obtained, and that one of these is the sum of the other two.

5. Several devices which may be employed to increase the definiteness of the resistances are discussed, and it is pointed out that a symmetrical arrangement of the current and potential connectors together with the use of branched potential connectors makes the resistance sufficiently definite for the most precise measurements, even in the case of conductors which are to carry very large currents.

6. Some of the ideas which are discussed regarding resistances and inductances are embodied in the design of a resistance standard to carry a fairly large alternating current.

7. The theory of the Thomson bridge using linear four-terminal conductors is given and some of the different ways of determining or eliminating the correction terms are discussed.

8. The way in which the adjustments are carried out in the precision resistance comparisons at the Bureau of Standards is described.

9. Expressions are derived for the sensitivity of the combination of Thomson bridge and D'Arsonval galvanometer, with the motion of the coil critically damped.

10. The theory of the Thomson bridge where the four-terminal conductors are not linear is discussed, and it is shown that with the adjustments which are regularly made the same simple relations exist between the resistances whether the four-terminal conductors are linear or nonlinear.

11. Where alternating current is used it is shown that if certain adjustments are made the bridge can be balanced, and if we can neglect certain of the mutual inductances we have definite relations between the resistances and inductances of the main ratio coils and the four-terminal conductors, and the frequency. Where the time constants of all four of these conductors are small it is shown that the relation between the resistances and inductances are practically independent of the frequency. In this case if the bridge is balanced first with direct current and then with alternating current, the change in the ratio of the resistances of the four-terminal conductors on changing from direct to alternating current is obtained.

WASHINGTON, March 8, 1912.

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DEPARTMENT OF COMMERCE AND LABOR

Vol. 8 No. 4

BULLETIN

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

MARCH 1, 1913

CONTENTS

		Page
Standardization of Potassium Permanganate Solution by Sodium Oxalate	*R. S. McBride*	611
Benzoic Acid as an Acidimetric Standard	*George W. Morey*	643
A Tubular Electrodynamometer for Heavy Currents	*P. G. Agnew*	651
Thermometric Lag	*D. R. Harper, 3d*	659
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STANDARDIZATION OF POTASSIUM PERMANGANATE SOLUTION BY SODIUM OXALATE

By R. S. McBride

CONTENTS

	Page
I. Introductory Part	612
1. Object of the research	612
2. Considerations affecting the choice of a standard	612
3. Reasons for the choice of sodium oxalate	613
4. Normal course of the reaction	614
II. Experimental Part	616
1. Reagents employed	616
2. Weighing of oxalate and permanganate used	617
3. Effect of conditions upon the result of a titration	618
(a) End-point corrections	619
(b) Rate of addition of the permanganate	621
(c) Volume of the solution	624
(d) Acidity and temperature	624
(e) Air access—oxidizing effect	628
(f) Presence of added manganous sulphate	631
III. Discussion and Conclusions	632
1. Review of possible errors	632
(a) Loss of oxalic acid by volatilization	633
(b) Decomposition of oxalic acid by water	633
(c) Decomposition of oxalic acid by sulphuric acid	633
(d) Oxidation of oxalic acid by the air	634
(e) Incomplete oxidation of the oxalic acid	636
(f) Loss of oxygen	637
(g) Incomplete reduction of the permanganate	639
(h) Impurities in the oxalate	639
(i) Abnormal oxidation products	639
2. Summary and conclusions	640
3. Method recommended for use	641
4. Accuracy and precision attainable	641

I. INTRODUCTORY PART

1. OBJECT OF THE RESEARCH

The standardization of potassium permanganate solutions has been the subject of much study and an excessive amount of controversy. There have been far too many standards proposed for this work to permit one not familiar with the subject to select the best; and, in fact, it is doubtful whether or not any one of those standards proposed can in all senses be considered the best. However, the work of the Bureau of Standards has demanded that some substance be selected for this use which could be employed with a certainty of a reasonably correct result. It was desired, if possible, that the standard selected should serve a threefold purpose, viz: First, as a primary standard of oxidimetry; second, as a working standard for regular use in our own laboratories; and third, as a substance which could be distributed by the Bureau with a guarantee both as to its purity and as to its reducing value when used under specified conditions.

Although the voluminous literature relating to the standardization of potassium permanganate solutions has been examined with considerable care, it is not thought desirable to give a history of the subject, or even a bibliography. None of the theories considered here are new; but it is hoped that the experimental facts presented will be of value as a guide to the proper use of sodium oxalate as a standard.

2. CONSIDERATIONS AFFECTING THE CHOICE OF A STANDARD

It is seldom that a single measure can be used both as a primary reference standard and as a regular working standard, but for some volumetric work this is possible. A substance to serve any such double function must be such that the following conditions will be fulfilled:

(*a*) Reasonable ease of preparation and accurate reproducibility of material must be assured.

(*b*) The purity must be determinable with sufficient accuracy, and the purified material must be stable under ordinary conditions of the laboratory.

(*c*) The use of the material in regular work must demand neither complex apparatus nor difficult manipulations.

(*d*) Such precision must be obtainable when it is used with ordinary care that one, or at the most a very few determinations suffice for the fixing of the value of a standard solution.

(*e*) The accuracy obtained under ordinary conditions of its use in standardization must be at least as great as that required in the use of the solution to be standardized.

3. REASONS FOR THE CHOICE OF SODIUM OXALATE

In the above six respects it appeared that sodium oxalate was probably best suited to our needs, and a detailed study of this standard was undertaken.

The methods of preparation and testing of sodium oxalate have been carefully studied by Sörensen.[1] In continuation of such study, Mr. J. B. Tuttle and Dr. William Blum, of this laboratory, have carried out several important lines of work and, at the request of the Bureau of Standards, several firms of manufacturing chemists have improved their methods of preparation of sodium oxalate on a large scale. At this point it is sufficient to state that all of this work indicates that it can be prepared in a suitable form at reasonable expense; that it is reproducible; that its purity can be tested readily; and that once purified it is satisfactorily stable under ordinary conditions.[2]

The discussion of the other three criteria as to the value of sodium oxalate, viz, convenience, precision, and accuracy, forms the subject of the present article.

In advance of the general discussion it is not amiss to state the conclusions drawn as to these three points. It appears that when the conditions for the use of sodium oxalate have been defined they may be conformed to easily and no unusual apparatus or complex procedure is necessary. The accuracy obtainable is, within the limits of our present knowledge, sufficient for even the most refined work (see p. 641), and the precision or agreement of duplicates is satisfactory.

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[1] Zs. anal. Chem., **36**, p. 639, **42**, p. 333, and p. 512.

[2] See Blum, J. Amer. Chem. Soc., **34**, p. 123; 1912.

Along with such striking advantages we find certain disadvantages, but none of these appear serious. The more important are as follows:

(*a*) The largest sample which can be used ordinarily (for 50 cc of N/10 $KMnO_4$) is only 0.3 gram. This necessitates an accurate weighing in order to gain a high degree of accuracy. It is not desirable to use a standard stock solution, unless freshly prepared. If such solution is kept for a considerable length of time, it acts upon the glass of its container; it is also slowly decomposed by the action of light.

(*b*) The detection of small amounts of allied organic compounds in the sodium oxalate is rather difficult, except by comparison of the reducing value with that of other samples of known purity.

(*c*) The initial drying of the oxalate is subject to slight uncertainty, but once dried it is practically nonhygroscopic.

4. NORMAL COURSE OF THE REACTION

In 1866 Harcourt and Esson [3] concluded from a study of the speeds of reaction under various conditions, that the steps of the reaction are as follows:

I. $2\,Mn(OH)_7+5H_2C_2O_4=2Mn(OH)_2+10CO_2+10H_2O$ (very slow)
II. $3Mn(OH)_2+2Mn(OH)_7=5Mn(OH)_4$ (very fast)
III. $Mn(OH)_4+H_2C_2O_4=Mn(OH)_2+2CO_2+2H_2O$ (fast, but less so than II.)

Among the more recent articles presenting either experimental or theoretical evidence on this subject, the more important are those by Schillow [4] and by Skrabal.[5]

From measurements of the speeds of reaction, the former presents the following system as representing the steps of the reaction:

I. $Mn(OH)_7+2H_2C_2O_4=Mn(OH)_3+4CO_2+4H_2O$ (very slow)
II. $Mn(OH)_3.2H_2C_2O_4+Mn(OH)_7=2Mn(OH)_3+4CO_2+4H_2O$ (measured)
III. $Mn(OH)_3+2H_2C_2O_4=Mn(OH)_3.2H_2C_2O_4$ (practically instantaneous)

The second of these can be divided into two parts:

II,a. $Mn(OH)_3.2H_2C_2O_4+Mn(OH)_7=Mn(OH)_6+Mn(OH)_4.2H_2C_2O_4$ (measured)
II,b. $Mn(OH)_4.2H_2C_2O_4+Mn(OH)_6=Mn(OH)_3+4CO_2+4H_2O$ (practically instantaneous)

[3] Philos. Trans., 156, p. 193; 1866. [4] Ber., 36, p. 2735; 1903. [5] Zs. anorg. Chem., 42, p. 1; 1904.

However, this author qualifies the proposed explanation by the following statements:

"This scheme is considered only as an approximate picture of the phenomenon and is true particularly for mean concentration of hydrogen ions and low temperatures (0–25°). Under other conditions, side reactions and disturbances enter in which can be partially observed or foreseen. * * *

"In addition to the reactions given above, the following two also require consideration:

IV. $2Mn(OH)_3+H_2C_2O_4=2Mn(OH)_2+2CO_2+2H_2O$
V. $4Mn(OH)_2+Mn(OH)_7=5Mn(OH)_3$"

These last reactions, however, are not assigned a definite rôle in the general scheme.

After an extended series of experiments on the speed of the reaction under various conditions, Skrabal (loc. cit.) advances the following scheme as representing the course of the reaction.

Incubation period:

(1) $H_2C_2O_4+KMnO_4=Mn^{\cdots}+CO_2$ (measurable)
(2) $H_2C_2O_4+Mn^{\cdots}=Mn(OH)_2+CO_2$ (practically instantaneous)

Induction period:

(3) $Mn(OH)_2+KMnO_4=Mn^{\cdots}$ (less rapidly)
(4) $Mn^{\cdots}+H_2C_2O_4=Mn(OH)_2+CO_2$ (practically instantaneous)
(5) $Mn^{\cdots}+H_2C_2O_4=Mn(OH)_3.H_2C_2O_4$ (practically instantaneous)
(6) $Mn^{\cdots}=Mn(OH)_2+Mn(OH)_4$ (practically instantaneous)

End period:

I. (7) $Mn(OH)_3.H_2C_2O_4=Mn^{\cdots}$ (measurable)
(8) $H_2C_2O_4+Mn^{\cdots}=Mn(OH)_2+CO_2$ (practically instantaneous)

II. (9) $Mn(OH)_2+Mn(OH)_4=Mn^{\cdots}$ (less rapid)
(10) $H_2C_2O_4+Mn^{\cdots}=Mn(OH)_2+CO_2$ (practically instantaneous)

Since this last system of reactions is based upon elaborate experimental work, it can be accepted as probably best representing the normal course of the reaction. For the present considerations any one of the systems would serve equally well to explain the observed facts.

The variations from a normal course will be considered after the experimental part of the present work has been described.

II. EXPERIMENTAL PART

1. REAGENTS EMPLOYED

The water and sulphuric acid employed were frequently tested for reducing matter and in each case were shown to be free from appreciable amounts of such impurities. The manganous sulphate was similarly tested for reducing and oxidizing influence and shown to be satisfactory. To purify the air used for the tests of Tables VI and VII, it was passed through cotton wool to remove dust and grease, bubbled through a solution of potassium hydroxide to take out any acid vapors present, and then through solutions of chromic acid and potassium permanganate. The purified air had no detectable reducing effect upon dilute solutions of permanganate under the conditions of titration.

The carbon dioxide used for the tests of Table VI was taken from a cylinder of the commercial liquid and passed through water, chromic acid, and permanganate solution before use. An analysis showed the presence of about 3 per cent of methane; but this gas seemed to have only a very slight reducing action on the permanganate in dilute solution, so that the results of the series in which it was used can be regarded as but slightly less reliable than the series in Table VII. For this latter group of tests, pure carbon dioxide was made from acid and soda. This source gave a gas which had no detectable reducing action under the conditions of its use.

The potassium permanganate used for most of the work was a sample of good quality which had been made up in normal solution for over six months before filtration and dilution to tenth normal strength for use. The diluted solution was filtered frequently through asbestos to insure freedom from precipitated manganese dioxide. For the series of tests reported in Table IV, b, a second permanganate was used. In this case the strong solution was boiled for a few minutes, cooled, filtered, and diluted to tenth normal strength. For the series of Table VII still a different permanganate was employed, this stock being prepared in the same manner as the main solution. About 40 grams of solution were used for each titration.

The sodium oxalate used for all of the experiments except those of Table IV, b, was a sample specially purified in this laboratory by Mr. J. B. Tuttle. It is sufficient to state here that all tests indicated a total impurity of not over 0.05 per cent. For the tests of Table IV, b, another sample of sodium oxalate, made by the Mallinckrodt Chemical Works especially for our use, was employed. This material was shown by test to be as pure as that prepared in our own laboratory.

2. WEIGHING OF OXALATE AND PERMANGANATE USED

Since in all of the work described in the following part of this article only comparative values were required, the absolute amount of oxalate employed in any one test was not of importance as long as the relative amounts present in the experiments of a series were accurately known. Therefore it was found desirable to use about 20-gram portions of a N/5 stock solution of the oxalate for each test. Each such sample was weighed from a burette to the nearest 5 mg; the relative wieght of each sample was thus determined to better than 1 part in 2000 in less time and with greater certainty than would have been possible by weighing out the dried powder.

In order to prevent any change in the strength of the stock solution from affecting the conclusions drawn from the results of any series of titrations, only those values obtained during a period of a few days are compared with each other. This plan avoided any uncertainty due either to slow decomposition of the oxalate in solution occurring through action on the glass of the container, or oxidation by the air through the aid of light, or to change in oxidizing value of the permanganate solution employed.

The measurement of the permanganate solution was accomplished by the use of a weight burette. For this work this form of instrument possesses the following advantages: (*a*) Correction for the temperature changes which affect the volume of the solution is not necessary; (*b*) completeness or uniformity of running down of the solution from the burette walls is unessential; and (*c*) the solution can be weighed readily to 0.01 g (1 part in 5000 on a 50-g sample), whereas measurement to 0.01 cc is exceedingly

uncertain. The freedom from errors due to temperature changes is of great importance in this work, as the burette is often very appreciably warmed by the steam rising from the hot liquid undergoing titration.

The burette used was made from a 50 cc cylindrical separatory funnel by drawing down the stem to the form of an ordinary burette tip. For exact work it has been found desirable to have this tip so drawn down that it will deliver about 10 cc per minute or 0.03 cc per drop.

3. EFFECT OF CONDITIONS UPON THE RESULT OF A TITRATION

The ordinary procedure for the use of sodium oxalate in the standardization of potassium permanganate, is as follows: Dissolve about a quarter of a gram of the oxalate in 250 cc of water, acidify with sulphuric acid, warm to 70°, and titrate to the first permanent pink.

It was desirable to determine the effect of the variation of the following conditions upon the result obtained, viz, temperature, acidity, volume of solution, rate of addition of the permanganate, access of air, presence of added manganous sulphate and in connection with these, the corrections necessary upon the apparent end points. In order to accomplish such determination, the factors were varied one at a time, noting the difference, if any, produced upon the apparent value of the permanganate. The results of the titrations are reported as the ratios of the oxalate used to the permanganate used multiplied by 10^{-2}. These values are proportional to, and, indeed, numerically almost equal to, the iron value of the permanganate. Therefore, for purposes of discussion the values are treated as if they were the iron value of the permanganate, expressed in grams of iron per gram of solution. It should be noted that an increase in the iron value represents a decrease in the permanganate consumed, and vice versa. A variation of 0.01 cc in the amount of permanganate used in a titration is approximately equivalent to a change of one unit in the last expressed figure of the iron value.

(a) END-POINT CORRECTIONS

In a recent article Dr. W. C. Bray [6] has suggested the necessity for the correction of the apparent end point obtained in the reaction of oxalic acid and permanganate, and has recommended that the correction be determined as follows: After reaching the end point the solution is cooled to room temperature, potassium iodide solution is added, and the iodine liberated is at once titrated with dilute (N/50) sodium thiosulphate.

In order to test the necessity and the accuracy of this method under the various conditions of titration, we examined experimentally the following points:

(*a*) Does the equivalent in oxidizing power of the permanganate excess remain in the solution long enough to allow the necessary cooling before the titration with thiosulphate?

(*b*) Does the thiosulphate titration give the total permanganate excess used to produce the end point, or does it indicate only the permanganate which remains in the solution as such?

(*c*) Does the depth of the pink color at the end of the titration show how great an excess of permanganate has been added in order to produce that end point?

Before applying the method of correction proposed by Bray, it is necessary to cool the solution to room temperature. Therefore, if, as is desirable, the permanganate end point is obtained while the solution is hot, some time must elapse during cooling and it is possible that a loss of oxidizing (or iodine liberating) power would occur, which loss would render the correction as subsequently determined valueless. This point was tested by determining the residual oxidizing power of solutions to which had been added known amounts of permanganate at various temperatures (30–90°), with varying acidities (2–10 per cent by volume H_2SO_4) and with the addition of various amounts of manganous sulphate (up to 1 g $MnSO_4.4H_2O$). These experiments showed that no appreciable loss occurred within a period of one-half hour, if only a small

[6] J. Amer. Chem. Soc., 32, p. 1204; 1910.

amount of permanganate (0.02–0.20 cc of N/10 solution) is added to any such solution. Moreover this condition was found to exist even when the permanganate was wholly decolorized by reaction with the manganous salt. The full oxidizing power was then retained in the solution, probably in the tri- or tetra-valent manganese salts ($Mn^{\cdots}$ or $Mn^{\cdots\cdot}$). The lapse of 5 to 10 minutes necessary for cooling the solution to room temperature, is thus shown to be without influence; and it is evident that the procedure described determines not only the permanganate remaining in the solution as such, but also that which was not completely reduced to the manganous condition.

The procedure described by Bray has been accepted on the above basis as giving a determination of the real excess of permanganate which was added to produce the end point. However, the question as to the difference between this end-point correction and that determined by the depth of the pink color at the end point, must be considered on a different basis.

For each of the titrations made in this investigation, two values were obtained, the one using the color method, the other the iodine method for correction of the total permanganate added. In several of the following tables both of these values are reported. By inspection of these data (Tables II, III, and V) it will be apparent that, in general, the color correction and the iodine correction differ by no more than 0.02 cc, except when the end point was reached with the solution at temperatures below 35°, or when the permanganate solution was added rapidly just at the end point. This was true in all but 6 of over 200 tests; and even in these the difference between the two corrections was not more than 0.04 cc. It is therefore certain that if the end point is approached slowly in a solution above 60°, the depth of color will be proportional to the total excess of permanganate added, i. e., no permanganate will be decolorized without at the same time being completely reduced to the manganous condition. However, at lower temperatures, in the presence of much sulphuric acid (more than 5 per cent by volume), and particularly with

rapid addition of permanganate or with insufficient stirring just preceeding the end point, the amount of permanganate decolorized but not wholly reduced to the manganous condition, easily may be as much as 0.1 cc.

(b) RATE OF ADDITION OF THE PERMANGANATE

The rate of the addition of the permanganate solution in the titration of iron in solutions containing chlorides has been shown to have considerable effect upon the result obtained.[7] The influence of this factor upon the oxalate-permanganate reaction was, therefore, studied.

The influence of varying the amount of permanganate added before the quick color change of rapid reduction began was first tested. This factor has an influence only at low temperatures (below 40–50°), as at any higher temperature the reduction proceeds rapidly from the very start. Tests made at 30° in 5 per cent sulphuric-acid solution gave results as follows:

Amount of $KMnO_4$ added before rapid decolorization was evident..	3 cc	10 cc	25 cc	40 cc
Results obtained (strength of $KMnO_4$).....	0.005936	0.005936	0.005932	0.005929
	37	35	33	
	34	37		

These results show that the addition of large amounts of permanganate before rapid oxidation of the oxalate begins tends to give lower values for the permanganate solution, i. e., to cause a larger consumption of permanganate. The influence of this factor is small, but it is of some significance, as will be apparent from the later discussion.

[7] A. Skrabal: Zs. anal. Chem., 42, p. 359; 1903. A. N. Friend: J. Chem. Soc. (London), 95, p. 1228; 1909. Jones and Jeffrey, Analyst, 34, p. 306; 1909.

TABLE I

Effect of Rate of Addition of Permanganate During Main Part of Titration

Experiment No.	Acidity, volume per cent H_2SO_4	Temperature	Time for titration (minutes)	Result, corrected iron value
118	5	80°	5	0.005945
120			5	45
119			[a] 27	46
125			[b] 120	56
129			[b] 120	56
121	5	30°	7	0.005942
123			6	42
122			[a] 37	44
124			[b] 120	44
230	2	30°	0.5	0.005592
231			0.5	92
206			5	92
223			4	93
207			5	91
225			[a] 60	94
226			[a] 60	95
232	2	90°	0.5	0.005589
233			0.5	88
208			5	98
209			5	5600
227			[a] 30	00
229			[a] 30	5598

[a] Dropwise with stirring.
[b] Intermittent, without continuous stirring.

After the initial reduction of 3–10 cc of permanganate the rate of addition of the main portion of the permanganate was varied, giving the results listed in Table I. These experiments indicate that except on long standing at a high temperature (after starting the reaction), or with a very rapid addition of permanganate to a weakly acid solution at a high temperature, the rate of the addition of the permanganate during the main part of the titration has no influence. The high values obtained in experiments 125 and 129 are probably due to oxidation of the oxalic acid by the air, and the low values of experiments 232 and 233 show a loss of oxygen, because of the excessive rate of the permanganate addition.

The effect of the rate of the addition of the permanganate just at the end point, has already been suggested (p. 620), but the following data will make this point clearer. In each pair of tests the only variable was the rate of the addition of the permanganate solution just before the end point. Only the two experiments in each group may be compared. (See Table II.)

In the first part of Table II some rather extreme cases (experiments done at 30°) have been chosen to show how large the error of the apparent end point may be, without any influence upon the corrected value. In cases where much smaller corrections were necessary (temperature 60° and above) the agreement of uncorrected values was better, regardless of the speed at which the end point was approached. The latter part of Table II shows this agreement of the uncorrected values at the higher temperature. Nevertheless, it is not desirable to approach the end point so rapidly as to make a correction larger than 0.1 cc necessary.

TABLE II

Effect of Rate of Addition of Permanganate at the End Point

[In the first of each pair of experiments the end point was approached slowly, in the second rapidly]

Temperature	Acidity, volume per cent H_2SO_4	Color blank	Iodine blank	Value of permanganate		
				Uncorrected	Color corrected	Iodine corrected
30°	2	0.01	0.03	0.005584	0.005586	0.005589
		.25	.34	37	71	87
30°	5	.03	.09	.005929	.005933	.005942
		.15	.39	5894	12	42
30°	5	.05	.085	.005030	.005034	.005037
		.08	.19	17	26	38
30°	5	.04	.08	.005923	.005929	.005935
		.—	.51	5871	—	37
30°	10	.06	.11	.005636	.005642	.005648
		.08	.14	33	43	50
60°	2	.05	.05	.005642	.005649	.005649
		.03	.05	45	49	51
60°	5	.05	.07	.005638	.005646	.005649
		.05	.06	40	44	46
83°	2	.06	.07	.005642	.005650	.005651
		.06	.06	45	52	52
96°	2	.05	.04	.005647	.005652	.005651
		.2	.15	33	61	54

(c) VOLUME OF THE SOLUTION

The volume of the solution in which the titration is made might affect the rate of the reaction and, therefore, possibly the result of the titration. Tests were made of this point by varying the initial volume in a series of titrations made at 90° in solutions containing 5 per cent by volume of sulphuric acid. The results were as follows:

Initial volume	50 cc	250 cc	500 cc	700 cc	1000 cc
Values obtained	0.005139	0.005139	0.005136	0.005133	0.005127
	36	39	34	32	29
		37			

By increasing the initial volume beyond 250 cc, the resulting value for the permanganate solution is markedly decreased. That this is due to the change in the initial concentration of the oxalate is indicated by the results obtained in two titrations in which the initial volume was 1000 cc but the amount of oxalate used was increased threefold, thus making its initial concentration about that ordinarily resulting in an initial volume of 350 cc. The results thus obtained were 0.005136 and 0.005136.

At 30° the initial rate of the permanganate reduction is so much less than at 90°, that it is not possible to note the effect of initial volume free from this influence. A few results were obtained as follows:

Initial volume	50 cc	250 cc	1000 cc
Values obtained	0.005943	0.005942	0.005945
	45	42	

In these experiments no decided effect of volume is to be found.

It should be noted that in each of these cases the change in volume had no effect upon the concentration of the sulphuric acid present.

(d) ACIDITY AND TEMPERATURE

The effects of acidity and of temperature were investigated by two series of experiments, the results of which are given in Tables III and IV. These data show that either higher temperature or less concentration of sulphuric acid tends to give higher values for the permanganate, i. e., to reduce the amount of permanganate required. These effects are not large and, indeed, the results

obtained in some cases show duplicates as discordant as experiments carried out under somewhat different conditions; nevertheless, the tendency in the two directions is rather convincing that these factors have a real influence upon the results obtained. The importance of even such slight variations is considerable, as will be apparent from the discussion later.

It will be seen from the data of Table IV that the first series exhibits a larger apparent influence of temperature than the second series reported. This is due to the greater care exercised in the latter series to stir the solution very vigorously and to add the permanganate more slowly, particularly at the beginning and end of each titration. However, the first series shows better the variation which may be expected if only ordinary titration precautions are observed.

TABLE III

Effect of Acidity and Temperature

Temperature	Acidity, volume per cent H_2SO_4	Color blank	Iodine blank	Value of permanganate		
				Uncorrected	Color corrected	Iodine corrected
30°	2	0.04	0.06	0.005129	0.005135	0.005137
	2	.04	.05	33	38	39
	5	.04	.05	27	32	34
	5	.03	.035	27	31	31
	10	.05	.10	18	25	32
	10	.02	.09	21	24	33
45°	2	.02	.03	.005134	.005136	.005138
	2	.03	.04	34	38	40
	5	.03	.04	30	34	35
	5	.03	.025	30	37	36
	10	.01	.03	35	36	39
	10	.03	.03	31	35	35
	10	.02	.035	37	40	42
60°	2	.02	.02	.005129	.005132	.005132
	2	.03	.03	36	40	40
	2	.035	.035	33	37	37
	2	.04	.05	35	41	42
	5	.04	.06	26	32	46
	5	.05	.05	32	39	39
	10	.05	.05	33	40	40
	10	.03	.05	33	37	40
	10	.02	.04	34	38	40

TABLE III—Continued

Effect of Acidity and Temperature—Continued

Temperature	Acidity, volume per cent H_2SO_4	Color blank	Iodine blank	Value of permanganate		
				Uncorrected	Color corrected	Iodine corrected
75°	2	0.03	0.02	0.005141	0.005144	0.005143
	2	.03	.03	37	41	41
	5	.04	.025	37	42	40
	5	.03	.03	39	43	43
	10	.05	.06	29	36	37
	10	.06	.04	34	43	40
80°	2	.05	.045	.005138	.005145	.005144
	2	.02	.03	36	39	40
	2	.04	.05	36	42	43
	5	.03	.05	38	42	44
	5	.04	.04	39	44	44
	10	.03	.03	34	38	38
	10	.03	.03	36	40	40
86°	2	.04	.03	.005142	.005146	.005144
	2	.02	.03	40	43	43
	5	.03	.035	38	42	43
	5	.02	.02	37	40	40
	10	.02	.02	38	41	41
	10	.03	.045	35	39	41
92°	2	.02	.03	.005143	.005146	.005148
	2	.02	.03	46	48	49
	5	.035	.04	39	43	44
	5	.05	.05	32	39	39
	5	.05	.06	30	37	39
	5	.06	.06	31	39	39
	10	.05	.05	32	38	38
	10	.06	.06	30	38	38

TABLE IV

Effect of Acidity and Temperature

(a) SERIES 1, REARRANGEMENT OF DATA OF TABLE III

[Each value reported is the average of at least two determinations]

Temperature	Acidity, volume per cent H_2SO_4		
	2 per cent	5 per cent	10 per cent
30°	0.005138	0.005133	0.005132
45°	39	35	38
60°	38	42	40
75°	42	42	38
80°	42	44	39
86°	44	42	41
92°	48	40	38

(b) SERIES 2

30°	0.005646	0.005646	0.005649
60°	50	48	48
80°	..	50	..
85°	51	50	44
92°	..	50	..
95°	53	52	46

In addition to the data of Tables III and IV, those in other tables will show the same influence of temperature and acidity as are here indicated. Taking these data all together, it is certain that, though small, these influences are appreciable.

The time in the titration at which the tempertaure has an influence upon the result is evident from the tests of Table V. These data show that, except in its influence upon the size of the end-point correction, the temperature of the solution after the first few cc of the permanganate have been decolorized is without appreciable effect. The significance will be apparent when considered in connection with the results of the tests given in Table VIII.

TABLE V

Time at which Temperature has Effect

[All titrations started in 250 cc of 5 per cent by volume sulphuric acid]

Temperature: (a) Initial, (b) during main part of titration, (c) at end point			Color blank	Iodine blank	Value of permanganate		
					Uncorrected	Color corrected	Iodine corrected
(*a*)	(*b*)	(*c*)					
90°	90°	90°	0.05	0.06	0.005130	0.005137	0.005139
			.06	.06	31	39	39
90°	90°	30°	.03	.07	29	33	38
			.03	.065	30	34	39
90°	30°	30°	.03	.09	25	29	37
			.04	.12	19	24	35
30°	30°	30°	.03	.05	.005126	.005130	.005133
			.05	.08	24	32	36
			.04	.085	22	27	32
30°	30°	90°	.05	.06	25	31	33
			.07	.09	25	34	36
			.05	.05	25	30	30
			.04	.05	29	34	34
30°	90°	90°	.03	.03	29	34	34
			.03	.03	31	35	35

(e) AIR ACCESS—OXIDIZING EFFECT

It has been suggested [10] that during the course of the reaction atmospheric oxygen, through the carrying action of the manganous salt, might cause the oxidation of appreciable amounts of oxalic acid. To test this point, a preliminary series of experiments was made as to the results obtained on the removal of the larger part of the air from the titration vessel. To accomplish this, the solution to be titrated was made up in a flask with recently boiled water and a small amount of sodium carbonate was added just before starting the titration. The carbon dioxide evolved carried out the bulk of the air, and the flask was kept stoppered and the permanganate run in through a small funnel, which passed through the stopper. From the results it appeared that no oxidizing action of the air was to be feared, but to further test this point the following method was employed:

[10] Who first suggested this is not known; Schröder has tested this point recently (Zs. öffentliche Chem., 16, p. 270; 1910). See p. 634 for discussion of his data.

The solution to be titrated was placed in a glass-stoppered flask of special form (see Fig. 1), and a strong current of air or carbon dioxide, as desired, was bubbled through the liquid before commencing the titration and during its progress. When carbon dioxide was used, the solution was heated to boiling and then cooled to the desired temperature in a stream of the gas, thus insuring the absence of all but the last traces of air.

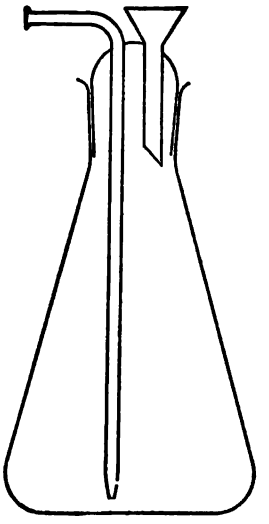

Fig. 1

The two series of tests made in this manner are summarized in Tables VI and VII.

TABLE VI

Effect of Access of Air on Titration

[All titrations started in 250 cc of 5 per cent by volume sulphuric acid]

Temperature	Titration in flask with air bubbled through solution	Titration in flask with carbon dioxide [11] bubbled through solution	Titration in beaker open to air, as in ordinary procedure
30°	0.005041	0.005037	0.005040
	40	38	39
60°	42	40	41
	42	43	40
90°	42	47	41
	41	51	44

[11] The carbon dioxide used in the experiments of Table VI was found to contain 3 per cent of methane. See p. 616.

TABLE VII

Effect of Access of Air on Titration

[All titrations started in 250 cc of 2 per cent by volume sulphuric acid]

Temperature	Titration in flask with air bubbled through solution	Titration in flask with carbon dioxide bubbled through solution	Titration in beaker open to air, as in ordinary procedure
30°	0.005591	0.005592	0.005591
	91	93	92
60°	94	99	97
	94	99	97
90°	94	5600	98
	95	01	5602

The data of Tables VI and VII show that no large error comes from atmospheric oxidation of the oxalate during titration. The largest difference between results obtained using air and using carbon dioxide are only 1 part in 500 (0.2 per cent), and the differences were usually less than one-half this amount. Moreover, if carried out in a beaker, as ordinarily done, the values obtained are not appreciably different from those where air is wholly excluded (cf. last two columns of Table VII); and even when air is bubbled through the solution during a titration at 90° the oxidation is scarcely appreciable. No such chance for atmospheric action is met with under the ordinary conditions of titration, since when following the usual procedure the carbon dioxide formed during the reaction prevents much oxygen from the air remaining in the solution during the reaction.

One other point tested was the effect of time elapsing before the beginning of the titration. Four samples were made up to 250 cc with 5 per cent by volume sulphuric acid and heated to 90°. Two were titrated at once, the other two after standing an hour at 90°. The four results agreed within 1 in 5500, showing that even under these rather severe conditions oxalic acid is sufficiently stable and nonvolatile to prevent loss during a titration from oxidation (in the absence of manganous salts, at least) from decomposition by the sulphuric acid, or from volatilization with the steam.

As a further proof that atmospheric oxidation is not an important factor in the oxalate titration, the effect of the rate of titration may be cited. In the data of Table I the titrations covering a period up to one hour show almost no tendency to give higher values. This is contrary to what might be expected if oxidation were taking place, as the oxidation occurring would naturally be, at least roughly, proportional to the time elapsed, and such is not the case. Experiments 125 and 129 are the only exceptions met with. (See Table I.)

(f) PRESENCE OF ADDED MANGANOUS SULPHATE

Since the presence of added manganous sulphate would probably influence the initial rate of the reaction, its effect upon the result of the titration was tested. A series of tests (all but the last two at 30°) was carried out with the addition of manganous sulphate at different points in the course of the reaction. One cc of the solution of manganous sulphate added was equivalent to the manganous salt formed by the reduction of 10 cc of the permanganate solution. The results of these experiments are given in Table VIII.

It appears from these data that unless the $MnSO_4$ be added before the beginning of the titration it has little effect upon the result obtained. However, if even a small amount is added before starting the reaction at 30°, the value obtained is the same as is ordinarily obtained at 90° without the addition of manganous salt. Addition before titration at 90° has no influence.

TABLE VIII

Effect of Added Manganous Sulphate

[All but last two tests at 30°, last two at 90°; all started in 250 cc of 5 per cent by volume sulphuric acid]

Amount $MnSO_4$ solution added	Time of addition	Value of $KMnO_4$ found
		0.005576
None		77
		77
		79
1 cc	Before start of titration	.005582
		81
20 cc		.005583
		83
1 cc	After decolorization of 10 cc of $KMnO_4$	.005578
		77
20 cc		.005578
		80
1 cc	Just before end point	.005579
		81
20 cc		.005578
		78
None		.005582
		83

III. DISCUSSION AND CONCLUSIONS

1. REVIEW OF POSSIBLE ERRORS

As the standardization procedure consists in solution of a weighed sample in dilute sulphuric acid and direct titration with permanganate, any error occurring must be due either to error in weighing of oxalate or measuring of the permanganate or to a variation of the reaction itself from the form usually given, viz:

$$5\ H_2C_2O_4+2KMnO_4+3H_2SO_4=K_2SO_4+2MnSO_4+10CO_2+8H_2O$$

Since, as already indicated (p. 617), there is no necessity for an error in either the weighing of the oxalate or the measuring of the permanganate greater than 1 part in 2000, only the irregularities of the reaction need be considered at length.

The variations of the reaction from its normal course may tend to cause the use of an excess of either permanganate or oxalate,

or both sorts of influences may operate at the same time, in equal or unequal degree. There are nine possibilities of such irregularity; the first five of these would cause too small a consumption of permanganate, the next two an excessive use of it, and the last two might cause either of these effects. These possible sources of error are: (*a*) Loss of oxalic acid by volatilization from the solution; (*b*) decomposition of oxalic acid by water; (*c*) decomposition of oxalic acid by sulphuric acid; (*d*) oxidation of oxalic acid by contact of the solution with the air; (*e*) presence of oxalic acid unoxidized at the end point; (*f*) liberation of oxygen during the reaction; (*g*) presence at the end point of unreduced permanganate or other compounds of manganese higher than manganous; (*h*) presence of impurities in the oxalate, either of greater or less reducing power power than the oxalate; (*i*) formation of other products of oxidation than carbon dioxide and water.

(a) LOSS OF OXALIC ACID BY VOLATILIZATION

The fact that oxalic acid is readily volatile alone suggests the danger of loss from its solution by vaporization with steam. However, from the dilute solutions employed for titration purposes, no such losses occur, as has already been shown. The results obtained after an hour's standing at 90° in 5 per cent sulphuric acid solution showed (p. 630) that no loss of oxalic acid had occurred. Similarly when titration extended over a period of one hour (Table I, p. 622) or when carbon dioxide or air was bubbled through the solution (Tables VI and VII, pp. 629–630) no such higher values of the permanganate were obtained as would have resulted had volatilization of the oxalic acid occurred. From these results it is apparent that no appreciable vaporization of oxalic acid will take place during the course of an ordinary titration which lasts, at the most, only 5 to 10 minutes.

(b), (c) DECOMPOSITION OF OXALIC ACID BY WATER OR BY SULPHURIC ACID

The same data which have shown that no volatilization occurs also prove that the water and sulphuric acid had not caused a decomposition of the oxalic acid. These possible errors can,

therefore, be considered improbable under any ordinary conditions of titration.

The experiments reported by Carles [12] from which he was led to state that at 100° oxalic acid in sulution is slowly decomposed with the formation of carbon dioxide and formic acid are not confirmed by our experiments. If such decomposition does take place, it is so slow as to have no appreciable influence upon the results of a titration made at 90° or below.

(d) OXIDATION OF THE OXALIC ACID BY THE AIR

Schroeder [13] has shown that in the presence of manganous sulphate oxalic acid solutions are oxidized by the air at an appreciable speed. This author presents a large amount of experimental data, from which he draws very decided conclusions. Those which are of importance in the present discussion are as follows (numbered as in original):

(3) In the oxidation of oxalic acid by potassium permanganate, an error (too little $KMnO_4$ used) enters through the presence of atmospheric oxygen, which error is greater in the presence of manganous sulphate or more especially titanium dioxide.

(5) By rapid titration in a stream of oxygen this error of No. 3 disappears, the oxygen acting on the oxalic acid to give hydrogen peroxide, which requires a corresponding amount of permanganate.

(6) In the presence of much manganous sulphate, with oxalic acid of high concentration, especially in the presence of titanium dioxide, this hydrogen peroxide of No. 5 disappears, very rapidly if hot, and the error of No. 3 is then evident, as the compensating action of the hydrogen peroxide is not possible.

(7) Rapid titration at 50° with 30 cc of 1 : 1 sulphuric acid in 200 cc of water gives results which agree with the iodine method of standardization.

Schroeder therefore recommends that the greater part of the permanganate be rapidly run into a strongly acidified solution of the oxalate and then the titration finished more slowly. Addition of manganous sulphate is not recommended.

The experimental data presented by Schroeder which are of importance to this discussion are those in his tables numbered 3, 4, 5, 6, 10, and 11. These data, with very few exceptions, seem to warrant the following conclusions (not expressed by Schroeder, but drawn from his data):

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[12] Bull. Soc. Chim., 14, pp. 142–144; 1870.

[13] Zs. öffentliche Chem., 16, p. 270; 1910.

(*a*) High temperature tends to cause a smaller permanganate consumption. (Tables 3, 4, 5, and 6.)

(*b*) High acidity tends to cause a high permanganate consumption. (Tables 3, 4, 5, 10, and 11.)

(*c*) Slow titration (10 minutes) requires less permanganate than fast (1 to 1½ minutes) titration.

(*d*) Presence of added manganous sulphate (1 to 5 g) tends to cause less permanganate to be used, even with rapid titration. (Tables 5 and 6.)

(*e*) Titration in an air current required slightly more permanganate than in a current of carbon dioxide, when made either rapidly or slowly or in the presence of large or small amounts of acid.

It must be noted that in his experiments the stirring was very ineffective, since Schroeder often refers to his titration as "frequently whirled" (umgeschwenkt) and "whirled from time to time"; and no end-point corrections were made even when the titration was conducted rapidly, at low temperature, in the presence of large amounts of acid and manganous salt.

These conclusions as to the influence of conditions are confirmed, in general, by the data obtained in our own work, but it does not seem that the conclusions expressed by Schroeder (p. 634) necessarily follow from the data which he presents. In fact, from his own experimental results, we would infer, as stated by us in our conclusion (*e*) (second preceding paragraph), that the error due to atmospheric oxidation claimed by him (see (3) p. 634) does not exist. Although it is admitted that on long standing at high temperature oxalic acid in solution in the presence of much manganous sulphate is oxidized by the air, such combination of conditions is not met in the standardization of permanganate, and such oxidation does not take place to any such extent as claimed by Schroeder under the conditions which are properly employed. On the above basis we believe that Schroeder's conclusion (3) is not warranted, even by his own data.

If, as seems necessary, the explanation that loss of oxygen and not atmospheric oxidation is responsible for the variation of the

permanganate values obtained under different conditions of titration, then the conclusions (5) and (6) expressed by Schroeder are not necessary for the explanation of the experimental data.

The fact that the use of the conditions chosen by Schroeder gave values which agreed well with those obtained by the iodine method of permanganate standardization (see (7) p. 634) would not prove that these conditions are correct, since no evidence is presented to show that the iodine standardization was carried out under conditions giving correct values. As a matter of fact, the iodine standardization is even more largely affected by conditions than is the oxalate method, and none of these influences have been systematically studied.

As has already been stated (p. 630), the experimental data obtained by the author do not indicate that any appreciable atmospheric oxidation of oxalic acid occurs during the course of a standardization. Moreover, Schroeder's data warrant the conclusion that when titrations are made in a stream of air, no less (and perhaps even more) permanganate is required than if made in a stream of carbon dioxide.

Therefore it appears certain that the oxidizing effect of the air during the permanganate-oxalate reaction will not have an appreciable influence upon the result obtained from such standardization.

(e) INCOMPLETE OXIDATION OF THE OXALIC ACID

In so far as our present knowledge goes there is no evidence to indicate any incompleteness in the oxidation of the oxalic acid. Since one of the products of the reaction, the carbon dioxide, is almost wholly removed from the sphere of action, there is little if any tendency to come to an equilibrium before the whole of the oxalate is oxidized. The reaction:

$$Mn^{\cdots} \text{ (or } MnO_4) + H_2C_2O_4 = Mn^{\cdot\cdot} + H_2O + CO_2$$

can not be reversible, under the conditions of titration, and unless it be reversible unoxidized oxalic acid would hardly remain

in the presence of an excess of permanganate. Moreover, if no free oxalic acid remains the complex $Mn(OH)_3.H_2C_2O_4$ could not be stable, since it would be completely decomposed according to the reaction:

$$Mn(OH)_3.xH_2C_2O_4 = Mn(OH)_3 + xH_2C_2O_4$$

Since the $H_2C_2O_4$ formed by this last reaction is destroyed by the $KMnO_4$ excess as fast as formed, the reaction would continue to the complete decomposition of the complex. The fact that the solutions at the end of a titration contain an excess of permanganate and do not lose oxidizing power, even after standing for half an hour or more at 90°, indicates that the oxidation of the oxalate must have been complete during the titration.

(f) LOSS OF OXYGEN

In a recent article Sarkar and Dutta[14] claim that unlimited amounts of permanganate can be reduced by small amounts of organic material at high temperatures (85 to 90°) and high acidity by the following cycle of reactions:

(*a*) $5H_2C_2O_4 + 2KMnO_4 + 3H_2SO_4 = 10CO_2 + K_2SO_4 + 2MnSO_4 + 8H_2O$

(*b*) $3MnSO_4 + 2KMnO_4 + 2H_2O = 5MnO_2 + K_2SO_4 + 2H_2SO_4$

(*c*) $2MnO_2 + 2H_2SO_4 = 2MnSO_4 + 2H_2O + O_2$

(*d*) $3MnSO_4 + 2KMnO_4 + 2H_2O = 5MnO_2 + K_2SO_4 + 2H_2SO4$

Similarly other authors have claimed that oxygen is evolved from the interaction of sulphuric and permanganic acids.[15]

In reply to Sarkar and Dutta's claim Skrabal[16] has proposed the following scheme as explaining the evolution of oxygen when

[14] Zs. anorg. Chem., **67**, pp. 225–233; 1910.

[15] Hirtz and Meyer: Ber., **29**, pp. 2828–2830; 1896. Gooch and Danner: Am. J. Sci., [3] **44**, pp. 301–310; 1892. Jones: J. Chem. Soc., **33**, p. 95; 1878.

[16] Zs. anorg. Chem., **68**, pp. 48–51; 1910.

permanganate and oxalic acid react. (Compare with this author's scheme for the normal reaction given on p. 615):

I. Incubation period:

$$Mn^{VII} \rightarrow Mn^{II} + O_2 \qquad (1)$$

II. Induction period:

$$Mn^{VII} + Mn^{II} \rightarrow Mn^{III} \begin{cases} \rightarrow Mn^{II} + Mn^{IV} & (2) \\ \text{or} \\ \rightarrow Mn^{II} + O_2 & (3) \end{cases}$$

III. End period:

$$Mn^{IV} \rightarrow Mn^{II} + O_2 \qquad (4)$$

Whether one wishes to accept this last scheme or that of Sarkar and Dutta, or even some entirely different explanation, the fact remains that oxygen may be evolved during the course of these reactions when they take place under certain conditions. Such oxygen loss would be expected under either of the following conditions, viz, an increased concentration of acid or an increased concentration of $HMnO_4$ or MnO_2 (and possibly also of Mn_2O_3) in the solution. The first of these conditions is produced by direct increase in amount of acid added, and the result has been shown to be an increased permanganate consumption. (See Tables III and IV.) The presence of $HMnO_4$, MnO_2, and perhaps Mn_2O_3, one or all, is caused by any one of the following influences: Low temperature, which decreases the rate of their reduction to the manganous condition. Large bulk of solution, which reduces the oxalate concentration, and therefore the rate of their reduction to manganous condition. Rapid addition of permanganate (especially at the start or just before the end point) or insufficient stirring, both of which allow the unreduced or partially reduced manganese to accumulate throughout or in parts of the solution.

Each of these conditions has been shown to cause an increase in the permanganate used for the titration. On the other hand, addition of manganous sulphate, which increases the rapidity of the reduction of the manganese, causes a decreased consumption of permanganate. (See Table VIII.)

To briefly summarize these points: The loss of small amounts of oxygen, which certainly does occur under some conditions,

would account for the effect noted in all of the experiments described where change of temperature, of acidity, of volume, of rate of addition of the permanganate, of rate of stirring, or the addition of manganous sulphate has an appreciable influence upon the result of the standardization. Since with proper precautions these variations can be reduced, indeed almost eliminated, over considerable ranges of temperature, acidity, and volume of solution, there is good reason to believe that the loss of oxygen is almost, if not wholly, prevented. In this connection, it must again be emphasized that vigorous stirring throughout the titration, and slow addition of the permanganate at the beginning and at the end, are essential if correct results are to be had.

(g) INCOMPLETE REDUCTION OF THE PERMANGANATE

It has been shown in the discussion of the end-point corrections (pp. 620–621) that in addition to permanganate there is often manganic ($Mn^{\cdots}$) or tetravalent manganese ($Mn^{\cdots\cdot}$) salt present at the end point. However, the error caused by their presence is corrected by the end point "blank" described above, and therefore need not be considered as influencing the results already given. In the ordinary titration, such error would be serious only if improper conditions of titration were chosen. The conditions recommended below entirely eliminate this source of error.

(h) IMPURITIES IN THE OXALATE

The question of purity of the soidum oxalate, water, and acid, used in the titration, must, of course, be considered; but there is no need for appreciable error to arise due to this source if the simple tests referred to above under "reagents used" are applied. As especially affecting the present discussion this source of error has been eliminated even beyond the accuracy of the tests for purity of the oxalate, since only comparative values were employed.

(i) ABNORMAL OXIDATION PRODUCTS

Several abnormal products of the oxidation can be imagined, for example, carbon monoxide, hydrogen, hydrogen peroxide, and oxides of carbon higher than carbon dioxide. It is not at all

probable that either hydrogen or carbon monoxide should result from such a reaction as that under consideration, since the former would be liberated in a nascent state very easily oxidized and the latter could be formed only by a decomposition of the oxalate molecule in a way never observed except under the action of a strong dehydrating agent, e. g., concentrated sulphuric acid.

The possibility of the formation of hydrogen peroxide or higher carbon oxides has been discussed by Schroeder.[17] This could lead to error under only the first of the following two conditions, viz, that they subsequently decomposed, setting free oxygen, or that they remained intact at the end point. The former condition has already been discussed (p. 637); the latter possibility, that they remain unacted upon to the end of the titration, is both improbable and immaterial, since, when the end-point correction is employed, the excess of oxygen which they might contain would undoubtedly be corrected for by the "blank" as determined with potassium iodide.

2. SUMMARY AND CONCLUSIONS

As is indicated in the discussion immediately preceding, there appear to be only two sources of variation from the normal course of the reaction which are at all probable, viz, loss of oxygen from the solution or oxidation of part of the oxalic acid by atmospheric action. Although either of these theories explains a large part of the experimental facts, only the former can be held in the light of the experiments of Table VII and the other facts discussed in connection with this table.

Therefore, if the main source of error is due to oxygen losses, the higher values obtained in the various series are more nearly correct. This conclusion has been accepted as a working basis for the recommendations regarding choice of conditions for titration. Indeed such choice can not be far in error, since the greatest discrepancies noted under wide ranges of conditions is not over 0.2 per cent (not including errors due to long standing, inefficient stirring, and excessively rapid permanganate addition just at the end point).

[17] Zs. öffentliche Chem., 16, p. 270: 1910.

3. METHOD RECOMMENDED FOR USE

Although it can be seen that under quite varied conditions of standardization, the values obtained for the permanganate will vary less than .1 part in 1000, it seems desirable to fix more definitely the conditions of titration, both in order to make the results obtained slightly more precise and also to insure the use of those conditions under which the difficulty of proper operation is a minimum. For this purpose the following detailed method of operation is recommended:

In a 400-cc beaker, dissolve 0.25–0.3 g of sodium oxalate in 200 to 250 cc of hot water (80 to 90°) and add 10 cc of (1:1) sulphuric acid. Titrate at once with $N/_{10}$ $KMnO_4$ solution, *stirring the liquid vigorously and continuously*. The permanganate must not be added more rapidly than 10 to 15 cc per minute and the last ½ to 1 cc must be added dropwise with particular care to allow each drop to be fully decolorized before the next is introduced. The excess of permanganate used to cause an end-point color must be estimated by matching the color in another beaker containing the same bulk of acid and hot water. The solution should not be below 60° by the time the end point is reached; more rapid cooling may be prevented by allowing the beaker to stand on a small asbestos-covered hot plate during the titration. The use of a small thermometer as stirring rod is most convenient in these titrations, as the variation of temperature is then easily observed.

4. ACCURACY AND PRECISION ATTAINABLE

The agreement of duplicates in the numerous tables given above shows the precision which can be expected in the use of sodium oxalate when the conditions or titration are regulated. It appears that agreement of duplicates to 1 part in 2000 should be regularly obtained if weight burettes are used. Failure to obtain this precision should be at once taken to mean that some condition has not been regulated with sufficient care.

Although the absolute accuracy of the values obtained under different conditions has not yet been studied, it seems probable

that if the method recommended is followed the error will not exceed 0.1 per cent and probably will be less than 0.05 per cent.

It is hoped to undertake a comparison of this method with other oxidimetric standards, particularly iron, silver, and iodine, in order to check its value. However, such work involves one uncertainty which renders the best results which could be obtained probably no more conclusive than those now at hand. The difficulty is that of end points, for in each case mentioned the uncertainty in the blank correction required for an end point is about of the same order as that of the uncertainty in the oxalate values, namely, 0.1 per cent (usually equivalent to 0.05 cc of N/10 $KMnO_4$). We therefore, at the present time, would assume no greater absolute accuracy for the values obtainable than one-tenth of 1 per cent.

WASHINGTON, June 1, 1912.

BENZOIC ACID AS AN ACIDIMETRIC STANDARD

By George W. Morey

The study of the suitability of benzoic acid as a primary standard in acidimetry and alkalimetry was suggested by experience gained in the purification of benzoic acid to be used as a calorimetric standard. During that work it was found that benzoic acid could be titrated with standard alkali to a high degree of accuracy, and that this titration afforded the most rapid and accurate method of comparing the purity of various samples. Since pure benzoic acid has been furnished for some time by the Bureau of Standards as a calorimetric standard, it would, of course, be advantageous to use it also for a standard in acidimetry if found suitable.

A search of the literature showed that Wagner[1] in a report presented to the Fifth International Congress of Applied Chemistry, in 1903, had mentioned benzoic acid among a number of other possibilities for the purpose named; and that Phelps and Weed[2] had later included it in a short study of the availability of several organic acids and acid anhydrides.

The method used in studying this problem was that of standardizing a hydrochloric-acid solution by several well-known and

[1] Proceedings of the Fifth International Congress of Applied Chemistry, Berlin, 1903, Vol. I, p. 323.
[2] Am. Jour. Sci., 26, p. 141; 1908.

standard methods, and comparing the results so obtained with those obtained by standardizing the same hydrochloric acid against benzoic acid.

The methods chosen for the work were the distillation method of Hulett and Bonner, the gravimetric silver-chloride method, comparison with a sulfuric-acid solution standardized gravimetrically by the barium sulfate method, and comparison with the same sulfuric acid standardized volumetrically by the sodium oxalate method.

PREPARATION OF MATERIALS

Two samples of benzoic acid were used in this work. Sample A was prepared from the "Kahlbaum" grade by recrystallizing twice from alcohol, once from water, and then fractionally subliming in vacuo. Sample B was prepared in the same manner from a 50-pound lot furnished on specification by the Baker & Adamson Chemical Co. The final product in both cases was free from chloride, and otherwise as pure as could be prepared.

A hydrochloric-acid solution was prepared by diluting hydrochloric acid made by Hulett's method with freshly boiled distilled water.

Two samples of sodium oxalate were used. Sample A was prepared by Mr. J. B. Tuttle, in the Bureau of Standards, from Merck's "Oxalsaures Natrium nach Sörensen." Sample B, was prepared by Dr. Wm. Blum, also in the Bureau of Standards, from a sample furnished by J. T. Baker Chemical Co., by crystallizing twice from water and precipitating once by alcohol. This had been tested by Dr. Blum [3] and found to be identical in composition with the best material which could be prepared. The above samples were compared with five others of special purity by converting to the carbonate and titrating, and all were found to be identical within the limits of accuracy of the method.

The sulfuric-acid solution was prepared by diluting C. P. acid with distilled water and boiling, after which the solution was transferred to a stock bottle, in which it was allowed to cool. Sodium hydroxide was prepared free from carbonate by adding a little barium hydroxide, precipitating the excess barium with

[3] J. Am. Chem. Soc., 34, p. 127; 1912.

sulfuric acid, and filtering. Barium hydroxide was likewise prepared from the C. P. salt by dissolving in water and filtering.

All these solutions were carefully protected from the carbon dioxide of the air by suitable guard tubes, and were tested for the presence of carbonate from time to time by titrating both in the cold and while boiling, using phenolphthalein as indicator. A difference between the two titrations would indicate the presence of carbonate, but the titrations were always within the limit of error of the method. Weight burettes were used for all titrations, which were made in a 300-cc flask, through which passed a stream of air free from carbon dioxide.

DIRECT STANDARDIZATIONS OF HYDROCHLORIC ACID

The hydrochloric acid was first standardized in its preparation, having been prepared by the method of Hulett and Bonner,[4] which is based on the constancy of composition, at a definite atmospheric pressure, of the constant boiling mixture formed by hydrochloric acid and water. A liter of HCl ($d_{25°} = 1.098$) was distilled and a portion of the last quarter of the distillate collected, the barometric pressure being 755.5 mm. Of this acid 89.293 g was diluted with 4855.9 g of freshly boiled distilled water. From the above data and those given by Hulett and Bonner,[5] the resulting solution was calculated to contain 0.0036396 g HCl per g solution, corresponding to a 0.1 N factor of 0.9980.

The next standardization was by the gravimetric silver-chloride method. About 50 cc of the solution was diluted with 150 cc of distilled water, precipitated with excess of silver nitrate, heated rapidly to boiling and allowed to stand over night in a dark closet. The precipitate was then collected on a weighed Gooch crucible, washed free from silver with nitric acid (1 per cent), then washed with water and dried at 130° to constant weight. The hydrochloric acid in the filtrate and washings was determined with the nephelometer, and the amount found added to that calculated from the weight of silver chloride. The results are given in Table I.

[4] Hulett and Bonner. J. Am. Chem. Soc., 31, pp. 390–393; 1909.
[5] Loc. cit.

TABLE I

No.	Wt. HCl	Wt. AgCl	HCl in filtrate	Total HCl	HCl per g solution
	g	*g*	*mg*	*g*	*g*
1	57.26	0.8188	0.2	0.2085	0.0036403
2	50.11	0.7167	0.1	0.1824	0.0036400
3	59.03	0.8440	0.3	0.2150	0.0036414
4	50.635	0.7245	0.1	0.1844	0.0036417

Mean of four determinations 0.0036411
Corresponding 0.1 N factor 0.9984

The hydrochloric acid was standardized from the sulfuric acid by comparing the two solutions through a solution of sodium hydroxide. The mean of six determinations of the ratio HCl : NaOH was 1.0464; of four determinations of the ratio H_2SO_4 : NaOH was 0.8454. These determinations were very concordant, the maximum difference between any two being 1 part in 2500. The ratio HCl : H_2SO_4 is therefore 1.2378. For the gravimetric standardization of the sulfuric acid, the following method was used. About 50 g of the sulfuric acid was diluted with 150 cc hot water, heated to boiling, and 15 cc of a 10 per cent barium chloride solution added slowly (1.5 minutes), with constant stirring. This solution was then evaporated to dryness on the steam bath, taken up with 100 cc hot water, filtered, and washed free from chlorine. The filtrate and wash water were united and evaporated to dry-

TABLE II

No.	Wt. H_2SO_4 solution	Wt. $BaSO_4$	H_2SO_4 per g solution
	g	*g*	*g*
1	47.74	0.6891	0.006065
2	52.62	0.7587	0.006059
3	49.21	0.7096	0.006059
4	51.13	0.7375	0.006061
5	55.37	0.7992	0.006065
6	54.06	0.7793	0.006058

Mean of six determinations 0.006061
Corresponding 0.1 N factor for H_2SO_4 ..... 1.2358
Corresponding 0.1 N factor for HCl 0.9984

ness, taken up again, etc., the precipitate was collected on a small filter and added to the main portion. After igniting and weighing in the usual manner, the precipitates were tested for occluded barium chloride by the method of Hulett and Duschak,[6] but the amount was always less than 0.1 mg, a negligible quantity. The results are given in Table II.

The standardization by means of sodium oxalate was made with great care, after a series of experiments to compare various samples of the salt. The samples were dried at 230° in an electric oven for an hour, a weighed amount was placed in a platinum crucible and the covered crucible heated with an alcohol lamp, first very gently, then more strongly, but in no case was the resulting sodium carbonate fused. Though an alcohol flame was used, it was found advisable to place the crucible in a perforated asbestos shield as recommended by Lunge,[7] to avoid loss of material due to mechanical carrying away by the gaseous current. The contents of the cooled crucible were carefully washed into a 300-cc flask, diluted to 200 cc, and a slight excess of the standard sulphuric acid (0.1 to 0.5 cc) was added. The solution was reduced by boiling to two-thirds its original volume, while a current

TABLE III

No.	Sample	Wt. $Na_2C_2O_4$	Wt. solution	H_2SO_4 per g solution
		g	*g*	*g*
1	A	0.5013	60.56	0.006059
2	A	0.5027	60.75	0.006057
3	A	0.5001	60.38	0.006063
4	A	0.5018	60.62	0.006059
5	B	0.5037	60.84	0.006061
6	B	0.5021	60.66	0.006059
7	B	0.5011	60.53	0.006060
8	B	0.5015	60.61	0.006058
9	B	0.5007	60.46	0.006062

Mean of nine determinations..... 0.006060
Corresponding 0.1 N factor for H_2SO_4..... 1.2356
Corresponding 0.1 N factor for HCl..... 0.9982

[6] Zs. anorg. Chem., 40, p. 196; 1904. [7] Zs. angew. Chem., 18, pp. 1520–1528; 1905.

of carbon dioxide-free air passed through to expel carbon dioxide, after which the excess acid was titrated with sodium hydroxide, using phenolphthalein as indicator. The end point chosen in this case, as well as in all other titrations, is that described under the standardization with benzoic acid. While concordant results were obtained by strict adherence to a certain procedure in the transformation of the oxalate to the carbonate, it was found that slight variations in the rate or manner of heating might produce very discordant results. The data are given in Table III.

STANDARDIZATION OF HYDROCHLORIC ACID BY BENZOIC ACID

The next step was the standardization of the hydrochloric-acid solution with benzoic acid. Because of the bulkiness of the sublimed benzoic acid, it was found convenient to fuse or compress it before weighing. Fusion has the advantage of diminishing the possibility of large surface effects. A platinum dish was filled with sublimed benzoic acid and the covered dish placed in an oven heated to about 140°. When melted, the liquid was poured into a test tube, and after solidifying the stick so obtained was broken into pieces of convenient size and preserved in a glass-stoppered bottle. Samples so prepared can be kept indefinitely and used without preliminary drying.

About a gram of this material was weighed and placed in a 300-cc flask which had been swept free from carbon dioxide; 20 cc of alcohol were added, the flask was stoppered and let stand until the sample had dissolved. Three drops of a 1 per cent solution of phenolphthalein were then added and the solution titrated directly with 0.1 N alkali, a current of air free from carbon dioxide bubbling through the solution until the titration was completed. The end point chosen was that of a 7 per cent transformation of the indicator added, that being the end point which should give the best results.[8] The effect of the alcohol on the end point was determined in a blank experiment and the titrations were corrected by this amount. This blank ranged from 0.06–0.08 cc.

In the first series of experiments a solution of barium hydroxide was used, this being the most convenient alkali to use when

[8] Noyes. J. Am. Chem. Soc., 32, p. 857; 1910.

exclusion of carbon dioxide is necessary. The mean of four concordant determinations of the ratio HCl : $Ba(OH)_2$ was 1.3790. The results are given in Table IV.

TABLE IV

No.	Sample	Wt. $C_6H_5CO_2H$	Wt. $Ba(OH)_2$	$C_6H_5CO_2H$ per g $Ba(OH)_2$ solution
		g	*g*	*g*
1	B	1.0425	62.02	0.016809
2	B	1.0798	64.27	0.016801
3	B	1.0290	61.25	0.016800
4	B	1.0020	59.61	0.016809
5	B	1.0350	61.57	0.016810
6	A	1.0090	60.04	0.016805
7	A	1.0250	61.01	0.016801
8	A	1.0580	62.96	0.016804
9	A	1.0348	61.58	0.016804
10	A	1.0114	60.18	0.016806
11	A	1.0048	59.92	0.016808
12	B	1.0249	61.00	0.016802
13	B	1.0225	60.82	0.016811
14	B	1.0228	60.84	0.016810
15	B	1.0224	60.84	0.016804

Mean of 15 determinations..... 0.016805
Corresponding 0.1 N factor for $Ba(OH)_2$..... 1.3769
Factor for HCl=1.3768÷1.3790=0.9984

In a second series of experiments the sodium hydroxide solution used in comparing the hydrochloric and sulfuric acids was used, for which the ratio HCl : NaOH was 1.0464. The data are given in Table V.

TABLE V

No.	Sample	Wt. $C_6H_5CO_2H$	Wt. NaOH	$C_6H_5CO_2H$ per g NaOH solution
		g	*g*	*g*
1	B	1.0067	78.97	0.012748
2	B	1.0108	79.29	0.012748
3	B	1.0208	80.08	0.012747
4	B	1.0440	81.89	0.012749

Mean of 4 determinations..... 0.012748
Corresponding 0.1 N factor for NaOH..... 1.0445
Factor for HCl=1.0445÷1.0464=0.9981

SUMMARY AND CONCLUSION

The results of the foregoing standardizations are summarized in the following table:

TABLE VI

Method	0.1 N factor for HCl
Direct by Hulett	0.9980
Direct by AgCl	0.9984
H_2SO_4-$BaSO_4$-NaOH-HCl	0.9984
H_2SO_4-$Na_2C_2O_4$-NaOH-HCl	0.9982
$C_6H_5CO_2H$-$Ba(OH)_2$-HCl	0.9984
$C_6H_5CO_2H$-NaOH-HCl	0.9981

The close agreement of these results proves the accuracy of the benzoic acid method. Moreover, benzoic acid has many advantages. Its high molecular weight permits the use of large samples, thus reducing the error of weighing; its stability and lack of hydroscopicity make it very convenient; and the method is rapid, since a single weighing and a titration are all the operations involved. These considerations, combined with the ease of obtaining it in a high state of purity, make benzoic acid an excellent material to use as a standard in acidimetry and alkalimetry.

WASHINGTON, May 4, 1912.

A TUBULAR ELECTRODYNAMOMETER FOR HEAVY CURRENTS

By P. G. Agnew

1. THEORY OF INSTRUMENT

To obtain the same current distribution for both alternating and direct current it is usual to strand the windings of the field coils of electrodynamometers, and the strands should also be throughly mixed, so that all shall have the same effective resistance and inductance. For very heavy currents this becomes a matter of extreme difficulty.

In the present instrument an entirely different means of obtaining the same end is employed. Evidently the ultimate requisite in an instrument for transferring from alternating to direct current is that the torque shall be the same, and it is immaterial whether we make the current distribution the same, or the field due to the current the same. The latter is the method adopted. There is no theoretical reason why the current distribution and the magnetic field could not both be allowed to vary if it could be shown that the torque did not change.

The field "coil" of the instrument consists of two coaxial copper tubes, thus giving a circular magnetic field in the space between the tubes. On direct current the distribution may be assumed to be uniform over the cross section of the tubes, but on alternating current, as is well known, the stream lines are crowded toward the

outside of the inside tube and toward the inside of the outside tube, and the amount of this change depends upon the frequency. But it may easily be shown that if we have axial symmetry the magnetic field at any point is independent of the current distribution. Expressed in the usual terms there is no magnetic field between the tubes due to the outside tube, and that due to the inside tube is independent of the skin effect, but it is better to consider the arrangement as a special case of a more general principle.

Let us consider a generalized toroid such as would be formed by rotating any closed circuit, such as A (Fig. 1) about an axis. We may think of the circuit A as consisting of a wire carrying a current, and similarly of the surface of revolution as being a current sheet; or we may think of the surface as being wound with a layer of wire with no space between turns and thus becoming an endless solenoid.

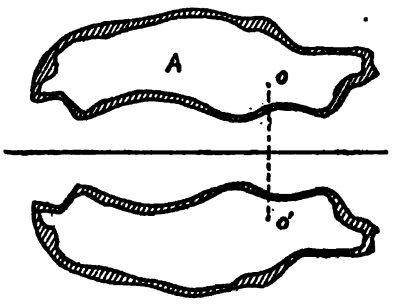

Fig. 1.—*Generalized toroid.*

It has been shown by Maxwell[1] and by Gray[2] that the field at any point, as at O, is independent of the size, shape, and thickness of the conductor and depends only on the total current and the distance of the point from the axis. This may be shown by a very simple process of reasoning. The all-important condition is axial symmetry.

Consider a circle perpendicular to the plane of the paper, and cutting it in the points O and O'. This circle links once with the current. Hence, the line integral of the magnetic field around this circle, or the work required to carry a unit pole around it, is 4π times the total current, or if we remember that H is uniform around the circle.

$$2\pi R H = 4\pi I$$
$$H = 2I/R$$

The value of H at any point within is thus entirely independent of the dimensions of the conductor, provided there is axial sym-

[1] Electricity and Magnetism, Vol. II, art. 681. [2] Absolute Measurements, Vol. II, p. 278.

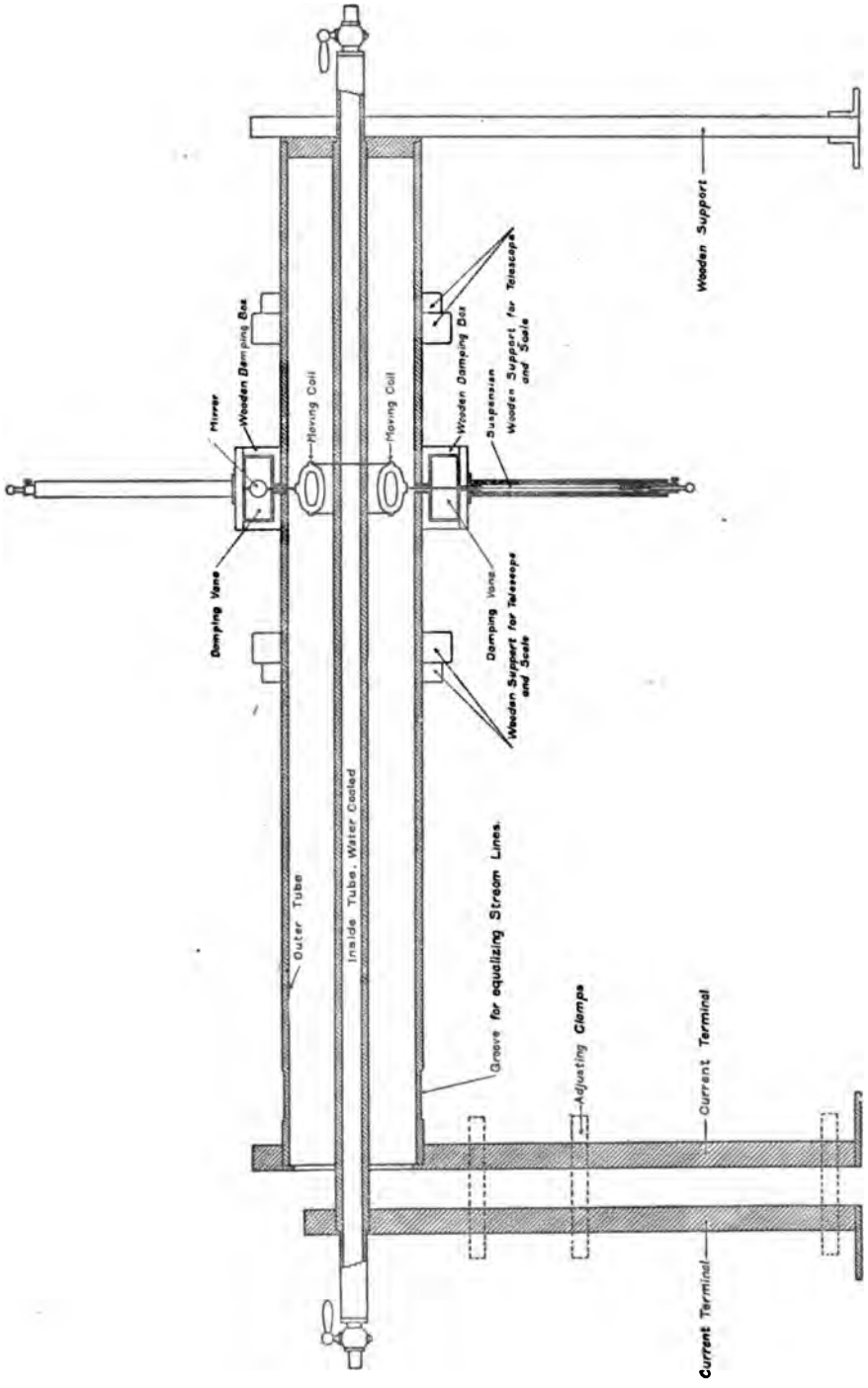

Fig. 2.—*Tubular electrodynamometer vertical section.*

metry. Hence, an approximation to such a form for the current coils is admirably adapted for alternating-direct-current transfer dynamometers, since it is independent of the current distribution.[3]

It will be seen that the concentric tubes form an approximation to a toroid or an endless solenoid of one turn. It can only be an approximation, since it must be open at one end in order to introduce the current.

2. CONSTRUCTION

The instrument was built in the instrument shop at the Bureau of Standards by Mr. G. C. Klein. Fig. 2, which is a vertical longitudinal section through the instrument, shows the general arrangement. The tubes are horizontal, in order to allow a vertical suspension, and the moving coils are placed astatically, one above and one below the inner tube. These are shown in the plane of the section instead of in the mean working position which is nearly perpendicular to the plane of the paper. This is done in order to show the shape of the coils, which are wound on ivory frames. They are held together by fine silver wires, one on either side of the inner tube, as indicated in the figure.

Phosphor-bronze strip suspensions are used, and two air damping vanes are provided. The damping boxes are of wood and are mounted directly on the outside copper tube. The instrument is read by telescope and scale.

Massive copper slabs placed parallel and 2.5 cm apart serve as current terminals. They are provided with five bolts so arranged that any number of parallel leads up to five may be connected symmetrically. These at the same time serve as supports for one end of the instrument, and the closed end is supported by an oak upright.

Water cooling is provided for the inside tube by passing water through it, but no special means are provided for cooling the outside tube.

While the very heavy copper slabs used for current terminals tend to equalize the stream lines in the larger tube about the

[3] For a detailed discussion of this principle and its application to electrodynamometers, see Agnew and Fitch, Elec. Rev. (Chicago) 60, p. 767; 1912.

axis as a line of symmetry, this was deemed insufficient for the most precise work. Accordingly a groove 5 cm wide was turned eccentrically in the large tube near the open end, so that the groove was 2 mm deep on the top of the tube and 3 mm deep on the bottom. When finally mounted a current was passed through the instrument and equipotential surfaces laid off on the tube. By a small amount of filing the eccentricity of the groove was adjusted, so that the equipotential surface was a plane perpendicular to the axis. It was found that the relation held very accurately throughout the length of the tube; that is to say, the distances between equipotential surfaces measured at different points on the circumference differed but 1 mm in 800.

From the principle of axial symmetry outlined above, it may easily be seen that no error due to skin effect in the closed end will enter. For this reason the moving coils were placed one-third of the length from this end, so as to minimize the effect of the leading-in cables, etc. Perhaps a fourth or even a fifth would have been better.

3. PERFORMANCE

It is of first importance to determine whether the instrument is affected by distribution errors. Rosa has described an indirect method for marking this determination by the use of capacity in the moving-coil circuit.[4] If the moving-coil circuit be short-circuited on a resistance shunted by a condenser having such a value that

$$L = C R^2$$

where L is the self-inductance, R the external resistance, and C the capacity shunted around it, then there should be no deflection on closing the circuit no matter what the position of the coil. But if there is such an error, then there will be a residual deflection.

When the instrument was first set up it showed no such error up to 300 cycles, but after it had been in service some time this same test showed a slight error. This was found to be due to a slight flexure of the inner tube, brought about by clamping

[4] This Bulletin, 3, p. 43; 1907. Reprint No. 48.

heavy cables and lugs to the massive terminals, and by differential thermal expansion. It was found that the trouble could be removed by slightly springing the inside tube. To get rid of the trouble the copper slab connected to the inner tube was cut in two and the two parts joined by means of a mercury contact. This gives a slight flexibility, enough to secure freedom from distortion of the inside tube. It is thus seen that a slight departure from the condition of axial symmetry introduces a distribution error; but this can easily be detected and corrected by a purely electrical test. Strong clamps and struts (shown in dotted lines in the figure) have been arranged to hold the copper slabs in proper position.

Measurements of large currents by means of a current transformer whose ratio had been carefully determined were in substantial agreement with the indications of the tubular dynamometer. It is believed that the instrument is accurate to 0.05 per cent.

It was found necessary to put loose-fitting diaphragms across the tube on each side of the moving coils to prevent air currents, for such currents of air caused considerable unsteadiness of the reading, even for comparatively small temperature differences in the mass of copper composing the instrument. The diaphragms effectually removed the trouble.

A more serious difficulty was encountered with magnetic impurities in the moving coil. The requirements here are even more exacting than in the sensitive moving-coil galvanometer, for the parts of the moving coils nearest the inside tube are in a field comparable in magnitude to the flux in the air gap in the permanent magnets of such galvanometers. The whole directive magnetic force exerted on the coil must be vanishingly small or the zero of the instrument will not be the same with current on that it is with current off. In the galvanometer, on the other hand, the requirement is merely that the magnetic force on the coil shall not vary appreciably through comparatively small angles.

Finally, special coils were wound by the Weston Electrical Instrument Co. which were entirely satisfactory. These were of

silver wire 0.2 mm in diameter, silk covered, and were stated by Mr. W. N. Goodwin, jr., of that company, to be actually diamagnetic.[5] They were mounted with great care, to prevent magnetic contamination from dust. The zero is precisely the same with current on or off.

At commercial frequencies no error whatever can be detected due to eddy currents being set up in the tubes by current in the moving system. Even with full load current of 900 cycles in the moving coils and with the field system short circuited, the deflection does not exceed 0.1 mm at any part of the scale.

It might have been better to have made the instrument with two independent sets of moving coils, one set for smaller currents, placed very close to the inside tube, where the field is strongest, perhaps even decreasing the diameter of the tube for the purpose, and a second set for heavier currents placed near the outer tube. This would extend the range of the instrument. In addition, current and power could be measured simultaneously.

4. CONSTANTS AND DIMENSIONS

Copper slabs for current terminals:	
Thickness	2.5 cm.
Width	20 cm.
Outer tube:	
Length	101 cm.
Radii	6.41 and 7.07 cm.
Inner tube:	
Length	126 cm.
Radii	0.50 and 1.27 cm.
Total weight of instrument (approximately)	80 kg.
Moving coils:	
Diameters (approximately), 116 turns of 0.2-mm. silver wire	2.5 and 5 cm.
Weight of each coil	7.3 g.
Moment of inertia, each	12 g-cm.^{2}
Resistance—	
Coils alone	12.2 ohms.
Total for system	14.4 ohms.
Self inductance	1.4 mh.

[5] Mr. Goodwin found the torque acting on the coils and tending to maintain them at right angles to a magnetic field having a strength of 1175 gausses to be 0.286 and 0.238 dyne-centimeters per radian, respectively.

Resistance:	
Outer tube	8 microhms.
Inner tube	56 microhms.
Complete instrument	67 microhms.
Power consumed (at 5000 amperes):	
Outer tube	200 watts.
Inner tube	1400 watts.
Complete instrument	1675 watts.
Magnetic field at center of coils (approximate)	300 gausses.
Sensitivity—100-cm deflection at 86-cm scale distance requires 100 amperes with 0.06 amperes in the moving coil.	
Capacity:	
With water cooling	5000 amp.
Without water cooling	1200 amp.
Of moving coil circuit	0.06 amp.

WASHINGTON, July 17, 1912.

THERMOMETRIC LAG

By D. R. Harper 3d

CONTENTS

	Page
INTRODUCTION	659
I. LAG OF A COMMON MERCURIAL THERMOMETER	660
Fundamental considerations	660
Physical meaning of "λ," the time lag of a thermometer	662
Falling meniscus	662
Principal equations	663
Methods of determining λ—illustrative typical values of same	667
Variation of λ with stirring	672
In water	672
In a kerosene oil	676
In air	678
II. LAG OF A BECKMANN THERMOMETER	682
Additional assumptions	682
Justification of assumptions made	688
Special cases	692
III. LAG OF ELECTRICAL THERMOMETERS	694
Galvanometer lag	695
Resistance thermometers	698
Jaeger-Steinwehr method of computing lag of above	699
Thermoelectric thermometers	702
IV. THERMOMETRIC LAG IN CALORIMETRY	702
V. LAG CORRECTIONS IN APPLIED THERMOMETRY:	
Résumé of practical instructions	711
VI. SUMMARY	713

INTRODUCTION

When a thermometer is immersed in a warmer or in a cooler medium for the purpose of ascertaining the temperature of the latter, it does not immediately indicate this temperature but exhibits a time lag in reaching it. A certain time must elapse after immersion before the reading is correct to within 0°.1, still longer before correct within 0°.01, etc., when the temperature of

the medium remains constant. If the latter be varying, the thermometer follows the variation in a definite way, maintaining a difference which may be large or small, according to the rate of the variation and the form of the thermometer.

It therefore becomes necessary under some conditions to apply a correction to a thermometer reading [1] in order to obtain the simultaneous value of the temperature of the medium in which it is immersed. Thiesen [2] in his elaborate treatise on thermometry devotes a section to the consideration of lag, and Guillaume [3] also, under the title of "Défaut de Sensibilité." Neither of these chapters conveys an adequate impression of the variation of the lag of a particular thermometer under different conditions of immersion, namely, nature of medium and rate at which it is stirred. Also electrical thermometers, thermoelectric and resistance, were not in use as precision instruments when these papers were published and are, of course, not considered. It has, therefore, seemed advisable to prepare a paper supplying some of these omissions, and in so doing to incorporate as much of the general theory as seems necessary for full comprehension of the subject.

I. THE LAG OF A MERCURIAL THERMOMETER

FUNDAMENTAL CONSIDERATIONS

The transfer of heat between a thermometer bulb and the medium in which it is immersed may be expressed in the form usually referred to as "Newton's law of cooling," namely, the rate of heat transfer directly proportional to the difference of temperature between the two. While direct experimental test of this fundamental assumption is difficult, it is very easy to test many of the equations deduced from it, some of which are so closely related to the parent equation that the test possesses all the force of a direct one. For the ordinary or "chemical" form of mercurial thermometer, the agreement between theory and experiment amply justifies the assumption, which analytically expressed is

[1] Throughout this paper the assumption is tacitly made, except where explicitly stated to the contrary, that all instrumental errors have been corrected for, i. e., calibration of bore, zero error, etc.

[2] Thiesen: Metronomische Beitrage, No. 3, p. 13; 1881.

[3] Guillaume: Thermometrie de Precision, p. 184; 1889.

$$\frac{\partial\theta}{\partial t}=\frac{1}{\lambda}(u-\theta) \tag{1}$$

where u = temperature of bath at instant t.

θ = temperature of thermometer at same instant.

λ = constant with respect to u, θ, or t.

The temperature of the bath, u, is assumed to be uniform throughout the neighborhood of the thermometer. No other case would be simple enough for convenient mathematical treatment. The temperature of the thermometer, θ, is the integrated or average temperature on which depends the average density of the thermometric liquid and the volume of the envelope. It defines the position of the meniscus in the capillary, i. e., the reading of the thermometer. λ requires no further definition at present than the placing of the limitation that it shall be independent of θ, u, and t. It will be found to be not independent of form of thermometer, medium surrounding latter, conditions of stirring, etc., a variation of a kind to be carefully distinguished from that denied to it in integrating equation (1).

Integration of (1) leads to relations between quantities all of which may be readily found by direct experiment, serving as a criterion of the validity of the assumption. For the ordinary form of thermometer a numerical value for λ, constant for a particular thermometer in any one medium stirred at a certain rate, may be found and used in applying corrections to any readings made under the same conditions, whenever a lag correction is necessary. For some forms of thermometer, e. g., those having a layer of air separating part of the bulb from the bath, the simple theory employing equation (1) fails.[4] If we endeavor to find numerical values for λ for such a thermometer by the equations deduced from (1), we are led to the conclusion that no single value is obtainable; the various values that might be computed from a few points on a curve of observations will differ according to the temperature or time corresponding to the points chosen. This is inconsistent with the limitation placed on λ in

[4] Thiesen, loc. cit.

deducing the equations, nullifying the whole theory for application to such an instrument. Modification of equation (1) by the introduction of additional assumptions leads to a treatment that is satisfactory for some such forms of thermometer; for instance, a calorimetric Beckmann, which has a considerable portion of its mercury in a large inclosed capillary, between the bulb and the fine capillary. (See Sec. II.)

PHYSICAL MEANING OF λ

Physical interpretation of the quantity λ is highly desirable. This is obtained from the equations (see pp. 664–665), but it may be well to anticipate slightly and give two such interpretations at this point. From the fundamental equation, which serves as the definition of λ,

$$\frac{\partial \theta}{\partial t} = \frac{1}{\lambda}(u - \theta) \tag{1}$$

it is seen by inspection that λ has the dimensions of time. Later equations give the following interpretations as a definite number of seconds:

(1) If a thermometer has been immersed for a long time in a bath whose temperature is rising at uniform rate, λ *is the number of seconds between the time when the bath attains any given temperature and the time when the thermometer indicates this temperature.* In other words, it is the number of seconds the thermometer "lags" behind such a temperature.

(2) If a thermometer be plunged into a bath maintained at a constant temperature (the thermometer being initially at a different temperature), λ *is the number of seconds in which the difference between the thermometer reading and the bath temperature is reduced to* ϵ^{-1} *times its initial value.*[5]

FALLING MENISCUS

Considerable attention has been paid to the lag of a thermometer when its temperature is decreasing,[6] and the results published seem liable to misinterpretation. Without entering the very dif-

[5] $\epsilon^{-1} = \frac{1}{2.718} = 0.4$, approximately.

[6] Thiesen, loc. cit.; Guillaume, loc. cit.

ficult field of analysis of the surface tension and capillary forces governing the motion of the meniscus, it is sufficient to point out that with falling temperatures one may observe sticking of the meniscus and even separation of the mercury column, indicating that there is no certainty that the forces tending to return mercury to the bulb accomplish the return at a rate proportional to the rate at which the transfer of heat from the bulb takes place. The usual equations of lag, being primarily equations of heat transfer, must not, therefore, be expected to cover all cases of falling meniscus. If a thermometer is plunged into a bath much cooler than itself, the drop of the meniscus is apt to be so erratic that a computation of the lag by the usual methods results in values widely variable under exactly the same conditions and oftentimes quite different from the lag measured with a rising meniscus. The pseudo λ so obtained may or may not be related to the temperature lag of the bulb, and as the conditions are impossible of specification the values thus determined fail of interpretation.

Because of the sticking of the meniscus and possible separation of the mercury column, the readings of a mercury thermometer with falling temperatures are less reliable than with rising ones, and in precision thermometry a falling temperature should be avoided whenever possible. When its use is unavoidable the difficulties are overcome to a considerable extent, in the case of a small rate of fall, by subjecting the thermometer to a series of rapid jars, as by an electric buzzer. There seems to be every reason for believing that under such conditions the position of the meniscus depends on the temperature of the bulb in the same relation as when the meniscus is rising, and the lag corrections to be applied may be computed from the value of λ determined with rising meniscus, for the medium and rate of stirring employed.

PRINCIPAL EQUATIONS

The fundamental equation

$$\frac{\partial\theta}{\partial t}=\frac{1}{\lambda}(u-\theta) \tag{1}$$

suffices to determine the function u if the function θ be known,

since $\frac{\partial\theta}{\partial t}$ may at once be determined. The converse problem of determining θ when u is known requires solution of the differential equation.

Solving $\lambda\frac{d\theta}{dt}+\theta=u$ under the conditions that the thermometer reads θ_0 at the time $t=0$, in a medium at temperature u_0; the reading θ at any time t when the medium is at temperature u, is

$$\theta=\theta_0\epsilon^{-\frac{1}{\lambda}t}+\frac{1}{\lambda}\epsilon^{-\frac{1}{\lambda}t}\int_0^t u\epsilon^{\frac{1}{\lambda}t}dt \tag{2}$$

An integration by parts puts this equation in a form showing directly the temperature difference, $\theta-u$.

$$\theta-u=(\theta_0-u_0)\epsilon^{-\frac{1}{\lambda}t}-\epsilon^{-\frac{1}{\lambda}t}\int_0^t\frac{\partial u}{\partial t}\epsilon^{\frac{1}{\lambda}t}dt \tag{3}$$

From this equation the solution for special cases is obtained upon substituting the proper expression for $\frac{\partial u}{\partial t}$. The cases of most importance are (a) constant temperature $u=u_0$; (b) linear rise, $u=u_0+rt$, usually at small rate, r; (c) exponential change of temperature according to formula $u=A+B\epsilon^{-\alpha t}$, the constant α being usually small.[7]

(a) When $u=u_0$, $\frac{\partial u}{\partial t}=0$

and equation (3) reduces to

$$(\theta-u_0)=(\theta_0-u_0)\epsilon^{-\frac{1}{\lambda}t} \tag{4}$$

Equation (4) states that an initial temperature difference (θ_0-u_0), between a thermometer and a constant-temperature bath in which it is immersed, decreases logarithmically with time, becoming in λ seconds, ϵ^{-1} times the original difference. From the approximate values, $\epsilon^{-7}=.001$ and $\epsilon^{-9}=.0001$, it will be seen

[7] The frequent occurrence of the conditions (a) and (b) is evident. The importance of (c) lies in the fact that whenever one body is exchanging heat with another according to Newton's law, its temperature is expressed by the exponential equation given. When α is very small, nearly always true in calorimetry for instance, the curvature of this exponential function may be so slight that only where high precision is sought need any account be taken of its departure from a linear function (case b)

that times of 7λ and 9λ seconds elapse before a difference of 10° is reduced to 0°.01 and 0°.001, respectively.

(b) $u = u_0 + rt$. —The value of $\frac{\partial u}{\partial t}$ is r, which is to be substituted in equation (3). Reduction and collection of terms gives the relation

$$\theta - u = -r\lambda + (\theta_0 - u_0 + r\lambda)\epsilon^{-\frac{1}{\lambda}t} \tag{5}$$

The interpretation of equation (5) is very simple, except the exponential term. Consideration of the numerical values which enter will show that this term is usually negligible. From two to six seconds, according to the size of bulb, is the value of λ for an ordinary thermometer in water stirred rather vigorously. Taking, as a mean, four seconds for substitution in the term $\epsilon^{-\frac{1}{\lambda}t}$, the value, for t = one minute, is ϵ^{-15}, about 3×10^{-7}. This multiplies a term $(\theta_0 - u_0 + r\lambda)$ not very large, so the product is insignificant. Accordingly, with a value of λ of four seconds or that order of magnitude, the lag of a thermometer, immersed in a bath the temperature of which is increasing at a constant rate r, is represented less than a minute after immersion, by the equation.

$$\theta - u = -r\lambda \tag{6}$$

A constant difference of temperature, numerically $r\lambda$ thus exists between the thermometer and the bath. Both rise at the rate r, so that in λ seconds either one increases its temperature by $r\lambda$. Accordingly, λ seconds after the bath attains a given temperature, the same is indicated by the thermometer.

The application of equation (6) to numerical examples will illustrate the magnitude of the corrections necessary for lag in applied thermometry. For a rate of rise of 0°.03 per minute, perhaps the maximum allowable when readings to single thousandths of a degree are taken, the lag correction to any reading if $\lambda = 4$ seconds is +0°.002.

Care must be taken in the application of equations (5) and (6) to any case where λ is large, e. g., in still air, where the value of λ may be 50 to 100 times the value for the same thermometer in

stirred water. Computation in this case requires consideration of the term $\epsilon^{-\frac{1}{\lambda}}$, which becomes very small only after a rather long time elapses.

(c) EXPONENTIAL CHANGE OF TEMPERATURE $u=A+B\epsilon^{-\alpha t}$.—Before proceeding to the solution for this case, it may be well to give an example of its occurrence and the interpretation of the symbols.

Let a calorimeter, whose temperature may be designated u, be exchanging heat with surroundings at constant temperature A according to Newton's law,

$$\frac{du}{dt}=-\alpha(u-A).$$

the solution of which is $u-A=B\epsilon^{-\alpha t}$, the equation written above. B is the initial value of $u-A$, i. e., the value of the temperature difference between the calorimeter and its surrounding at any arbitrary time at which we may choose to start applying the equation. Instead of B, therefore, may be introduced another constant, u_0, which, if defined as the value of u for the time zero, will be in harmony with all the foregoing equations and permit of using them.

If $u=A+B\epsilon^{-\alpha t}$; $u_0=A+B$, $u-A=(u_0-A)\epsilon^{-\alpha t}$.

Substituting the value of $\frac{\partial u}{\partial t}$, namely, $-\alpha(u_0-A)\epsilon^{-\alpha t}$ in equation (3) and collecting

$$\theta-u=\left(\theta_0+\frac{\alpha\lambda A-u_0}{1-\alpha\lambda}\right)\epsilon^{-\frac{1}{\lambda}t}+\frac{\alpha\lambda}{1-\alpha\lambda}(u-A). \tag{7}$$

Every term in the coefficient of the exponential is a constant, so that some value of t can be found, after which the term will be negligible to any required order. For the values of α, λ, A, and u_0 commonly met with in practice, this time is comparatively short. The "steady state" is then said to be established, and the behavior of the thermometer is given by

$$\theta-u=\frac{\alpha\lambda}{1-\alpha\lambda}(u-A). \tag{8}$$

The difference $(\theta - u)$ is, therefore, dependent on u; but when α is quite small, the change, with time, in u is small and $u - A$ is almost constant. The temperature of the thermometer then follows that of the bath in a manner almost similar to that expressed by equation (6), namely, with a constant difference.[8]

METHODS OF DETERMINING λ

The equations of which the derivations and applications have just been discussed may be easily transformed into some which can be conveniently employed in the determination of λ.

Absolute Determination.[9]—Equation (4) may be written in the form

$$\frac{\theta_0 - u_0}{\theta - u_0} = \epsilon^{\frac{1}{\lambda}t}$$

which, in logarithmic form, is

$$\lambda = \frac{t}{\log \dfrac{\theta_0 - u_0}{\theta - u_0}} \tag{9}$$

This linear relation between time and logarithm of temperature differences is very convenient in getting the best mean value from a large number of readings of θ and t, because a graphical plot (time against logarithms of temperature difference) should be a straight line of slope λ.

The experimental details for carrying out the determination are simple. The thermometer is cooled and plunged into a bath

---

[8] From the expansion $\epsilon^{-\alpha t} = \left(1 - \alpha t + \frac{(\alpha t)^2}{2} - \ldots\right)$, one can conveniently study the function for values of t less than $1/\alpha$. The expression for u is

$$u = A + B\left(1 - \alpha t + \frac{(\alpha t)^2}{2} - \ldots\right)$$

If the precision of the work in hand permit of neglecting αt with respect to unity we have $u = A + B$, or the change in u is insignificant during the time t considered.

If αt be appreciable, but its square and higher powers are negligible, we have $u = A + B(1 - \alpha t) = (A + B) - \alpha B t$, which is the linear change $u = u_0 + \gamma t$.

Calorimetric work, generally speaking, never permits of neglecting the first power. Ordinary work does permit of neglecting the second and higher powers and the temperature changes, which are really logarithmic, are treated as linear. Precision work requires retention of at least one more term in the series and it may be as convenient to keep the exponential as such as to employ a quadratic expansion for it.

[9] Method given here is the classical procedure, described by Thiesen, Guillaume, and others.

which is stirred at the desired rate, and maintained at constant temperature throughout the experiment. This condition may be secured by the use of a bath of large heat capacity and at room temperature. As the meniscus of the thermometer rises, the times of passing chosen graduations are read off on a watch or recorded on a chronograph. The later thermometer readings serve to give the bath temperature u_0. Any reading may be taken as θ_0 and time reckoned from that of this reading. Corresponding to each reading θ, a value of log $\frac{\theta_0 - u_0}{\theta - u_0}$ is computed and plotted against time. The slope of the best mean straight line is the numerical value of λ. If logarithms to base 10 instead of the base ϵ be used, the modulus must be taken account of in computing the slope.

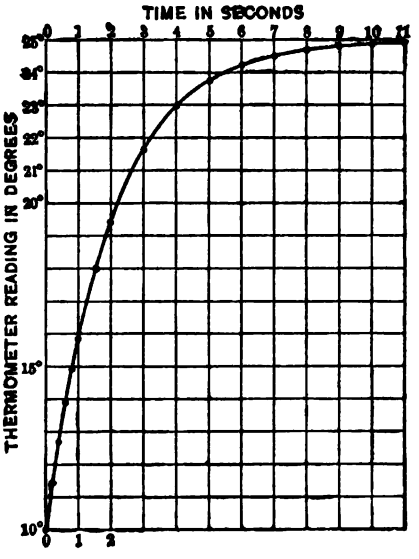

Fig. 1.—*Rise of thermometer cooled below 10° and plunged into bath maintained at 25° (for λ=2 seconds)*

The chronograph should be employed if the λ to be determined is small. Fig. 1 illustrates the rate at which the meniscus of a thermometer rises from 10° to 25° for $\lambda = 2$ seconds (small bulb chemical thermometer in well-stirred water). It makes evident the desirability of securing time readings to a fraction of a second.

The agreement of results in a well-stirred water bath when a chronograph is employed has been found to be well within 5 per cent. Observations taken with a watch depart from a straight line, on a logarithmic plot, to an extent making it difficult to determine the slope within this limit of certainty.

Departures of points either side of a straight line (logarithmic plot) in an irregular fashion, indicate the degree of experimental error in making readings, but departures in the nature of a consistent curvature are a warning that the thermometer is one to

which the equations just deduced should not be applied other than as first approximations. Slight curvature may be considered to define limits between which lies a mean value that may be used as λ in the equations above, for the approximate computation of lag corrections to readings. Additional assumptions, modifying equation (1), lead to a more perfect representation of the behavior of many such thermometers. (See Sec. II.)

Relative Determinations.—The difference in the values of λ for two thermometers, under the same conditions, can be obtained in a bath whose temperature is uniform and rising linearly. If one be known under the conditions of the intercomparison, the value of the other is thus obtained.

Applying equation (6) to two thermometers read at the same instant

$$\theta_1 - u = -r\lambda_1;\ \theta_2 - u = -r\lambda_2$$

Subtracting

$$\theta_2 - \theta_1 = -r(\lambda_2 - \lambda_1) \tag{10}$$

$$\lambda_2 - \lambda_1 = \frac{1}{r}(\theta_1 - \theta_2) \tag{11}$$

The rate of rise, r, is the same for both thermometers and for the bath (p. 665), and therefore readily determined by observing the progress of either thermometer. The difference $(\theta_1 - \theta_2)$ [10] at any

[10] θ_1, θ_2, refer here, as elsewhere, to readings corrected for bore, zero error, etc.; i. e., assume a perfect thermometer. Since $\theta_1 - \theta_2$ is generally small and since only the comparative instrumental correction is required, much greater precision can be obtained by direct comparison of the thermometers at the time of the experiment for lag, than by the use of tables of corrections pertaining to the instruments individually.

Convenient procedure is to carry out the lag experiment with the required rate of rise of temperature, and then so diminish the supply of heat to the bath employed that the condition of constant temperature is as nearly attained as may conveniently be, and intercompare the thermometers. The departure from constancy of temperature must be in the direction of a slow rise, never a fall, because the sticking of a falling meniscus vitiates thermometer comparisons.

The working equations will be:

Let θ=reading of thermometer if "perfect."

R=observed reading.

ϵ=correction for all instrumental errors.

$$\theta_1 - \theta_2 = (R_1 + \epsilon_1) - (R_2 + \epsilon_2)$$

so that equation (11) is used in form

$$\lambda_2 - \lambda_1 = \frac{1}{r}[(R_1 - R_2) + (\epsilon_1 - \epsilon_2)]$$

where $\epsilon_1 - \epsilon_2$ is determined directly from the mean of a number of pairs of readings taken in the inter comparison at a very small rate r. Designating such readings with primes, equation (10) is

$$(R'_2 + \epsilon_2) - (R'_1 + \epsilon_1) = 0$$

or

$$\epsilon_1 - \epsilon_2 = R'_2 - R'_1$$

because when r is small enough, no matter what λ_1, λ_2, may be, the term $r(\lambda_2 - \lambda_1)$ may be neglected.

instant may be repeatedly obtained to give a reliable mean, and if λ_1 be known the data suffices to give λ_2 with the simplest of computations. λ_1 may be known by a previous application of the method described on p. 667, or it may be computable from other constants of the thermometer (e. g., see Section III on platinum resistance thermometers).

Frequently λ_1 may be neglected, and the method becomes an approximate absolute one for "slow" thermometers and more convenient than the other. For example, if the result of an experiment gives $\lambda_2 - \lambda_1 = 30$ seconds, and λ_1 is surely less than 3 seconds, λ_2 is determined to be 30 seconds ($\pm$ the experimental error) within 10 per cent. A little familiarity with thermometers and their behavior in baths stirred at various rates enables one to place by inspection an upper limit for λ for many forms of thermometers and thus apply the method just described to "slow" thermometers.

Illustrative Values of λ.—To illustrate the concordance of both final results and the individual readings pertaining to a single result, a short series of experiments performed in vigorously stirred water is tabulated in Table I and plotted on Fig. 2. The method employed was the logarithmic one described on page 667 et seq.

TABLE I

Date	Thermometer number	Brief description of same	Lag in well-stirred water
1909			Sec.
Aug. 30	Chabaud, 77874	0°–50° thermometer divided in 0°1 (1° about 6.6 mm long). Convenient to read 0°01. Bulb approximately 4.5 mm diameter and 25 mm long.	2.00 2.13 2.15
Do.	Golaz, 4192	Open-scale calorimetric thermometer divided in 0°02 (1° about 31 mm long). Convenient to read 0°001. Bulb approximately 9 mm diameter and 52 mm long.	4.82 4.95 4.96 4.78
Do.	5951	"Einschluss Faden." Bulb a long narrow thread of mercury, 100 mm × 2.5 mm (diameter) surrounded by an air space of about 3 mm, outer envelope of glass approximately 8 mm in diameter inclosing the whole.	52
Do.	1787	Callendar type platinum resistance thermometer. Fine platinum wire coil wound on a cross of sheet mica and inclosed in a porcelain tube about 30 cm long, 1 cm diameter, and 0.15 cm thick.	15.3 15.7

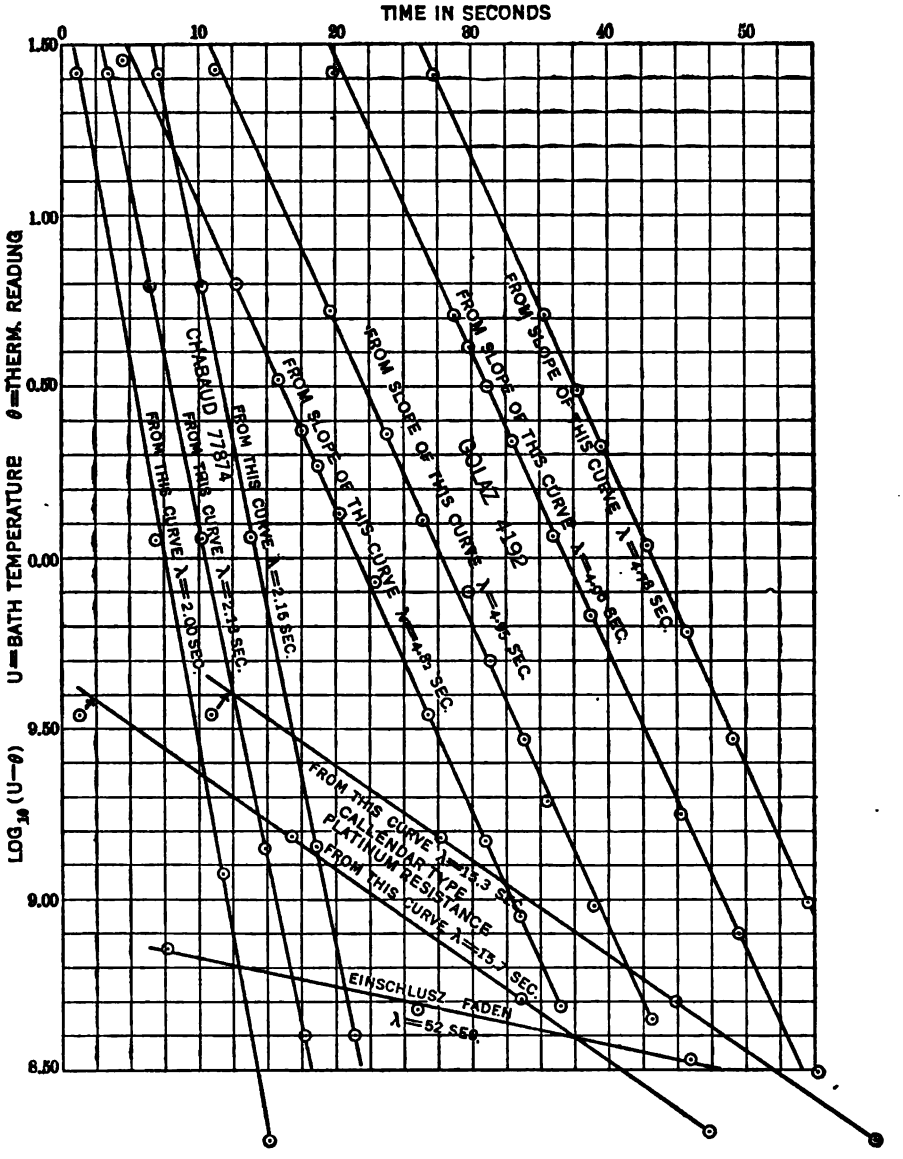

Fig. 2.—*Typical curves obtained in determining λ (vigorously stirred water). To illustrate degree of concordance obtained for individual readings forming one determination; and also for slopes of the several determinations with a given thermometer*

A brief description of the thermometers is included, giving an idea of the value of λ for different types. The lag of a platinum resistance thermometer of the Callendar type is tabulated with those of the mercurial instruments to show its comparative magnitude.

VARIATION OF λ WITH STIRRING

With a given thermometer of the usual type in a given medium stirred at a certain rate, a definite numerical value for λ in the equation (1)

$$\frac{\partial\theta}{\partial t}=\frac{1}{\lambda}(u-\theta)$$

may be found and applied in this equation, or integrated forms deduced from it. But this same value of λ must not be employed if the medium be stirred at a different rate.

LAG IN WATER.—As an interesting example arising under conditions of use, the following values were measured in a calorimeter with a propeller stirrer rotated as slowly and as rapidly as the convenience of the calorimeter stirring arrangement would permit.

TABLE II

Thermometer Golaz 4191

	λ	Mean λ
	Sec.	Sec.
Vigorous stirring	4.22 4.34 4.42	4.3
Slow stirring	6.46 6.40 6.57	6.5
No stirring	15.5 14.5 14	14.5

An increase of 50 per cent in the value of λ is thus observed upon decreasing from the normal vigorous stirring of calorimetric use to a propeller speed of one-third the normal. Such an experiment gives no definite relations, however, because the velocity of

flow past the thermometer bulb is not known. Accordingly, a special apparatus was assembled in which the velocity of flow could be measured. The schematic drawing, Fig. 3, renders description unnecessary.

The velocity of the current of water past the thermometer bulb was computed from the quantity of water delivered in a definite time and the area of the annular space $\left[\frac{\pi}{4}(\overline{1.19^2} - \overline{0.54^2}) = 0.88_8 \text{cm}^2\right]$ between the thermometer bulb and the surrounding glass tube.

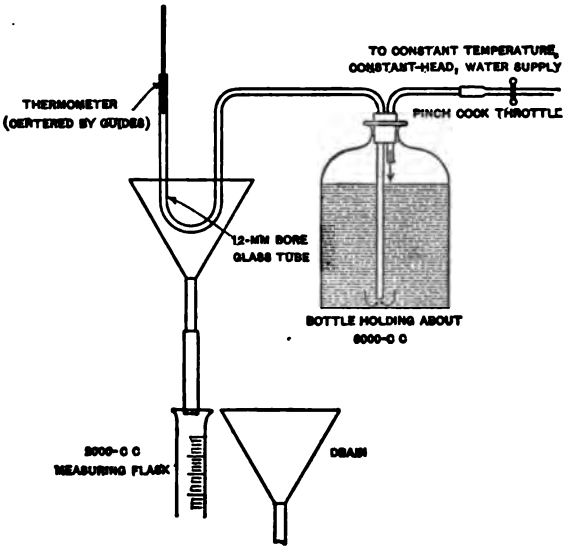

Fig. 3.—*Apparatus for determining lag of thermometer in a stream of liquid flowing at a definite rate*

This does not take account of the "drag" near the walls, but the nature of the problem in hand does not warrant such refinement.

With the stream flowing at the desired rate, and its temperature that of the room (between 30° and 35°) the thermometer was cooled in ice and dropped into place in the tube, centering by guides. As the meniscus passed appropriate graduations between 10° and room temperature the observer made a record chronographically. This was repeated several times for each velocity tested. The results are summarized in the accompanying Table

III and plotted on Fig. 4. The method of computing results was the logarithmic plot method described on page 667, and the individual points lie on a straight line very closely. The results of the several determinations at each rate of stirring are in good agreement, as shown by the table.

TABLE III

Value of λ for Various Velocities of Water Flow Past Bulb (in Apparatus of Fig. 3)

Thermometer Chabaud 80659

Date	Experiments	Mean flow cc per sec.	Av. dev. from mean	Velocity cm per sec.	Mean λ sec.	Av. dev. from mean
1911						
July 13	4	0.62	±0.01$_5$	0.70	5.6	±0.1$_0$
De	4	1.14	0.01$_0$	1.28	4.6	0.1$_5$
De	4	1.6$_7$	0.01$_0$	1.9$_0$	4.2	0.1$_3$
De	6	1.8$_7$	0.02	2.1	4.1	0.0$_5$
De	5	3.3	(1 obs.)	3.8	3.5$_4$	0.0$_5$
July 5	4	4.9	(1 obs.)	5.5	3.2$_4$	0.0$_5$
July 13	6	5.3	0.0 (2 obs.)	6.0	3.22	0.04
July 6	5	9.6	0.1	10.9	2.89	0.03
De	6	20.$_9$	0.2	23.$_5$	2.59	0.06
De	5	33.$_6$	0.1	37	2.37	0.02
De	5	40.$_0$	0.1	45	2.41	0.03

As the velocity decreased, the logarithmic plots curved a little, making accurate determination of λ impossible. It is to be noted, however, that considerable latitude in the value of λ (vertical displacement of a point) at the lower velocities would not greatly affect the shape of the curve.

For the determination of lag under conditions approximating infinite velocity the thermometer was plunged into steam and read as above described. The heat supplied instantaneously as steam condenses on the bulb maintains its surface at the temperature of the steam, less the drop through the layer of water formed there. The thickness of this, if uniformly distributed over the bulb, would be less than 0.05 mm at the conclusion of the experiment, a film of the order of one-tenth the thickness of the glass in the bulb. It may be concluded that the behavior of the thermometer under the conditions of this experiment is probably not very different

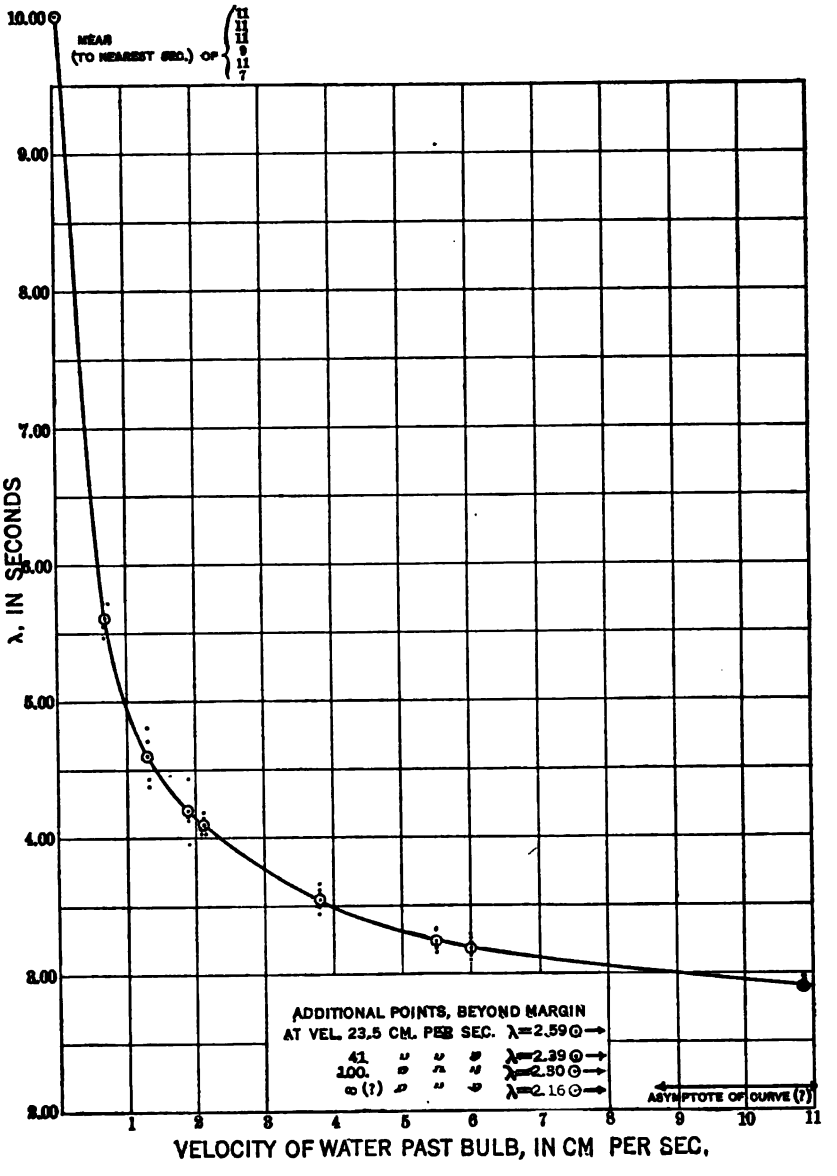

Fig. 4.—*Variation of λ with stirring*
Thermometer Chabaud 80659 in water

from what its behavior might be if an infinite supply of heat could be instantaneously brought to the surface of the bulb.[11]

The values of λ obtained in steam are given in Table IV. Here are also the values obtained in water at rest except for its own natural convection.

TABLE IV

Limiting Values of λ for Curve of Fig. 4 (Zero Velocity of Water Past Bulb; Infinite Velocity (?) of Any Medium Past Bulb)

Thermometer Chabaud 80659

Date	Immersion	Remarks	Experiments	Av. dev. from mean	Mean λ
1911					Sec
July 10...	Slightly superheated steam[12]..	Equivalent of infinite stirring (any medium)?	6	±0.04	2.16
Do...	Still water in calorimeter can..	Lag curves not straight lines. Cf. pp. 668-9.	3	0	11
July 13...	.....do.....	.....do.....	3	1	9

[12] Steam determinations in International Bureau form of steam-point apparatus (due to Chappuis). Pressure of steam about 5 mm of water in excess of atmosphere.

The determinations in unstirred water give wide latitude of variation, probably due to differences in convection currents. A mean value of λ of 10 seconds was obtained, using a large can of water at constant temperature as the immersion bath.

The minimum value of λ obtained in the experiments summarized in Table III being 2.4 seconds, it seemed desirable to test higher velocities and see whether the steam value of 2.2 seconds was more closely approached. Modifications of the apparatus for the purpose of securing a larger flow somewhat impaired the accuracy of measurement. One hundred cm per second was attained and the mean λ found was 2.3 seconds (the statement of a third figure not being warranted by the results).

LAG IN KEROSENE OIL.—On account of the great variation in the rate of change of the lag with the velocity as shown on Fig. 4, a short investigation with another liquid was undertaken to see

[11] If this be true, the λ determined in this way should be independent of the medium; i. e., a characteristic constant of the thermometer. Any other vapor of high latent heat should give the same result. An experiment in alcohol gave a value of λ (2.5— seconds) somewhat larger than in steam (2.2 seconds), being about equal to the λ found in water at the highest velocity measured with the apparatus of Fig. 3 (λ=2.4 seconds at 45 cm per second, Table III).

whether the same general form of curve (when the same units of measurement were employed) would be found. A kerosene oil was employed with the same thermometer and gave a very similar plot. As might be expected, the more viscous liquid gives a greater value of λ for a given velocity (Fig. 5). It must be noted that the values given were all obtained in one tube (internal bore 12 mm), and the question of change in the relations at that velocity where the flow through a tube changes from a steady drift to a turbulent motion, involving experiments in tubes of several sizes, would enter into a more complete study of the subject.

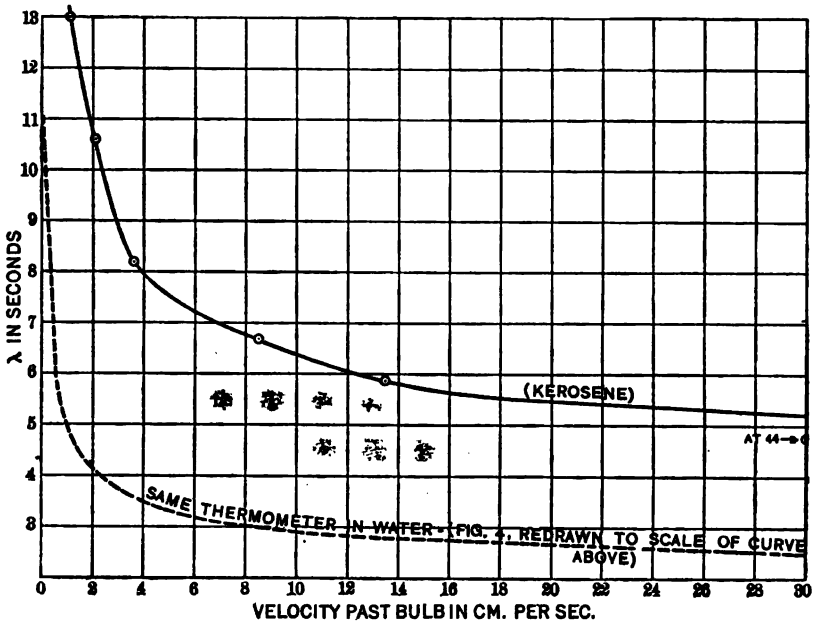

Fig. 5.—*Variation of* λ *with stirring*
Thermometer Chabaud 80659 in a kerosene oil

The data obtained for kerosene oil, including the density and viscosity of the oil at 20° C, are summarized in Table V, and the results plotted as Fig. 5. The discrepancies between individual observations are more likely due to lack of care in obtaining the data than inherent in the method or the heat-convecting properties of the oil. No great pains were taken concerning temperature regulation of oil, and the observations were taken more hastily

than for the work in water, as an accuracy of 5 per cent seemed quite sufficient to illustrate what was desired. No observations are discarded in making Table V.

TABLE V

Lag of a Thermometer in Oil. (Apparatus of Fig. 3)

Thermometer Chabaud 80659

[Mixture of kerosene oils: Density, at 20° C, 0.870 g per cc; viscosity, at 20° C, 5° Engler,[13] equivalent to 0.31 dynes per cm²]

Date	Experiments	Mean flow cc per sec.	Av. dev. from mean	Velocity cm per sec.	Mean λ	Av. dev. from mean
1911					Sec.	
Nov. 13	6	0	.....	0	40-50	[14](20)
Nov. 11	3	0.96	±0.04	1.08	13.0	0.1
Nov. 13	4	1.80	0.15	2.03	10.6	0.2
Nov. 11	4	3.2	0.15	3.6	8.2	0.0_4
Nov. 10	6	7.5	0.3_8	8.5	6.7	0.0_7
Nov. 11	4	11.9	0.2	13.5	5.9	0.0_7
Do	4	38	1.5	44	4.8	0.1_3

[13] A viscosity of 5° Engler means that a fluid of that viscosity runs through the efflux tube of an Engler viscosimeter at a rate one-fifth that for water at same temperature.

[14] Determinations of the lag in this oil unstirred showed large variation, owing probably to the relatively high viscosity of the oil interfering with convective interchange of heat between the liquid near the bulb and the mass of the liquid.

LAG OF A THERMOMETER IN AIR.[15]—The lag of a thermometer employed to measure the temperature of gas has considerable practical interest because it is great enough to affect the results quite appreciably. It is not, however, easy to make proper correction in the usual case, because the thermometer is subject to

[15] To this subject a large number of papers have been contributed, chiefly in the meteorological journals. Most of these must be characterized as little more than qualitative investigations, because the experimenters do not more closely specify the velocities employed than as "strong wind," "lightly moving wind," "quiet room," etc. The most complete quantitative papers are by Hergesell, Meteorologische Zeitschrift, 14 pp. 121 and 433, 1897. Simultaneously with this appeared the paper of J. Hartmann, Zeitschrift für Instrumentenkunde, 17, p. 14 (1897), which is qualitative only.

Wilhelm Schmidt, Meteorologische Zeitschrift, 27, p. 400 (1910), contributes data for over a dozen thermometers, mercurial, alcohol, toluene, and metallic expansion, under a variety of conditions, but without quantitative specification thereof.

de Quervain, in the same journal, 28, p. 88 (1911), reviews previous work and deduces therefrom some formulæ for which it appears that he claims great generality. Certainly they are not universal in their application, for they fail to agree with the observed behavior of the thermometer used as an example in the present paper, a thermometer of the ordinary "chemical" type in a current of air. Brief reviews by de Quervain summarize the salient features from the following papers:

Dufour in 1864 (see Meteorologische Zeitschrift, 14 p. 276, 1897); Hartmann, loc. cit.; Hergesell, loc. cit.; Valentin, Meteorologische Zeitschrift, 18 p. 257, 1901; Maurer, ibid., 15 p. 182, 1898, 21 p. 489, 1904.

Rudel, in the same issue of the Meteorologische Zeitschrift, 28, p. 90, 1911, gives the data for some thermometers and reviews the work of Krell, Zeitschrift für Heizung, Lüftung, und Beleuchtung, 11, 1906-7.

Marvin, Monthly Weather Review, 27, p. 458, 1899, gives some data pertaining to the lag of kite thermographs.

drafts of widely different velocities and no single value for λ holds for more than a few moments. To illustrate the magnitudes involved, it may be well to anticipate the data tabulated below and discuss the lag in air of the thermometer whose constants in water and oil have just been given. In still air the mean λ found was 190 seconds. The time of 7λ seconds for an initial difference of 10° between the thermometer and still air in which it might be immersed, to be reduced to 0°.01 (see p. 665) is accordingly over 20 minutes. The reading of a gas temperature to hundredths of a degree with a mercurial thermometer must therefore be undertaken with due lapse of time permitted after immersion. Then, too, in a space warming a degree in 15 minutes (if

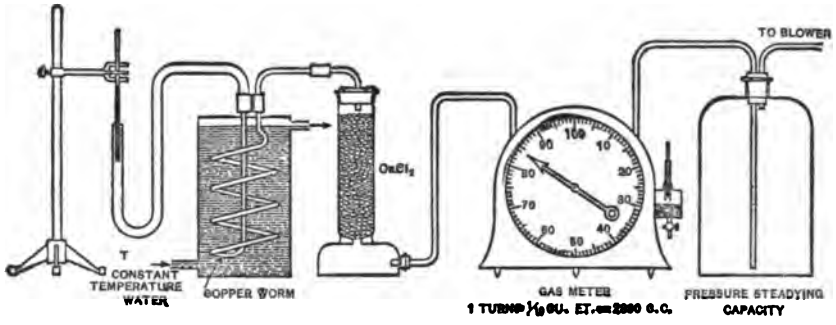

Fig. 6.—*Apparatus for determining lag of thermometer in a stream of gas flowing at a definite rate*

there be no drafts) the error in any reading of this thermometer after reaching the equilibrium state, would be over 0°.20. In the presence of drafts the numbers cited are considerably reduced, but they are worth consideration as examples of the error under the worst conditions.

To study the variation of the lag with drafts in air, thermometer Chabaud No. 80659 was immersed in currents of various velocities in the same U tube as that employed for water and oil (Fig. 3). The current of air was measured by a gas meter in series with the tube. The area of the annular orifice past the thermometer being the same as for the experiments with water, the computation on page 673 of 1.13 cm/sec velocity for each cc per second current holds good here. The accuracy claimed for the tabulated velocities is but 5 per cent.

Since the wet meter raised the humidity of the air above the point where dew condensed on the thermometer bulb, when cooled sufficiently for the experiment, a drying tower of $CaCl_2$ was necessary. A copper worm in a thermostatic water bath was introduced to steady temperature fluctuations. The apparatus is shown diagrammatically in Fig. 6.

The resistance of the piping employed placed the upper limit, for the apparatus shown, at a current of about 90 cm per second. To obtain some points on the curve at higher velocity the apparatus was modified by substituting a dry meter, of larger capacity, dispensing with the drying tower and also the copper worm. With this arrangement, velocities up to 1000 cm per second were obtained. The value of λ of the thermometer is, at this velocity, 23 seconds, whence it appears that an almost inconceivably large velocity of gas past the bulb of a thermometer would be necessary to supply heat as fast as the surface can transmit it to the interior, corresponding to a value of 2.2 seconds for λ, the value in steam (p. 674).

The logarithmic method described on page 667 was modified slightly to a more convenient form to avoid plotting lines and computing logarithms. This is possible when the motion of the thermometer meniscus is comparatively slow throughout the scale; i. e., when λ is large. Every λ seconds a given difference of temperature between medium and thermometer is reduced to ϵ^{-1} times its initial value. From tables of ϵ^{-n} we obtain the following data for an initial difference of 10°.

TABLE VI

(ϵ^{-n})	At time t,	Temperature difference equals—	For bath at 30°, thermometer reads—
$\epsilon^{0} = 1$	0	10°00	20°00
$\epsilon^{-\frac{1}{2}} = .6065$	$\frac{\lambda}{2}$	6.06	23.94
$\epsilon^{-1} = .3679$	λ	3.68	26.32
$\epsilon^{-\frac{3}{2}} = .2231$	$\frac{3\lambda}{2}$	2.23	27.77
$\epsilon^{-2} = .1353$	2λ	1.35	28.65
$\epsilon^{-\frac{5}{2}} = .0821$	$\frac{5\lambda}{2}$	0.82	29.18
$\epsilon^{-3} = .0498$	3λ	0.50	29.50
$\epsilon^{-\frac{7}{2}} = .0302$	$\frac{7\lambda}{2}$	0.30	29.70
$\epsilon^{-4} = .0183$	4λ	0.18	29.82

Cooling a thermometer and plunging it into a bath at 30° one would read off the times corresponding to the thermometer readings tabulated in the last column. These should occur at equal intervals of half λ. Most of the thermometer readings will be odd valued and not easily carried in mind, no matter what temperature range is chosen, whence the method is hardly to be recommended unless the observation interval be great enough to permit of reference to notes between readings. Also with a rapidly moving meniscus odd values can not be timed as accurately as can coincidences with even graduation lines.

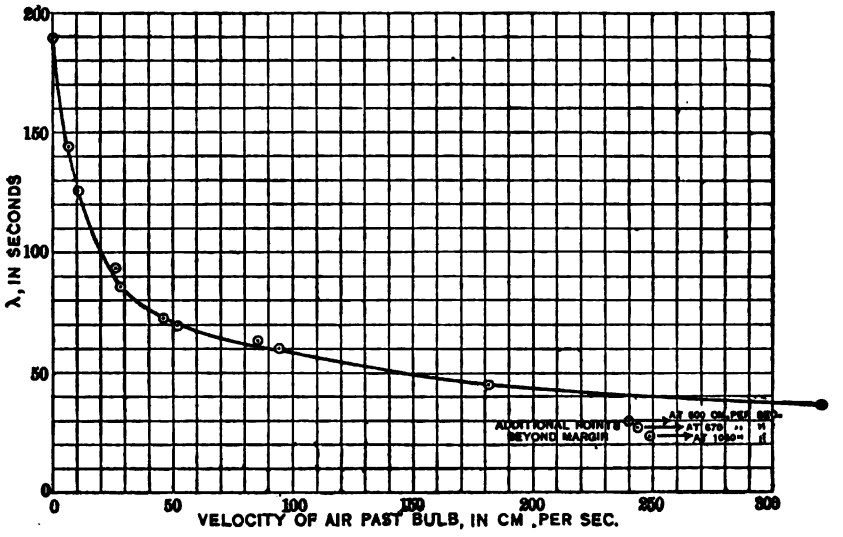

Fig. 7.—*Variation of λ with current, in air. Thermometer Chabaud 80659*

It is very easy to see that a slight change in the bath temperature affects the time-differences for the later readings by a large percentage; consequently the first three or four differences are the more dependable, and it is desirable to employ a table of readings made out for every λ/2 seconds rather than for any greater interval, to secure a number of intervals before the temperature difference grows too small.

The results of the experiments in air are summarized in Table VII and shown graphically on Fig. 7.

TABLE VII

Values of λ in Air Passing Thermometer Bulb at Different Rates

Thermometer Chabaud 80659

Date	Apparatus, etc.	Experiments	Velocity[16] cm per sec.	Mean λ	Av. dev. from mean
1911				Sec.	
July 25	Inclosed space—air at rest except convection	3	0	190	±11
July 27	U tube, etc. (no current)	1	0	190	.....
Sept. 14	U tube, etc., Fig. 6 (wet meter)	3	6.3	144	5
Do	do	3	11	126	2
Do	do	3	26	94	4
Aug. 19	do	1	$28._5$	86	.....
Do	do	2	47	73	1
July 27	Dry meter (tower and worm removed)	1	52	70	.....
Aug. 17	Fig. 6 (wet meter)	4	86	64	2
July 27	Dry meter, etc	2	95	60	0.5
July 26	do	2	18_0	46	0.0
Do	do	2	32_0	37	0.5
Do	do	2	60_0	30	0.5
Do	do	3	67_0	28	0.0
July 28	do	2	106_0	24	0.5
July 25	Suspended vertically in greatest draft of high-speed horizontal fan.	3	?	24	0.5

[16] The average deviation from the mean velocity is without significance; as to the precision read, different observations were in exact agreement among themselves. Readings were taken to about 1 per cent, but the gas-meter calibration was slightly uncertain, making a systematic error greater than this possible.

II. LAG OF A BECKMANN THERMOMETER

ADDITIONAL ASSUMPTIONS

It is found that the behavior of the ordinary type of Beckmann thermometer is not completely represented by the equations that have been developed, whence the assumption $\frac{\partial\theta}{\partial t}=\frac{1}{\lambda}(u-\theta)$, equation (1) (see p. 661) is not justifiable. A consideration of the form of the instrument suggests a reason for this. The main bulb is like the bulb of the type of instrument considered above. But in addition the large capillary, which is common in such instruments, between the bulb and the zero of the scale, acts as a second smaller

bulb.[17] Inclosed in a tube with an air layer between it and the bath, it is quite slow to assume the bath temperature, yet the amount of mercury in this secondary bulb is sufficient to appreciably affect the position of the meniscus in the small capillary.

There are thus two bulbs, the temperature of each of which may be expressed by the law $\frac{dA}{dt}=\frac{1}{\lambda}(u-A)$ where u is the temperature of the surrounding medium and A is the average temperature of the bulb, as previously defined (p. 661).

$$\frac{dB}{dt}=\frac{1}{\lambda_B}(u-B) \text{ for main bulb} \tag{12}$$

$$\frac{dC}{dt}=\frac{1}{\lambda_c}(u-C) \text{ for large capillary} \tag{13}$$

B and C together define the position of the meniscus in the bore of the thermometer, this being the reading θ. If the relation of θ to B and C be stated (this being equation 14 below) it will be possible to eliminate from the three equations (12), (13), (14) the two quantities B and C, which are not directly determinable, and leave a relation connecting θ with u, quantities in which interest centers.

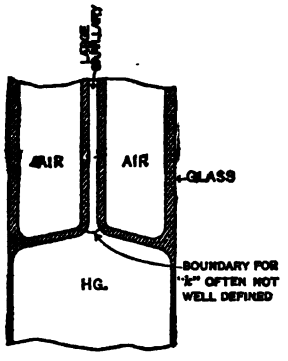

Fig. 8.—*Section through Beckmann thermometer*

In stating the dependence of θ upon B and C one has to bear in mind—

Firstly, that the volume of the mercury originally in the bulb will change proportionately to the change in the temperature B, and the volume of that originally in the large capillary will be likewise related to C.

Secondly, the position of the meniscus in the bore will vary as the sum of these two volumes changes.

[17] Total immersion is assumed in this paper, along with the assumption previously mentioned that the thermometer is instrumentally perfect. The difference in the behavior of any thermometer between total and partial immersion must be considered as part of the theory of emergent stem corrections and can not be included here. In the experiments tabulated in this section the thermometer was immersed to the top of the large capillary, the results given being on the basis of total immersion.

Whence $d\theta$ will be the sum of two quantities proportional to dB and dC, respectively, and, since no constant of integration except zero could satisfy the obvious relation $\theta=0$ when both B and C are 0, θ must be related to B and C exactly as $d\theta$ to dB and dC, although this does not imply that the temperature of the thermometer as a whole is to be thought of as an addition of the temperatures of the several parts.

If unit quantity of mercury be distributed between two bulbs, k in the first and $1-k$ in the second, the effect of each in defining the position of the meniscus will be in the proportion k to $1-k$, besides the effect of the respective temperatures. We may take for the unit of quantity the total amount of mercury in any thermometer and so omit factors of proportionality and state the relation in the simple form

$$\theta=kC+(1-k)B \tag{14}$$

where k is the fraction of the total volume contained in the large capillary.

Omitting the steps of the elimination of B and C from (12), (13), and (14) and collecting, the resulting equation is

$$\lambda_B\lambda_C\frac{d^2\theta}{dt^2}+(\lambda_B+\lambda_C)\frac{d\theta}{dt}+\theta=[k\lambda_B+(1-k)\lambda_C]\frac{\partial u}{\partial t}+u \tag{15}$$

If equations deduced from it are verified by experiment this equation, which can not be directly tested, will be justified. We shall find that a sufficient agreement obtains to do this, the equations developed from it proving to be fairly close approximations to exact statements of the behavior of this type of thermometer. It must be noted that k can not be accurately determined, since the end of the bulb and beginning of the large capillary is not thermally a definite location, even if mechanically it were so. (Fig. 8.)

To obtain the primitive of (15) requires considerable mathematical manipulation, but presents no difficulties as the steps follow common textbook suggestions in [18] order, being type-form

[18] See, for instance, A. R. Forsyth: A Treatise on Differential Equations—"General linear equations with constant coefficients," p. 64 (3d ed., 1903); Notes on particular integral and complementary function in the section on General Linear Equations of the Second Order, p. 98; "Method of variation of parameters," p. 110.

processes. The result is

$$\theta = u + A_1\epsilon^{-\frac{1}{\lambda_B}t} + A_2\epsilon^{-\frac{1}{\lambda_C}t} - k\epsilon^{-\frac{1}{\lambda_C}t}\int\frac{\partial u}{\partial t}\epsilon^{\frac{1}{\lambda_C}t}dt - (1-k)\epsilon^{-\frac{1}{\lambda_B}t}\int\frac{\partial u}{\partial t}\epsilon^{\frac{1}{\lambda_B}t}dt \quad (16)$$

where A_1 and A_2 are the arbitrary constants of integration and must be fixed by assigning two definite conditions. One of these may well be to assign simultaneous values to all the variables, defining for the time 0 that the corresponding bath temperature be u_0 and thermometer reading θ_0. The most convenient second condition to impose is that all parts of the thermometer shall be at the same temperature at this time zero, i. e., $B_0 = C_0$. Since $\theta = kC + (1-k)B$, whenever $B = C$, either of these equals θ, so that this second condition is expressed by the relation

$$\theta_0 = B_0 = C_0 \quad (17)$$

Whenever the thermometer remains in a medium of constant temperature for a considerable time all parts come to this temperature and the condition (17) is then fulfilled at every instant; so that by taking as $t = 0$ the instant of the transfer to any medium at a different temperature (for immersion in which latter medium the equation is to be applied) both conditions outlined can be readily satisfied in practice.

A_1 and A_2 are given by the two equations

$$\theta_0 = u_0 + A_1 + A_2 - k\int^0 \epsilon^{\frac{1}{\lambda_C}t}\frac{\partial u}{\partial t}dt - (1-k)\int^0 \epsilon^{\frac{1}{\lambda_B}t}\frac{\partial u}{\partial t}dt \quad (18)$$

from the substitution of initial values in (16), and

$$\left(\frac{k}{\lambda_c} + \frac{(1-k)}{\lambda_B}\right)(\theta_0 - u_0) = \frac{A_1}{\lambda_B} + \frac{A_2}{\lambda_c} - \frac{k}{\lambda_c}\int^0 \epsilon^{\frac{1}{\lambda_C}t}\frac{\partial u}{\partial t}dt - \frac{1-k}{\lambda_B}\int^0 \epsilon^{\frac{1}{\lambda_B}t}\frac{\partial u}{\partial t}dt \quad (19)$$

from substitution in the first derivative of (16) of the first derivative of (14) and subsequent reduction of the result by the use of (12) and (13) to a form where the values at the time $t = O$ (relation 17) can be substituted to give the form as written.

Equations (18) and (19) are not worth solving explicitly for A_1, A_2 for substitution in (16), because the special cases arising in practice are more easily referred to these implicit forms. The cases arising more frequently in laboratory practice will be treated

in a later section, but one of these must be developed at this point. By considering the behavior of a Beckmann thermometer plunged into a bath maintained at constant temperature, and the behavior predicted by the theory here developed, we shall find that an excellent test of the validity of the theory is afforded.

(a) **Constant Temperature.**—If $u=U_0$, $\frac{\partial u}{\partial t}=O$ and equation (16) becomes

$$\theta=U_0+A_1e^{-\frac{1}{\lambda_B}t}+A_2e^{-\frac{1}{\lambda_C}t} \tag{20}$$

Equations (18) and (19) have the forms

$$\theta_0=U_0+A_1+A_2$$

$$\left(\frac{k}{\lambda_C}+\frac{1-k}{\lambda_B}\right)(\theta_0-U_0)=\frac{A_1}{\lambda_B}+\frac{A_2}{\lambda_C}$$

From which A_1 and A_2 may be readily obtained.

$$A_1=(1-k)(\theta_0-U_0) \tag{21}$$

$$A_2=k(\theta_0-U_0) \tag{22}$$

Equation (20) in its complete form is [19]

$$\frac{\theta-U_0}{\theta_0-U_0}=(1-k)e^{-\frac{1}{\lambda_B}t}+ke^{-\frac{1}{\lambda_C}t} \tag{23}$$

under the conditions, at $t=o$, imposed above (p. 685).

When $k=0$ or $k=1$ or $\lambda_C=\lambda_B$, any one of which conditions corresponds to a single bulb instead of a compound one, this equation reduces to (4) of p. 664 as is necessary.

The properties of functions of e^{ax} depend so entirely on the numerical value of the exponent that it is rather difficult to generalize, but a few remarks may assist to a clearer picture of

[19] An equation of this general form was proposed by Thiesen (loc. cit.), with a bare statement that the idea had occurred to him of separate, independent lags for two parts of those thermometers which failed of expression by the more common single lag equations. No derivation of the equation is given, and it is left with undetermined coefficients in the form $\theta-U_0=Ae^{-\frac{1}{\lambda_1}t}+Be^{-\frac{1}{\lambda_2}t}$, with the suggestion that this form be tried in cases where $\theta-U_0=Ae^{-\frac{1}{\lambda_1}t}$ obviously fails to express the behavior of the instrument.

the curve theoretically representing the behavior of a thermometer of the usual Beckmann type plunged into a constant temperature bath. To illustrate the discussion, Fig. 9 is inserted, although it has been necessary to exaggerate greatly the value of k from the usual size, in order to separate sufficiently the curves, on the scale which must be here employed, to make clear which function is which.

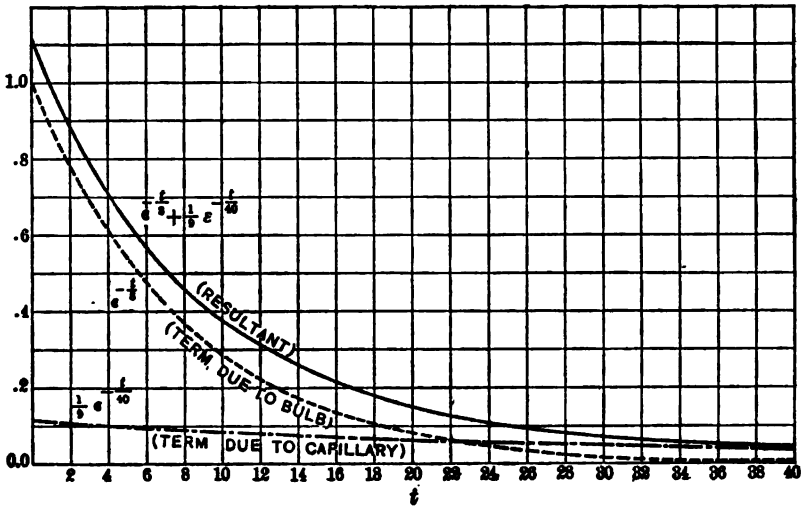

Fig. 9.—*Curves to explain theory of Beckmann lag (exaggerated value of k)*

$\lambda_B = 8$ *sec.* $\lambda_c = 40$ *sec.* $k = 0.1$ $y = e^{-\frac{1}{\lambda_B}t} + \frac{1}{9}e^{-\frac{1}{\lambda_c}t}$

Let us confine our attention to the two terms separately, that due to the capillary, $\frac{ke^{-\frac{1}{\lambda_c}t}}{1-k}$, and then that due to the bulb, $e^{-\frac{1}{\lambda_B}t}$. K is small with respect to unity, and λ_c is generally several times λ_B. Fixing in mind λ_B as the interval of time for use as a convenient comparison unit, t must be large before $e^{-\frac{1}{\lambda_c}t}$ reduces very greatly from unity; whence for a considerable time the term $\frac{k}{1-k}e^{-\frac{1}{\lambda_c}t}$ is very little different from k. On Fig. 9, with the numbers employed, it starts at the value 1/9 and very gradually approaches the axis.

The term $\epsilon^{-\frac{1}{\lambda_s}t}$ decreases at a rate very much more rapid than that of $\epsilon^{-\frac{1}{\lambda_c}t}$, λ_s being but a fraction of λ_c, so that although its initial value unity is very large in comparison with k, it reduces to this value, k, before very long, and continuing to decrease disappears numerically while the other term is appreciable.

The sum of the terms in equation (23) is accordingly almost exactly the first for small values of t and the second after considerable time. This point is illustrated by Fig. 9, and the actual magnitudes which are involved in a practical case can be pictured by supposing k diminished perhaps as much as thirty times. At the one end the difference between the resultant and its first term would be reduced from 10 per cent of either, as there shown, to a very small fraction of it, or for most purposes the curves might be considered identical. At the other end the total magnitude of the resultant would of course be much smaller, but it would, at some time, bear exactly the same relation to the second term that it does on the plot, namely, become practically equal to it, because the first term is diminishing at all points at a rate much greater than that of the second. It may also be well to call attention to the fact that as k is smaller, the point where the first and second terms are the same size, about 22 seconds on Fig. 9, displaces more and more toward the right.

The conclusions that have been reached predict that if a thermometer of this type be plunged into a bath 10° warmer than it is, it will cover the first 9°, say, of its rise in almost the same way as would a thermometer with a single lag constant λ_s; and will cover (about) the last 0°.1 in nearly the same way as would a thermometer of single lag constant λ_c; the interval between corresponding to neither.

JUSTIFICATION OF ASSUMPTIONS MADE

The method of testing out the theory proposed is at once suggested. The value of k may be approximately determined for a thermometer with a "secondary" bulb, such as the large capillary below the scale in the usual type of Beckmann. Then λ_s and λ_c may be approximately found by using the two ends of the observed "lag curve" of that thermometer, obtained by the logarithmic

method described on page 667. If, then, the middle of the curve computed from these values of λ_B, λ_c, and k in the equation (23) be the same as the middle of the curve obtained by direct observation, it is fair evidence that the function written is a proper one to represent the behavior of the thermometer. The test was made and the results with one thermometer are given in full to make clear the procedure.

Beckmann No. 5952 has, in its large capillary (Fig. 8) about 20° of mercury inclosed by the outer glass tube so as to have only poor thermal contact with the medium of immersion. 6300° being approximately the volume[20] of the mercury in a thermometer bulb, the value of k is $20/6300 = 0.003_2$; and $1-k$ is unity within the limits of accuracy of this computation. Substituting these numbers in equation (23),

$$\frac{\theta - U_0}{\theta_0 - U_0} = \epsilon^{-\frac{1}{\lambda_B}t} + .003_2\epsilon^{-\frac{1}{\lambda_c}t}$$

should be the equation to give the reading (θ) of this thermometer, at any time (t) after immersion in a bath maintained at constant temperature (U_0), provided all parts of the thermometer were at the same temperature (θ_0) when the instrument was introduced into the bath. If θ_0 be below U_0 the first 0.9 of the rise must follow very closely the equation $\frac{\theta - U_0}{\theta_0 - U_0} = 1.00\epsilon^{-\frac{1}{\lambda_B}t}$, and applying the methods of page 667 to simultaneous readings of thermometer and time an approximate value of λ_B may be determined. A number of such experiments made with thermometer No. 5952 gave a mean value of 8.7 seconds for λ_B.

By the time the quantity $(U_0 - \theta)$ is reduced to 1 per cent of $(U_0 - \theta_0)$ the term $\epsilon^{-\frac{1}{\lambda_B}t}$ is smaller than $0.0032\ \epsilon^{-\frac{1}{\lambda_c}t}$ if λ_c be about five times λ_B, and a little later may, for first approximations be neglected. The equation

$$\frac{\theta - U_0}{\theta_0 - U_0} = 0.0032\epsilon^{-\frac{1}{\lambda_c}t}$$

[20] The unit of volume being the volume that forms one degree in the stem of the particular thermometer of which the bulb is a part. The number 6300 depends somewhat on the glass, but the relative expansion coefficient is seldom far from 0.00016 = 1/6300 (approximate).

applied for a number of readings after θ has almost reached U_0 gives λ_c. Since the temperature differences to be read are extremely small, little more than the order of magnitude of λ_c may be deduced. This however is all that is necessary, as an error of 20 to 30 per cent would not greatly influence numerical values in the complete equation. The mean of a number of experiments indicated 50 seconds to be the best value to use for λ_c.

In Table VIII is summarized the computation of

$$F \equiv (1 - .0032)\, \epsilon^{-\frac{1}{\lambda_B} t} + .0032\, \epsilon^{-\frac{1}{\lambda_c} t}$$

for given values of t; λ_B taken as 8.70 seconds, λ_c, 50 seconds.

TABLE VIII

t	$(1-.0032)\,\epsilon^{-\frac{t}{8.70}}$	$.0032\,\epsilon^{-\frac{t}{50}}$	F	$\log_{10} F$	$\log_{10} \epsilon^{-\frac{t}{8.70}}$
10	0.3160	0.0026	0.3186	9.503	9.501
20	.1002	.0021	.1023	9.010	9.002
30	.0317	.0018	.0335	8.524	8.502
40	.0101	.0014	.0116	8.064	8.003
50	.0032	.0012	.0044	7.643	7.505
60	.0010	.0010	.0020	7.301	7.005
80	.00010	.00065	.00076	6.881	6.007
100	.00001	.00043	.00044	6.643	5.008
120	.00000	.00029	.00029	6.462	4.010

The second and fourth columns show at a glance the extent of the agreement of the function F with its first term, the fifth and sixth columns showing the same for the logarithms, which are more apt to be employed in computing a lag experiment. The sixth column is of course a straight line, when plotted against time; the fifth is so up to about 40 seconds, after which it rapidly assumes marked curvature. This length of time after immersion was accordingly the interval available for the determination of λ_B.

This curvature of the logarithmic plot, due to the term involving the lag of the capillary, distinguishes it sharply from the plot due to a single lag, which is linear. The functions themselves are of the same general shape as illustrated on Fig. 9, so that in comparing the function F to the corresponding curve obtained by experiment, it is better to compare logarithms than direct values. This is done in Fig. 10.

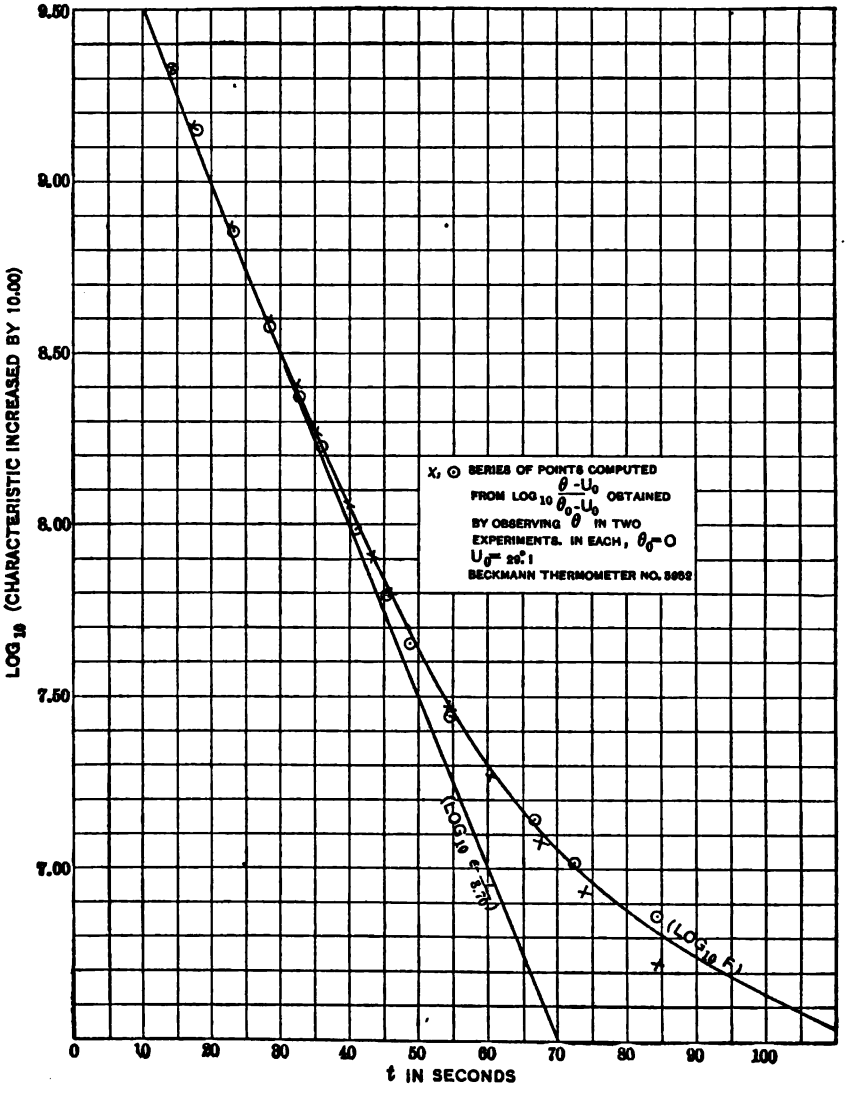

Fig. 10.—$Log_{10} F \quad F \equiv (1-0.0032)e^{-\frac{t}{8.70}} + 0.0032e^{-\frac{t}{50}}$

If equation (23) represents the facts, log F, of Table VIII, and $\log \frac{\theta - U_0}{\theta_0 - U_0}$, as obtained experimentally for this thermometer, must coincide. A number of experiments were made by cooling it to 0° throughout and plunging it into a bath maintained at 29°.10, reading as it warmed up. The data from two representative experiments gave the circles and crosses plotted on Fig. 10. The agreement with $\log_{10} F$ (the curve as shown) seems to justify the theory proposed for this type of thermometer.

SPECIAL CASES

The more important special cases for which the theory will be developed somewhat in detail are three, (a) constant temperature, (b) linear change, (c) exponential change, according to the law $u = A + B\epsilon^{-at}$.

The application of the equations to practical problems proceeds along the lines already given in the section devoted to common or "chemical" thermometers.

(a) CONSTANT TEMPERATURE.—The equations for this condition have been derived above (p. 686) and need not be repeated. The only question of lag which arises is: How long after immersion of the thermometer in a constant temperature bath must an observer wait to secure a given accuracy in reading? The time is computed from equation (23), following the principle outlined for an ordinary thermometer in the discussion following equation (4). However, as the time for a Beckmann to attain an "equilibrium condition" when plunged into a liquid bath is greater than that for a common-type chemical thermometer, and it may be advisable to summarize the computations of an example to show the order of magnitude. The quantities involved are

$$(1-k)\epsilon^{-\frac{1}{\lambda_B}t} \text{ and } k\epsilon^{-\frac{1}{\lambda_c}t}$$

For the thermometer No. 5952 (in stirred water).

$$k = 0.0032 \qquad \lambda_B = 8.70 \text{ sec.} \qquad \lambda_c = 50 \text{ sec.}$$

The term $(1-k)\epsilon^{-\frac{1}{\lambda_B}t}$ becomes 10^{-4} after about $9\lambda_B$ seconds, or 78 seconds, but at that time $k\epsilon^{-\frac{1}{\lambda_c}t}$ is 0.0032×0.211, or 0.00067.

This term does not diminish to 10^{-4} until t reaches the value of 174 seconds. At this time the first term is so small that the sum is identical with the latter value. Accordingly, about three minutes is the time for a temperature difference of 10° to be reduced to 0°.001 when this Beckmann thermometer is plunged into a bath of water vigorously stirred.

(b) LINEAR RISE OF TEMPERATURE.—$u = U_0 + rt$, $\frac{\partial u}{\partial t} = r$. Substituting this value of $\frac{\partial u}{\partial t}$ in equation (16), (p. 685), and performing the integrations, the result is

$$\theta = u + A_1 \epsilon^{-\frac{1}{\lambda_B} t} + A_2 \epsilon^{-\frac{1}{\lambda_c} t} - rk\lambda_c - r(1-k)\lambda_B \tag{24}$$

A_1 and A_2 may be determined, under the conditions imposed (p. 685), by equations (18) and (19). Replacing $\frac{\partial u}{\partial t}$ of these equations by r and carrying out the solution for A_1, A_2,

$$A_1 = (1-k)(\theta_0 - U_0 + r\lambda_B) \tag{25}$$

$$A_2 = k(\theta_0 - U_0 + r\lambda_c) \tag{26}$$

These values may be put into equation (24) to give its complete form. A considerable time after immersion the terms containing them reduce to negligible size because of the factor $\epsilon^{-\frac{1}{\lambda} t}$, and the equation has the simple form.

$$\theta = u - r(1-k)\lambda_B - rk\lambda_c \tag{27}$$

whence the thermometer follows the bath in which it is immersed with a constant difference of temperature existing between them of

$$r[(1-k)\lambda_B + k\lambda_c] \tag{28}$$

Although λ_c is generally several times λ_B, k is usually so small a fraction of $1-k$ that the last term is quite small with respect to the first, being negligible more often than not.

For the thermometer previously discussed, $\lambda_B = 8.70$ seconds and $(1-k)\lambda_B + k\lambda_c = 8.83$ seconds, a difference less than 2 per

cent and of the order of the uncertainty of λ_B as determined. The lag of this thermometer in the equilibrium condition when immersed in a medium whose temperature rises linearly is practically that of the bulb alone, the value of k and the lag of the capillary not entering the result to an appreciable extent. While the equilibrium condition is being established, however, their significance is not to be overlooked. From (24), (25), (26), neglecting small terms, it is seen that the ratio $\frac{\theta - u}{\theta_0 - U_0}$ is diminished with time at a rate dependent on $(1-k)\epsilon^{-\frac{1}{\lambda_B}t} + k\epsilon^{-\frac{1}{\lambda_c}t}$. This exact function was carefully examined for the case of a bath maintained at constant temperature, (p. 692), and the computations need not be repeated. About three minutes after immersion this thermometer would differ from the steady state by 10^{-4} times the initial temperature difference.

(c) LOGARITHMIC TEMPERATURE CHANGE, $u = A + B\epsilon^{-\alpha t}$.—The occurrence of this case has been explained in the footnote to p. 664, and the detailed development of pages 666–7 permits us to dismiss the form with a bare statement of the solution. Placing the value of $\frac{\partial u}{\partial t}$, namely $-\alpha B\epsilon^{-\alpha t}$ in equation (16), and dropping the terms containing an exponential, in accordance with the discussion following equation (7), we get for the solution in the steady state,

$$(\theta - u) = \alpha\left[k\frac{\lambda_c}{1-\alpha\lambda_c} + (1-k)\frac{\lambda_B}{1-\alpha\lambda_B}\right](u - A) \qquad (29)$$

III. LAG OF ELECTRICAL THERMOMETERS

Electrical thermometers in common use fall into one of two classes, thermoelectric or resistance. Some form of galvanometer is necessary as an indicator for either, and the lag of this galvanometer is to be added to the lag exhibited by the thermocouple or resistance coil in acquiring the temperature under measurement. It will often be found that the galvanometer lag is the greater portion of the whole; in fact, that frequently it is the only portion which need be considered at all.

GALVANOMETER LAG

An expression for the lag of a galvanometer is easily derived from the familiar equations governing the behavior of the instrument, whence it is quite unnecessary to treat the subject in detail, but the general method of deducing the required expression may well be summarized for the most common case, that of a D'Arsonval instrument, under the condition of critical damping. The fundamental equation for the motion of the coil

$$I\frac{d^2\theta}{dt^2}+K\frac{d\theta}{dt}+\tau\theta=N \tag{30}$$

N = applied moment (in general, a function of t)
θ = displacement
t = time
I = moment of inertia of moving system
K = damping coefficient
τ = elastic coefficient

has three different solutions according as K^2 is greater than, equal to, or less than $4I\tau$, leading respectively to the equation of motion if overdamped, critically damped, or underdamped. Critical damping occurs when $K^2=4I\tau$, and the solution [21] is

$$\theta=Ae^{-\frac{K}{2I}t}+Bte^{-\frac{K}{2I}t}+e^{-\frac{K}{2I}t}\left[t\int^t\frac{N}{I}e^{\frac{K}{2I}t}dt-\int^t\frac{N}{I}te^{\frac{K}{2I}t}dt\right] \tag{31}$$

For nearly all thermometric work, the function N is linear. A Wheatstone bridge or potentiometer is approximately balanced, and the unbalanced emf causes a deflection of the galvanometer according to the equation just written. This unbalanced emf changes in direct ratio to the temperature change, and may be taken as linear over the range of any one reading.

Let the moment N_0 be impressed on the galvanometer coil by closing the circuit at time zero, and let this moment decrease at

[21] See, for instance, A. R. Forsyth: A Treatise on Differential Equations—"Linear equation with constant coefficients—Case of 'equal roots,'" p. 64 (3d ed., 1903). Notes on particular integral in the section on the General Linear Equation, p. 98, or "Method of variation of parameters," p. 110.

constant rate r (passing through nil and increasing in opposite direction). The function N will be

$$N = N_0 - rt \tag{32}$$

The moment is zero at the instant $t = \frac{N_0}{r}$, which is accordingly the time of the temperature corresponding to exact balance of the bridge or potentiometer. If the deflection, θ, of the galvanometer be not zero until λ seconds later, or at time $\frac{N_0}{r} + \lambda$, it is evident that an error, due to lag, is made in the usual manner of reading such instruments when measuring changing temperatures.

Omitting all the steps of substitution of $(N_0 - rt)$ for N in the equation (31) and of determination of A and B for the initial conditions, stated below,

$$\theta = \frac{4I}{K^2}\left\{N_0 - rt + \frac{4I}{K}r - \epsilon^{-\frac{K}{2I}t}\left[\left(\frac{K}{2I}N_0 + r\right)t + N_0 + \frac{4I}{K}r\right]\right\} \tag{33}$$

if, at time zero, moment N_0 be suddenly impressed on the coil by closing the circuit when the coil is at rest $\left(\left.\frac{d\theta}{dt}\right]_0 = 0\right)$ in its equilibrium position $(\theta]_0 = 0)$.

For most galvanometers the value of $\frac{K}{2I}$ when critically damped will be found to be such that in a few seconds the term $\epsilon^{-\frac{K}{2I}t}$ is very small with respect to the other terms, and the factor multiplying it is not large, so the equation simplifies to

$$\theta = \frac{4I}{K^2}\left\{N_0 - rt + \frac{4I}{K}r\right\} \tag{34}$$

When the deflection, θ, is zero,

$$t = \frac{N_0}{r} + \frac{4I}{K} \tag{35}$$

The lag, λ, of the galvanometer behind an emf changing linearly is accordingly, after a short time, $\frac{4I}{K}$ seconds (see eq. (32)). As

an illustration of the magnitude of this quantity, a galvanometer much used by the author has the value $\frac{I}{K}=0.4$ second, whence its lag is 1.6 seconds. The circuit must be closed at least 5 seconds before taking a reading, if an accuracy of 0.1 per cent of the deflection is desired, as shown by the following computation, which gives the time that must elapse before the term in $\epsilon^{-\frac{K}{2I}}$ becomes 0.001.

If $\frac{I}{K}=0.4$ second, $\epsilon^{-\frac{K}{2I}t}=\epsilon^{-1.25t}=$ (about) 0.001 for $t=5$ seconds.

For the determination of $\frac{K}{I}$ for any particular galvanometer, many methods might be divised from the common equations discussed in the numerous papers on galvanometers. One may be outlined here. The free period, T, of a system of moment of inertia I and elastic coefficient τ is $2\pi\sqrt{\frac{I}{\tau}}$, from which relation it follows that

$$\tau=\frac{4\pi^2 I}{T^2}$$

The condition imposed by critical damping is that

$$K^2=4I\tau$$

from which

$$K^2=\frac{16\pi^2 I^2}{T^2}.$$

$$\frac{K}{I}=\frac{4\pi}{T}$$

The free period of a galvanometer system is very little different from that in which it vibrates under any conditions not closely those of critical damping, so that the period observed when swinging as little damped as possible, will usually suffice to give T with high accuracy. $\frac{K}{I}$ is thus given very directly; the lag of the galvanometer, being $\frac{4I}{K}$, is of the extremely simple form $\frac{T}{\pi}$ seconds.

RESISTANCE THERMOMETERS

The resistance thermometers studied were found to be either very fast or very slow in comparison with mercurial thermometers. The well-known type made after the design of Callendar, consisting of a platinum coil wound on a mica frame and inclosed in a glass, quartz, or porcelain tube, is quite slow. Immersed in well stirred water, the values of λ measured usually lay between 15 and 30 seconds, though even this value was exceeded. A departure from the straight-line plot by the logarithmic method described on page 667 was evident when the temperature difference was small, indicating that the equation $\frac{d\theta}{dt}=\frac{1}{\lambda}(u-\theta)$ is only a first approximation to the statement of the behavior of such thermometers. The deviation was quite marked in some instances and always in the direction and of the general curvature exhibited by the Beckmann thermometers discussed in an earlier section. It appears that the two-term formulæ there developed are better equations to apply empirically to a Callendar type resistance thermometer than the simpler equations. This may perhaps be explained by the fact that the temperature of the platinum coil is partly determined by that of the inner surface of the containing tube, for which the lag is relatively small, and partly by that of the support, for which the lag is relatively large. The conditions, therefore, resemble those considered in the section on Beckmann thermometers.

The very fast resistance thermometers were of the type in general use in this Bureau, in the range 0–100°C, an improved form of the instrument described in this Bulletin in 1907 by Dickinson and Mueller,[22] and will be more fully described in a future paper. The essential features respecting the lag of these instruments are the small heat capacity of the enveloping sheath and the intimate thermal contact between this and the resistance coil. Attempts to measure the lag (in liquids) gave no results, merely indicating it to be smaller than the method would admit of determining, namely, considerably smaller than the galvanometer lag which was about one and one-half seconds.

[22] Calorimetric Resistance Thermometers and Transition Temperature of Sodium Sulphate: This Bulletin, 8, p. 641, Reprint No. 68.

JAEGER-STEINWEHR METHOD OF COMPUTING THE LAG OF A RESISTANCE THERMOMETER

A method of computing the lag of such thermometers was proposed by Jaeger and Van Steinwehr, but in the form in which they published [23] it, only a lower limit is placed upon the value of the lag. The method depends upon measuring the heating of the coil by different intensities of current.

Writing the equation governing the transfer of heat between the coil and the medium in which the thermometer is immersed as

$$\frac{d\theta}{dt}=\frac{1}{\lambda}(u-\theta) \tag{1}$$

θ = temperature of coil
u = temperature of medium
λ = lag (in seconds)

the rate at which heat is transferred will be

$$M\frac{d\theta}{dt}=\frac{M}{\lambda}(u-\theta)$$

M = heat capacity of the system cooling (or warming).

If θ be greater than u, the thermometer coil will lose heat and in time dt will lose a quantity dH

$$dH=\frac{M}{\lambda}(\theta-u)dt \tag{36}$$

If there be any electric current, i, in the coil, heat will be generated at the rate Ri^2, where R is the resistance. This will tend to raise the temperature of the coil above that of the medium in which the thermometer is immersed. It will rise until the dissipation, which is proportional to $(\theta-u)$, equation (36), equals the generation. This heat generated in time dt, (J being number of joules in a calorie) is

$$\frac{Ri^2}{J}dt$$

[23] Jaeger and Von Steinwehr: Zeitschrift für Instrumentenkunde, 26, p. 241; 1906.

and so in the equilibrium state of the thermometer

$$\frac{M}{\lambda}(\theta-u)dt=\frac{Ri^2dt}{J}$$

$$\frac{1}{\lambda}(\theta-u)=\frac{R}{MJ}i^2 \tag{37}$$

Direct measurement of $(\theta-u)$ is rather difficult, if at all possible, but indirect determination of the value by employing two or more values of i is quite easy. Applying (37) to such a series, u being kept constant

$$\frac{1}{\lambda}(\theta_1-u)=\frac{R_1}{MJ}i_1^2$$

$$\frac{1}{\lambda}(\theta_2-u)=\frac{R_2}{MJ}i_2^2$$

etc.

The difference in θ does not change R appreciably for the second members of the two equations, whence subtracting

$$\frac{1}{\lambda}(\theta_2-\theta_1)=\frac{R}{MJ}(i_2^2-i_1^2)$$

$$\lambda=M\frac{J}{R}\,\frac{\theta_2-\theta_1}{i_2^2-i_1^2} \tag{38}$$

The value of M proposed by Jaeger and Von Steinwehr is the water equivalent of the platinum resistance coil, computed from dimensions, density, and specific heat. Obviously, this gives merely a minimum value. The water equivalent of a portion of the silk and shellac wrapping about the wire of their thermometer should have been included. This being doubtless several times [24]

[24] When a cylindrical heat source is surrounded by an annular covering (inner radius a and outer radius b), whose outer surface is maintained at a definite constant temperature (0), the equilibrium distribution of temperatures is expressed by

$$U=C\frac{\log\frac{r}{b}}{\log\frac{a}{b}}$$

Integrating through the cylindrical shell, to determine the position of the boundary for considering the heat transfer as between two bodies, one the core and a portion of the annular covering, the other the

that of the wire alone, the value of λ as 1/33 of a second, obtained in the computation, is many times too small. Nevertheless, the lag of such a form of thermometer is small compared with that of any ordinary galvanometer.

The thermometers of the Dickinson-Mueller type (improved form) possess the following constants: Heat capacity of platinum coil, 0.010 cal. per degree C. Heat capacity of mica in region of coil, 0.16 cal. per degree C.

The mica between the head of the thermometer and the resistance coil, supporting and insulating the leads (with a heat capacity of 0.30) and the platinum sheath (with a heat capacity of 0.60) can not be supposed to be heated by the coil appreciably above the temperature of any liquid in which the thermometer is immersed, and play no part in the computation under such a condition. This is equivalent to the statement that the whole temperature drop between the coil and the bath is to be found in the mica and air spaces separating these two. The value 0.17 for M places a safe upper limit, and the value 0.01 is the certain minimum for such a form of thermometer.

The factor

$$\frac{J}{R}\cdot\frac{\theta_2-\theta_1}{i^2-i_1^2} \qquad \text{(see eq. 38)}$$

has been found, for the thermometer investigated when immersed in well-stirred water, to be 8.0 $\left(\frac{\text{sec.} \times \text{degrees}}{\text{calories}}\right)$, giving to λ, for the values of M just stated, the limits 1.4 seconds and 0.08 seconds. For a close winding, such that the middle mica plate is quite inclosed, the water equivalent should include all of this plate and one-half of each of the outer plates, separating coil from sheath. This is, in all, two-thirds of the mica; the corresponding value of

---

outside medium and remainder of covering, this boundary is located so that a fraction of the covering

$$\left[\frac{1}{\log\left(\frac{b}{a}\right)^2}-\frac{1}{\left(\frac{b}{a}\right)^2-1}\right]$$

belongs to the core.

If $b=10a$, as in the case of a 0.1 mm wire covered to a total diameter of 1 mm, the fraction evaluates to 0.20; and for $b=5a$, 0.27 of the total heat capacity of such a covering adds to that of the core. (In either case it should be borne in mind that the heat capacity of the whole covering is many times (99 or 24) that of the core if volume specific heats be the same, i. e., the total heat capacity is 20 times, or 7 times the heat capacity of the core.)

M is 0.11 and λ would be 0.9 second. However, the thermometers investigated were of very "open" winding (see Fig. 11), pitch 0.7 mm. and wire 0.1 mm. diameter. The mica to be included with the wire, roughly estimated, is that inclosed in the dotted circles, about one-third the total. The most probable value of λ in a well-stirred liquid is thus less than one-half second.

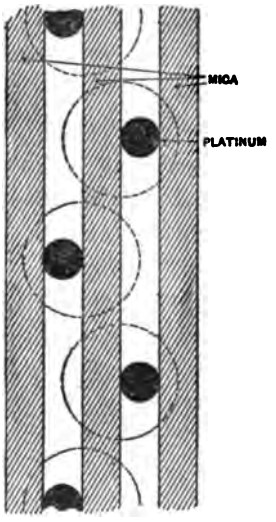

Fig. 11.—*Section through resistance thermometer of the Dickinson-Mueller type (greatly magnified)*

THERMOELECTRIC THERMOMETERS

The lag of a thermocouple in acquiring the temperature of a medium in which it is immersed is, like the lag of a resistance thermometer, principally a question of the form of mounting. Probably there are almost as many forms in use as there are makers of thermocouples, for there seems to be plenty of latitude for variation in this respect without impairing the usefulness of the finished instrument. Consequently it was deemed unimportant to test any particular forms of thermoelectric thermometer for lag. Dependent on the mounting of the junction one would no doubt find lags ranging from a small fraction of a second to perhaps 30 seconds for immersion in a well-stirred water bath. The important point is that for all work in that part of the temperature scale where high precision is attainable, so that lag corrections might be appreciable, it is possible to design a thermocouple with a lag as small or smaller than that of an ordinary galvanometer and generally, if not always, quite negligible.

IV. THERMOMETRIC LAG IN CALORIMETRY

In view of the different conclusions that have been reached by authors [25] who have considered the effect of thermometric lag on

[25] Jaeger and von Steinwehr: Verh. Deut. Phys. Gesells., 5, p. 353; 1903. Richards, Henderson, and Forbes: Proc. Amer. Acad., 41, p. 1; 1905; or Zs. für Phys. Chem., 52, p. 551; 1905. Jaeger and von Steinwehr: Zs. für Phys Chem., 54, p. 428; 1906. White: Physical Review, 27, p. 526; 1908.

calorimetric measurements, it was deemed of sufficient importance to consider the question in some detail. The conclusions reached have been arrived at, by a different analysis, by W. P. White.[26]

Classic procedure is to divide the ordinary calorimetric experiment into three parts, designated, respectively, as preperiod, middle period, and after period. A precise measurement of the calorimeter temperature, before adding the supply of heat whose determination constitutes the object of the experiment, defines the instant which separates the first two periods, and similarly a precise temperature measurement after the heat is added marks the dividing line between the second and third periods. Considerations of lag might therefore be said to pertain wholly to the middle period, but it is more convenient to treat the first temperature mentioned as the close of the preperiod rather than the beginning of the middle one, and to assign the second temperature similarly to the afterperiod. In this way the investigation is split up into three parts. Two of these concern the error, due to lag, in determining the temperatures mentioned, and are treated by investigating the conditions pertaining to the exchange of heat between the calorimeter, the thermometer, and the jacket, in the steady state, during preperiod and afterperiod. The third part involves the lag errors in the temperature readings used in the computation of the cooling correction, or is a middle-period function.

Numerous methods, differing radically in many ways, have been devised for performing calorimetric computations, but the fundamental relations underlying the "cooling correction" are the same, by whatever method the details be accomplished. It is generally assumed that the calorimetric aggregate exchanges heat with its envelope according to the law commonly called Newton's law of cooling:

$$-\frac{\partial u}{\partial t}=\alpha(u-U)$$

where u is the calorimeter temperature at the instant t and U the jacket temperature. In the simplest case the latter is constant throughout the three periods ($U=A$). α, the cooling constant,

[26] White: Physical Review, 31, p. 562; 1910.

and A are assumed possible of determination in the course of the run and will be discussed later.

After assembling the calorimeter any necessary time may be allowed to elapse for the effect of the initial conditions to be obliterated, so that during the preperiod the calorimeter temperature approaches that of the jacket according to a curve really logarithmic but of curvature so small it may frequently be considered linear. At a given instant, t, is commenced a supply of heat to the calorimeter and the middle period begins. If this supply could be distributed instantaneously, its measurement could be accomplished with no further process than the exact determination of the temperatures just before and just after the addition of the heat. No matter how long the middle period, the same result would ensue if the calorimeter could be perfectly insulated from external sources or sinks. In taking account of the loss or gain of heat in the middle period it is quite customary to compute, not a quantity of heat, but rather a temperature correction, to apply to the observed difference. The result is thereby stated in terms of the perfectly insulated calorimeter of the same heat capacity.

After the supply of heat to the calorimeter ceases, the temperature of the latter tends to a steady state of approaching that of the envelope at an almost constant rate. At any time after this state is attained, the precise measurement of the temperature is made, of which the time, t_3, marks the close of the middle period and beginning of the afterperiod.

After proper instrument corrections are applied, the thermometer readings give θ_1 and θ_2, which in turn, by appropriate lag corrections would give the observed rise of temperature, u_2-u_1. The additional increment of temperature resulting from the "cooling correction" is given by applying

$$du=-\alpha(u-A)dt$$

to each instant of the middle period, t_1 to t_3; whence the total "correction" is

$$K=+\alpha\int_{t_1}^{t_3}(u-A)dt$$

The equivalent rise of temperature, as if there were no heat losses during the middle period, is accordingly

$$u_2 - u_1 + \alpha \int_{t_1}^{t_2} (u - A)dt$$

for which it is customary to substitute a similar formula containing thermometer readings (corrected for instrument errors) on the hypothesis that they represent actual temperatures without lag. If any error due to lag occur, let it be designated ϵ

$$\epsilon = \left[\theta_2 - \theta_1 + \alpha' \int_{t_1}^{t_2} (\theta - A')dt\right] - \left[u_2 - u_1 + \alpha \int_{t_1}^{t_2} (u - A)dt\right] \quad (39)$$

and on the assumption that $\alpha' = \alpha$, $A' = A$. (See p. 706).

$$\epsilon = (\theta_2 - u_2) - (\theta_1 - u_1) + \alpha \int_{t_1}^{t_2} (\theta - u)dt \quad (40)$$

If the thermometer used in the middle period obey the equation (1), $\frac{d\theta}{dt} = \frac{1}{\lambda}(u - \theta)$ (p. 661), equation (40) is very easily reduced. If it do not, as, for instance, when a Beckmann thermometer is employed, the mathematical manipulation is more complex. This case will be discussed after taking up the one first mentioned. From (1) we obtain directly.

$$\int_{t_1}^{t_2} (\theta - u)dt = \int_{t_1}^{t_2} -\lambda \frac{d\theta}{dt} dt = -\lambda(\theta_2 - \theta_1)$$

so that equation (40) has the form

$$\epsilon = (\theta_2 - u_2) - (\theta_1 - u_1) - \alpha\lambda(\theta_2 - \theta_1) \quad (41)$$

The values θ_1 and θ_2 must be obtained in the preperiod and afterperiod in the steady state of heat exchange between the calorimeter, its surrounding jacket and the thermometer, for which condition the relations are given [27] by equation (8) as explained in the derivation of that equation, case (c) page 666.

[27] This relation is also derived by W. P. White, loc. cit.

Applying equation (8) to the conditions occurring at t_1 and t_2.

$$\theta_1 - u_1 = \frac{\alpha_1\lambda_1}{1-\alpha_1\lambda_1}(u_1 - A)$$

$$\theta_2 - u_2 = \frac{\alpha_2\lambda_2}{1-\alpha_2\lambda_2}(u_2 - A)$$

between which equations the jacket temperature A is to be eliminated. It is almost universal to employ the same thermometer in preperiod and afterperiod so that $\lambda_2 = \lambda_1$. Let us assume that the heat capacity of the calorimeter remains unaltered, as, for instance, by adding some substance in the course of a method of mixtures experiment; then $\alpha_2 = \alpha_1 = \alpha$. The elimination of A gives

$$(1-\alpha\lambda_1)(\theta_2 - \theta_1) = (u_2 - u_1) \tag{42}$$

But (41) is $$(1-\alpha\lambda)(\theta_2 - \theta_1) = (u_2 - u_1) + \epsilon$$

Therefore $$\alpha(\lambda_1 - \lambda)(\theta_2 - \theta_1) = \epsilon \tag{43}$$

The error due to thermometric lag is thus proportional to the difference of lag of the two thermometers used in the end periods and the middle period, vanishing if the same thermometer be employed in both.

The discussion is not quite complete without a consideration of the assumption $\alpha' = \alpha$ and $A' = A$ made at the time of defining ϵ and passed over almost without comment (p. 707). Examination of the magnitudes involved makes it quite evident that the assumption is always true to the extent that deviations from it are numerically insignificant for the highest precision work yet accomplished. For the Regnault-Pfaundler [28] and allied methods of computation it may be shown to hold to the extreme limit of being an exact statement of the relations involved. The proof follows:

From the primary equation

$$-\frac{du}{dt} = \alpha(u - A)$$

[28] Synopsis of this method may be found in Berthelot "Traite Pratique de Calorimetrie Chimique," 2d ed., p. 117.

applied to the instants t_1 and t_2, may be obtained two equations containing α and A, which may be solved for either. Solving,

$$\alpha = \frac{\left.\frac{du}{dt}\right]_1 - \left.\frac{du}{dt}\right]_2}{u_2 - u_1}$$

whereas we use a value computed from a similar expression connecting thermometer readings,

$$\alpha' = \frac{\left.\frac{d\theta}{dt}\right]_1 - \left.\frac{d\theta}{dt}\right]_2}{\theta_2 - \theta_1}$$

Now by equation (8)

$$\frac{d\theta}{dt} = \left(1 + \frac{\alpha\lambda}{1 - \alpha\lambda}\right)\frac{du}{dt} = \frac{1}{1 - \alpha\lambda}\frac{du}{dt} \tag{44}$$

$$\left.\frac{d\theta}{dt}\right]_1 - \left.\frac{d\theta}{dt}\right]_2 = \frac{1}{1 - \alpha\lambda}\left(\left.\frac{du}{dt}\right]_1 - \left.\frac{du}{dt}\right]_2\right)$$

and by equation (42)

$$\theta_2 - \theta_1 = \frac{1}{1 - \alpha\lambda}(u_2 - u_1)$$

from which it is evident that α' is identically α. Solving for A, for instance in the form,

$$A = u_1 + \frac{1}{\alpha}\left.\frac{du}{dt}\right]_1$$

an exactly similar process to the above shows that A' is identically A[29]

---

[29] $A' = \theta_1 + \frac{1}{\alpha}\left.\frac{d\theta}{dt}\right]_1$

$= u_1 - \lambda\left.\frac{d\theta}{dt}\right]_1 + \frac{1}{\alpha}\left.\frac{d\theta}{dt}\right]_1$ by equation (1)

$= u_1 + \frac{1}{\alpha}(1 - \alpha\lambda)\left.\frac{d\theta}{dt}\right]_1$

$= u_1 + \frac{1}{\alpha}\left.\frac{du}{dt}\right]_1$ by equation (44)

$\equiv A$ Q. E. D.

When a thermometer obeying the law $\frac{d\theta}{dt}=\frac{1}{\lambda}(u-\theta)$ *is employed in a calorimetric experiment, no account is to be taken of lag, if the same thermometer be used for preperiod, middle period, and after-period.* The important restriction on the generality of this proposition that the upper temperature measurement be postponed until the steady state of heat exchanges is well established, need hardly be included in the italicized conclusion because no measurement taken before the restriction was complied with would satisfy the more fundamental requirements of the calorimetric determination.

A thermometer which does not obey the law above stated must be examined by a process similiar to that outlined in the preceding pages. The only type we will consider here is the Beckmann, general equations for which have been deduced in Section II. The conclusions of the investigation show also that not merely is any error due to lag quite inappreciable for ordinary conditions, but that it is mathematically zero as for the case above. The expression for the error is a little tedious of derivation, although not so very difficult.

Let us start from equation (40) which is an entirely general expression for ε, the total error due to thermometric lag, with no limitation as to the form of thermometer, except in so far as might be said to lie in the assumption that α' and A' computed from lagging thermometer readings may be used interchangeably with α (the real cooling constant of calorimeter) and A (real jacket or "convergence" temperature), the values which would be obtained with a lagless thermometer. This assumption, briefly discussed before, is quite evidently true for a Beckmann thermometer.

Equation (40) is $\varepsilon=(\theta_2-u_2)-(\theta_1-u_1)+\alpha\int_{t_1}^{t_2}(\theta-u)dt$, and the difficulty in reducing it, when a Beckmann thermometer is employed, is the determination of the value of the last term for a thermometer obeying the law expressed by the relation (15). (The explanation of which quite uninterpretable equation, and the notation employed for this type of instrument, are to be found in the paragraph on p. 683, containing equations (12) and (13).)

The expression for $(u-\theta)$ developed in the general treatment contained in Section II (equation (16)), unfortunately contains u under an integral sign, presupposing a knowledge of the function before further reduction of the relation. It is very desirable to make this investigation entirely independent of the form of the curve which the calorimeter temperature rise follows, so that this expression can not be employed as there written. Tracing it back to its genesis we are obliged to use equations (12), (13), and (14) as follows·

$$\theta = kC + (1-k)\ B \tag{14}$$

$$u = ku + (1-k)\ u \qquad \text{(Identity)}$$

$$C - u = -\lambda_c \frac{dC}{dt} \tag{13}$$

$$B - u = -\lambda_B \frac{dB}{dt} \tag{12}$$

from which

$$\theta - u = k\left(-\lambda_c \frac{dC}{dt}\right) + (1-k)\left(-\lambda_B \frac{dB}{dt}\right)$$

and

$$\alpha \int_{t_1}^{t_2} (\theta - u)dt = -\alpha k \lambda_c \int_{t_1}^{t_2} \frac{dC}{dt} dt - \alpha(1-k)\lambda_B \int_{t_1}^{t_2} \frac{dB}{dt} dt$$
$$= -\alpha[k\lambda_c(C_2 - C_1) + (1-k)\lambda_B(B_2 - B_1)]$$

and equation (40), with the integration performed, becomes

$$\varepsilon = (\theta_2 - u_2) - (\theta_1 - u_1) - \alpha[k\lambda_c(C_2 - C_1) + (1-k)\lambda_B(B_2 - B_1)] \tag{45}$$

The Bs and Cs can not be united to θs on account of the factors λ_c and λ_B multiplying them, so the θs must be split into parts to permit of collecting terms. This is accomplished by means of (14) and the identity just beneath it, a few lines above. Applying to the instants t_1 and t_2,

$$\theta_1 - u_1 = k(C_1 - u_1) + (1-k)(B_1 - u_1)$$
$$\theta_2 - u_2 = k(C_2 - u_2) + (1-k)(B_2 - u_2)$$

from which a value for $(\theta_2 - u_2) - (\theta_1 - u_1)$ may be substituted in equation (45) giving

$$\varepsilon = k[(C_2 - C_1)(1 - \alpha\lambda_c) - (u_2 - u_1)]$$
$$+ (1-k)[(B_2 - B_1)(1 - \alpha\lambda_B) - (u_2 - u_1)] \tag{46}$$

We turn now to the equilibrium state in the preperiod and after-period and will pass by the case where thermometers of different lag are employed in these and the middle period. The relations of C and B to u are given by equations (12) and (13).

$$\frac{dB}{dt}=\frac{1}{\lambda_B}(u-B) \quad (12); \quad \frac{dC}{dt}=\frac{1}{\lambda_c}(u-C) \quad (13)$$

and u is determined by the relation

$$\frac{du}{dt}=-\alpha(u-A)$$

so that from the discussion of case (*c*) in Section I (p. 666) it may be seen that equation (8) states the relation of B or C to u when the steady state obtains.

$$B-u=\frac{\alpha\lambda_B}{1-\alpha\lambda_B}(u-A) \quad C-u=\frac{\alpha\lambda_c}{1-\alpha\lambda_c}(u-A) \qquad (8)$$

from which, applied at instants t_1 and t_2 come the relations

$$(1-\alpha\lambda_B)(B_2-B_1)=u_2-u_1$$
$$(1-\alpha\lambda_c)(C_2-C_1)=u_2-u_1$$

and equation (46) is at once

$$\varepsilon=O$$

No error is therefore made by employing thermometer readings, uncorrected for lag, throughout the computation when a Beckmann instrument of the usual form is used. The restriction mentioned before that this conclusion is true only when both θ_1 and θ_2 are measured in the steady state of heat exchanges requires more emphasis than in the case of an ordinary thermometer, because the Beckmann type is so much slower to reach this state, as shown on pages 692–694.

When thermometers with different lag, whether Beckmann or "chemical" type or electrical, are employed for the middle period and the end periods, when the heat capacity of the calorimeter is greatly changed in the course of a run, or when the adiabatic method of calorimetry is used, with thermometers of differ-

ent lag in the calorimeter and its jacket, the corrections because of lag may be appreciable and may be computed from the most convenient of the foregoing equations.

V. LAG CORRECTIONS IN APPLIED THERMOMETRY

Statements and conclusions relating to thermometric lag of importance in applied thermometry, are here summarized for convenience. As has been previously suggested, a determination of λ to an accuracy of only 50 or 100 per cent is quite often sufficient, as lag corrections are usually exceedingly small.

RÉSUMÉ OF PRACTICAL INSTRUCTIONS

1. **Constant Temperature.**—In a bath at constant temperature no correction for lag is to be made if the thermometer reading be taken a sufficient interval of time after introducing the instrument into the bath. A convenient interval to remember is 10λ seconds, in which time the initial difference of temperature is reduced to $\epsilon^{-10}(=0.00004)$ times itself. For all ordinary chemical thermometers in liquid baths stirred even slowly, 10λ will be rather less than a minute, and decrease from this value if the stirring be increased. In a gas where λ may have any value between 15 seconds and 10 minutes or more, depending upon conditions of stirring and form of thermometer, it may not be convenient to wait 10λ seconds for a reading. Computation on the basis of reduction every λ seconds of any temperature difference between bath and thermometer to a value ϵ^{-1} times that at the beginning of the period of λ seconds will show the interval which must elapse before the thermometer reading is correct within the allowable error.

2. **Linear Change of Temperature.**—In a bath whose temperature is rising at a uniform rate of r units per second, the correction to a thermometer reading at any time, to get the bath temperature at the same instant is $+r\lambda$ units. As a concrete example of the largest error likely to occur were lag neglected, may be reviewed the numbers in an intercomparison of a 3-second and a 15-second thermometer in a comparison bath rising at about the most rapid rate with which it would be practicable to obtain readings reliable to single thousandths, about 0°.03 per minute. One thermometer

would lag behind the bath by (0.0005×3) degrees, the other by $(0^\circ.0005 \times 15)$, making a difference of $0^\circ.006$ in their indications due to lag. So rapid a rate of rise when reading single thousandths and so great a difference in the values of λ for two thermometers in the same bath is of quite infrequent occurrence, whence it is not often that a lag correction will be found necessary.

3. **General Case of Changing Temperature.**—In a bath whose temperature is changing according to any more complicated law, but so as to be a continuous single valued function of time, the lag correction to any reading of a thermometer immersed in it is found by reading temperatures at sufficiently short intervals to plot the function θ with time so as to have the value $\frac{\partial\theta}{\partial t}$ at any point. The fundamental equation (1) (p. 661) then gives $u=\theta+\lambda\frac{\partial\theta}{\partial t}$, the lag correction being $+\lambda\frac{\partial\theta}{\partial t}$.

4. **Calorimetry.**—No correction for thermometric lag is to be made if all of the following conditions hold:

(1) The thermometer be one whose behavior accords with

$$\frac{\partial\theta}{\partial t}=\frac{1}{\lambda}(u-\theta)$$

(Symbols explained on p. 661), or is a Beckmann instrument of the usual form with a large inclosed capillary just above the bulb.

(2) The same thermometer (or different ones with the same lag, λ) be employed for preperiod, after-period, and middle-period temperature readings.

(3) The lower and upper temperatures be read after the calorimeter, its jacket, and the thermometer have certainly "attained the steady state" of heat exchange (usually a close approximation to linear rise or fall of temperature of calorimetric aggregate). "Attaining steady state" must, of course, be interpreted in the sense used physically for any mathematically asymptotic relation. There is seldom any doubt about this respecting the lower temperature reading; it is only the upper one concerning which the observer must be cautioned.

(4) The heat capacity of the calorimeter be the same during the three periods of the experiment.

VI. SUMMARY

1. Thermometric lag is conveniently expressed by employing a quantity λ, whose significance may be stated as follows:

(1) If a thermometer has been immersed for a long time in a bath whose temperature is rising at a uniform rate, λ is the number of seconds between the time when the bath attains any given temperature and the time when the thermometer indicates this temperature. In other words, it is the number of seconds the thermometer "lags" behind the temperature; or [30]

(2) If a thermometer be plunged into a bath maintained at a constant temperature (the thermometer being initially at a different temperature), λ is the number of seconds in which a difference between the thermometer reading and bath temperature is reduced to $1/\epsilon$ of its initial value.

The fundamental equation of heat transfer, commonly referred to as Newton's law of cooling, is stated in terms of λ, for application to problems in thermometric lag, and the principal working equations derived therefrom are reviewed.

2. To express analytically the lag of the common form of Beckmann thermometer, the simpler theory was modified to take into account the fact that the lag of the bulb and that of the large capillary, between the bulb and the fine capillary, are different.

3. Methods of determining lag are discussed and experiments are cited to test the theories as applied to ordinary "chemical" thermometers and to Beckmann thermometers.

4. The large variation in the lag of a given thermometer with the nature of the medium in which it is immersed, and with the rate of stirring of this medium, is brought out by experiments in water, in a viscous kerosene, and in air, in which these media were forced past the bulb of the thermometer at different measured rates.

Values from the curves obtained are:

[30] These two interpretations of λ are mutually consistent. The *definition* of the quantity is most logically made by designating it as the "constant" of the fundamental equation, and then deducing the interpretations here given.

λ in seconds. Small-Bulb "Chemical" Thermometer

Vel. past bulb in cm/sec.	0	1	5	10	50	100	500	1000	∞
Water	10.0	5.1	3.3	2.9	2.4	2.3	.....	.....	2.2 sec. (any medium)
Oil	40 to 50	13.4	7.5	6.4	4.8	.....	.....	.....	
Air	190	170	148	128	71	58	33	25	

5. Part, and frequently the largest part, of the lag of a thermoelectric or electrical resistance thermometer is the lag of the galvanometer. A d'Arsonval galvanometer, critically damped, is shown to lag $4 \times \frac{\text{Moment of Inertia of Moving System}}{\text{Damping coefficient.}}$ seconds behind an emf changing linearly with time, after the steady state of motion is attained. A close approximation to this value is lag $= T/\pi$ seconds, where T is the complete period of the moving system, oscillating much underdamped.

6. Types of resistance thermometers were tested for lag. The Callendar type, in a liquid bath, was found to lag greatly in comparison with an ordinary "chemical" mercurial thermometer, and the empirical expression of the lag is of the same form as that developed for a Beckmann thermometer. The Dickinson-Mueller type of resistance thermometer bulb in a liquid bath was found to lag much less than the fastest of mercurial thermometers.

7. The Jaeger-Steinwehr method of computing the lag of a resistance thermometer, from the heating effect of the measuring current and the heat capacity of the thermometer, is critically discussed. The lag of a Dickinson-Mueller thermometer in a well-stirred liquid bath is shown to be about one-half second.

8. The necessary corrections that must be applied to the observed readings of a thermometer to correct for the effect of its lag under the usual conditions of use are discussed in some detail.

9. An analytic proof is given of the fact that in an ordinary calorimetric experiment, in which the same thermometer is used to determine temperatures in the "preperiod," the "middle period," and the "afterperiod," the correction for lag in the middle period neutralizes the corrections for lag in the preperiod and the afterperiod.

WASHINGTON, March 5, 1912.

DETERMINATION OF MANGANESE AS SULPHATE AND BY THE SODIUM BISMUTHATE METHOD

By William Blum

CONTENTS

Page

I. Introduction 716
1. Importance of accurate manganese determinations 716
2. Sources of error in gravimetric methods 716
3. Sources of error in volumetric methods 717
II. The Bismuthate Method 718
1. Outline of method 718
2. History of method 718
3. Standardization of solution 719
(a) Discussion of methods 719
(b) Evidence based on reduction and reoxidation 719
4. Preparation of materials and solutions 720
(a) Water 720
(b) Air 720
(c) Asbestos 720
(d) Potassium permanganate 721
(e) Permanganate solutions 721
(f) Stability of permanganate solutions 722
(g) Manganous sulphate 723
(h) Sodium oxalate 724
(i) Ferrous sulphate solutions—Stability 724
(j) Nitric acid 724
(k) Bismuthate 725
(l) Ferric nitrate 725
(m) Use of weight burettes 725
5. Standardization of permanganate with sodium oxalate 726
6. Standardization of permanganate with manganous sulphate 727
(a) Standardization of manganous sulphate solutions 727
(b) Effect of conditions upon the standardizations with manganous sulphate 730
(1) Ferrous sulphate-permanganate ratio 730
(2) Amount of manganese present in a determination 732
(3) Acidity, volume, time of standing, etc 734
(4) Use of phosphoric acid 734
(c) Probable course of reactions 735
(d) Conditions recommended 736
7. Agreement of values derived from sodium oxalate and manganese sulphate 736
8. Analysis of pure permanganate crystals 737
9. Analysis of manganese ores 738
10. Summary 739

I. INTRODUCTION

1. IMPORTANCE OF ACCURATE MANGANESE DETERMINATIONS

In spite of the large number of methods in use for the determination of this important element, results by different methods and different chemists seldom show satisfactory agreement. While differences of as much as a few per cent of the manganese present have little commercial significance in iron and steel containing 1 per cent or less of manganese, the highest possible accuracy is demanded in the analysis of high-grade materials such as manganese ore and ferromanganese, of which large amounts enter into commerce, at prices dependent upon the results of analysis. For example, imports of manganese ore by this country in 1911 amounted to 176 852 long tons, valued at $1 186 791. It can readily be seen that a constant error of 1 per cent in the analyses of such material may cause a considerable difference in the total amount paid for the ores. That constant errors of such magnitude are possible with our present methods will be shown in this paper.

2. SOURCES OF ERROR IN GRAVIMETRIC METHODS

Even with the greatest care the gravimetric results are not necessarily accurate, due not alone to the possibilities of losses by solubility of precipitates; and of too high weight due to contamination from vessels or reagents; but also to uncertainty in the composition of the precipitates as weighed. The three forms in which manganese is most commonly determined gravimetrically are Mn_3O_4, $Mn_2P_2O_7$ and $MnSO_4$. It is generally admitted that the first of these is unsatisfactory, as the composition depends directly upon the temperature of the ignition and the nature of the atmosphere surrounding the precipitate. Even under carefully regulated conditions Raikow and Tischkow[1] could not obtain results which agreed to better than 1 part in 200. Gooch

[1] Chem. Ztg., 35, p. 1013; 1911.

and Austin[3] have shown that the composition of manganese "pyrophosphate" depends upon the content of ammonium salts and ammonia, and the temperature, volume, and method of precipitation of the manganese ammonium phosphate. Even under the conditions which they recommend, their errors amounted in some cases to 1 per cent of the manganese present, and were in general too high. The method can not, therefore, be considered satisfactory for highly accurate work, and certainly not for obtaining a known amount of manganese to serve as a primary standard. Experiments described in this paper have led to the conclusion that manganous sulphate, obtained under proper conditions, is the most accurate form in which this element can be weighed, both in gravimetric analysis and in securing a known amount of manganese.

3. SOURCES OF ERROR IN VOLUMETRIC METHODS

The difficulty of securing a known amount of manganese to serve as a primary standard has hindered the accurate investigation of the great number of volumetric methods which have been proposed. In most cases they have been tested by comparison with methods, gravimetric or volumetric, which had not been shown to be intrinsically accurate. This fact, together with the usual dependence of the results of such methods upon the precise conditions of operation, has led to the publication of a large number of contradictory papers upon these methods. For example, the Volhard method and its various modifications has been the subject of over 50 investigations since its publication in 1879. It is generally admitted that the results by this method are low unless an empirical factor dependent upon the conditions of operation, is employed, though some investigators have obtained theoretical results under certain conditions. In view of the above situation, it is highly desirable to find some method which will yield results of known accuracy, which are not closely dependent upon the exact conditions of operation, and which may serve to test other methods.

[3] Am. J. Sci., 6, p. 233; 1898.

II. THE BISMUTHATE METHOD

1. OUTLINE OF METHOD

Of various methods considered, the bismuthate appeared most promising, and has been found to entirely fulfill the above requirements. In this method the manganese in nitric acid solution is oxidized with sodium bismuthate, of which the excess is removed by filtration through asbestos. To the resulting permanganic acid is added a measured excess of ferrous sulphate solution, which is then titrated with permanganate of known strength and of known ratio to the ferrous solution. The investigation therefore resolved itself into a study of the methods of standardizing the permanganate employed in the final titration and the influence of the various conditions of operation upon the results obtained by the bismuthate method.

2. HISTORY OF METHOD

The method as originally prepared by Schneider [3] depended upon the use of bismuth tetroxide as the oxidizing agent and titration of the permanganate acid with hydrogen peroxide. In this form the method was employed by Campredon,[4] Mignot [5] and Jaboulay.[6] Reddrop and Ramage [7] modified it by employing sodium bismuthate, which was more readily obtained free from chlorine, and suggested filtration of the permanganate acid directly into the hydrogen peroxide. On account of the instability of the latter reagent Ibbotson and Brearley [8] replaced it by ferrous ammonium sulphate, in which form the method has been since used, being described in detail by Blair,[9] whose directions for this method are generally followed in this country.

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[3] Ding. Poly. J., 269, p. 224; 1888.
[4] Rev. Chim. Indust., 9, p. 306; 1898.
[5] Ann. Chim. Anal., 5, p. 172; 1900.
[6] Rev. gen. chim., 6, p. 119; 1903.
[7] J. Chem. Soc., 67, p. 268; 1895.
[8] Chem. News, 84, p. 247; 1901.
[9] J. Am. Chem. Soc., 26, p. 793; 1904 and Chemical Analysis of Iron, 6th and 7th ed.

3. STANDARDIZATION OF SOLUTION

(*a*) **Discussion of Methods.**—As above stated the accuracy of any such volumetric method depends principally upon the method of standardization employed. Blair [10] mentions in his book three methods, viz, (a) calculation from the iron value, (b) use of a steel of known content, and (c) use of a known amount of manganous sulphate, without expressing any preference, or opinions as to their relative accuracy. Standardization by means of sodium oxalate may be included under (a) since values found with this standard under proper conditions [11] have been found at this Bureau to agree with iron values within 1 part in 1,000.[12] Method (b) is a secondary method and is evidently unsuitable for work of high accuracy. As will be shown later, the standardization of manganous sulphate is a tedious operation and subject to considerable errors. For this reason sodium oxalate was considered at this Bureau to be the most convenient and accurate standard for this method. Brinton,[13] however, stated that there was a difference of 1 per cent (at first stated as over 3 per cent) between the values based upon sodium oxalate and manganese sulphate, respectively. In a paper from this Bureau by Dr. W. F. Hillebrand and the author [14] the reasons for our belief in the accuracy of the sodium oxalate standard were expressed in the form of a preliminary paper, the conclusions of which have been verified by subsequent investigations described in this paper.

(*b*) **Evidence Based on Reduction and Reoxidation.**—The original basis of our use of the sodium oxalate standard for this method was the fact that if a definite amount of a permanganate solution be reduced and then reoxidized by means of the bismuthate method, it is exactly equivalent in oxidizing power to the original permanganate. This experiment was based upon a

[10] In the appendix of the 1912 edition, p. 330, Blair recommends the standardization with sodium oxalate, under approximately the conditions given by McBride, as the most accurate method; a conclusion based upon the work described in this paper.

[11] McBride: J. Am. Chem. Soc., 34, p. 415; 1912.

[12] See Bureau of Standards Certificate for Sibley Iron Ore, Standard Sample 27.

[13] J. Ind. Eng. Chem., 3, pp. 237 and 376; 1911.

[14] J. Ind. Eng. Chem., 3, p. 374; 1911.

similar one suggested by Wolff [15] and employed by de Koninck [16] for testing the accuracy of the Volhard method for manganese. Its significance as applied to the bismuthate method is that the manganese is oxidized to the same state of oxidation as was originally present in the permanganate, theoretically Mn^{VII}. In the absence of evidence to the contrary it seems highly improbable that any appreciable manganese can be present in a filtered permanganate-solution in a form other than Mn^{VII}, and still less probable that in an entirely different medium the manganese should be oxidized by bismuthate to the same state of oxidation, other than Mn^{VII}. Since, however, at least two persons in addition to Brinton had observed a discrepancy of the order of 1 per cent between the sodium oxalate and manganese sulphate values, the subject deserved further investigation; not alone from the standpoint of the manganese determination, but also as possibly throwing light upon the composition of permanganate solutions and their action as oxidizing agents. At this point it may be mentioned that the original observations regarding the reduction and reoxidation of the permanganate have been confirmed entirely, with solutions A_1, B, E and G, prepared as shown on page 722.

4. PREPARATION OF MATERIALS AND SOLUTIONS

(*a*) **Water.**—Water used in the purification of permanganate and in the preparation of all the permanganate solutions except I and K was distilled three times, the last two being from alkaline permanganate. Water used for the rest of the work was ordinary distilled water of good grade.

(*b*) **Air.**—The air used to deliver the solutions from the stock bottles was washed with acid bichromate solution and alkaline permanganate followed by a column of glass wool.

(*c*) **Asbestos.**—The asbestos used in the filtration of the permanganate solutions and in the bismuthate method, was digested for several days with hydrochloric acid, which was finally removed by thorough washing with hot water. It was then suspended in water and the finest portions separated and used in this work. For a few of the experiments this asbestos was ignited, without

[15] Stahl u Eisen, 11, p. 373; 1891. [16] Bull. Soc. Chim. Belg., 18, p. 56; 1904.

making, however, any appreciable difference in the results. A 2-inch platinum cone, arranged as suggested by Blair,[17] was used for preparing the filter.

(*d*) **Potassium Permanganate.**—Two commercial samples of potassium permanganate were employed, Baker and Adamson's C. P. salt, and Kahlbaum's "K" grade. A portion of the former was purified by two recrystallizations in Jena glass flasks, the solutions being electrically heated, and filtered through ignited asbestos just before being allowed to crystallize. The fine crystals so obtained were sucked dry on a platinum cone and were then exposed in a thin layer in the dark for four weeks in a vacuum dessicator over concentrated sulphuric acid, the vacuum being maintained at approximately 2 cm. In spite of this long drying, the material was found to contain 0.38 per cent water as determined by heating to decomposition and collecting the water in a weighed calcium chloride tube. When dissolved in pure water and immediately filtered through asbestos, the solution left a slight stain upon the filter. After thorough washing this stain was dissolved off with sulphurous acid, and its manganese content determined colorimetrically, being equal to about 0.01 per cent, i. e., a negligible quantity. Numerous attempts to prepare a permanganate solution which would leave absolutely no stain upon asbestos proved unsuccessful. Whether such stains were due to the action of the asbestos itself as claimed by Tscheishvili [18], or to reduction of the permanganate by traces of dust or other reducing substances, could not be determined. The amount of such reduction was, however, negligible, and far less than that observed by Tscheishvili.

(*e*) **Permanganate Solutions** were prepared by dissolving a weighed amount of the salt in water; and in the case of the commercial samples, filtering through asbestos to remove manganese peroxide, etc. They were then made up to a definite weight of solution, since the subsequent analyses were conducted entirely with weight burettes.

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[17] Chemical Analysis of Iron, 7th ed., p. 123.

[18] J. Russ. Phys. Chem. Soc., 42, p. 856; 1910.

The following solutions were employed in the investigation:

TABLE I

Permanganate Solutions Used

Solution	Approximate strength	$KMnO_4$	H_2O	Preserved
A_1	0.03 *N*	Purified	Purified	Dark
A_2	0.03 *N*	"	"	"
B	0.03 *N*+1% KOH	"	"	"
E	0.1 *N*	"	"	"
G	0.1 *N*	B & A	"	"
I	0.1 *N*	"	Ordinary	Light
K	0.1 *N*	Kahlbaum	"	"

These solutions were preserved in stock bottles provided with an inlet and exit tube with ground glass joint as in an ordinary gas wash bottle. To the inlet tube was sealed a U-tube containing some of the same solution as was in the bottle, thereby preventing changes in concentration of the latter. The exit tube was provided with a three-way stopcock and a tip by which the solution could be delivered to the weight burette by means of purified compressed air.

(*f*) **Stability of Permanganate Solutions.**—At first it was thought necessary to protect these solutions with black paper, but later experiments showed that in the course of several months no appreciable decomposition took place in the solutions exposed to diffused daylight, provided they were first freed from peroxide and were protected from dust and other reducing substances, and that only purified air entered the bottles. Solution I, for example, prepared from ordinary distilled water, and permanganate containing appreciable peroxide, which was removed by a single filtration through asbestos, did not suffer decomposition within the limits of observation (1 part in 2000) on standing for two months without protection from the light; even though it was intentionally exposed to bright sunlight for several hours soon after it was prepared. In connection with this observation, which simply confirms previous work of others,[19] it is desirable

[19] Morse, Hopkins and Walker: Am. Chem. J., 18, p. 401; 1896. Gardner and North: J. Soc. Chem. Ind., 23, p. 599; 1904. Warynski and Tscheishvili: J. Chim. Phys., 6, p. 567; 1908.

to call attention to another point in connection with the stability of permanganate solutions, which so far as I know has not been previously noted, or published. Under conditions which rapidly reduce neutral permanganate solutions, e. g., the presence of dust, reducing gases, or precipitated peroxide, decomposition is greatly retarded by the addition of a small amount of alkali. It was upon the basis of this observation, first noted qualitatively, that Solution B was prepared with 1 per cent of potassium hydroxide. Results with this solution were entirely satisfactory, but since the other solutions, when protected from reducing substances, were perfectly stable, the use of alkaline solutions for this work was found unnecessary. Under commercial conditions, however, where it is not always practicable to protect the solutions, the addition of a small amount of alkali will add to their stability.

(*g*) **Manganese Sulphate.**—Pure material was prepared from 300 grams of Kahlbaum's crystallized manganese sulphate ("Zur analyse"), the operations being conducted entirely in platinum. It was dissolved in water and filtered to remove a small amount of insoluble matter. It was next saturated with hydrogen sulphide, producing a small amount of a black precipitate which was found to contain copper. Additional hydrogen sulphide and a small amount of ammonia produced a precipitate entirely pink, which was filtered out. The hydrogen sulphide was expelled, a few drops of sodium hydroxide were added and the solution was boiled and filtered; the precipitate being found to contain iron. This last operation was twice repeated, the third precipitate being free from iron. An excess of pure, freshly prepared ammonium carbonate was then added and the precipitate of $MnCO_3$ washed with hot water, by decantation and suction, till free from sulphate. It was dissolved in a slight excess of hydrochloric acid and crystallized twice as $MnCl_2$, $4H_2O$ (at $-5°$). The latter crystals were treated with an excess of sulphuric acid, and heated in a double walled platinum dish till almost all the excess sulphuric acid was expelled. The product was entirely soluble in water, and contained a slight excess of sulphuric acid as determined in subsequent tests (Table II, A, p. 728).

(*h*) **Sodium Oxalate.**—Two samples were employed, one which had been especially purified by the author for a previous investigation,[20] and a larger sample prepared especially for this Bureau, and which was found to have a reducing value equal to the former, within the limits of 1 part in 2000.

(*i*) **Ferrous Sulphate** and ferrous-ammonium sulphate were employed indiscriminately after it was found that the solutions possessed about the same stability. The C. P. salts as purchased were employed, since their exact composition was not important. For use with 0.03 *N* permanganate, the solution was prepared according to Blair, with 12.4 g ferrous-ammonium sulphate (or 8.8 g crystallized ferrous sulphate) and 50 cc concentrated sulphuric acid per kilogram of solution. For use with 0.1 *N* permanganate, a solution containing 39.2 g ferrous-ammonium sulphate (or 27.8 g ferrous sulphate) and 50 cc concentrated sulphuric acid per kilogram was prepared. If phosphoric acid was employed, as recommended by Dudley,[21] it replaced half of the sulphuric acid in the 0.03 *N* solutions; but was added in addition to the regular amount of sulphuric acid in the 0.1 *N* solutions.

Stability of the ferrous sulphate solution.—Incidental observations upon the change in strength of 0.03 *N* ferrous ammonium sulphate indicated that the rate of oxidation, though slow, was erratic, due no doubt to variation in the extent of its exposure to air. With 0.1 *N* ferrous sulphate and ferrous ammonium sulphate, the daily rate of oxidation under the conditions used was approximately 1 part in 500, i. e., about 1 per cent in five days, over considerable periods. This rate will depend no doubt upon the conditions of its preservation, and is of interest only as indicating how often its strength should be checked up for work of any desired degree of accuracy. Ratios obtained at the beginning and end of various series of determinations showed that no appreciable change took place in a period of a few hours, thus confirming the observation of Baskerville and Stevenson.[22]

(*j*) **Nitric Acid** of regular C. P. grade was employed; in the concentrated form, and diluted to 25 per cent and 3 per cent by vol-

[20] Blum: J. Am. Chem. Soc., 34, p. 123; 1912.
[21] Blair: Chemical Analysis of Iron, 7th ed., p. 125.
[22] J. Am. Chem. Soc., 33, p. 1104; 1911.

ume. The former two solutions were preserved in the dark, since it has been recently shown by Reynolds and Taylor [28] that nitric acid as weak as 10 per cent is decomposed by light, but that recombination takes place in the dark.

(*k*) **Bismuthate.**—Two samples of C. P. sodium bismuthate were employed, one from Baker and Adamson and one from Eimer and Amend. These two samples differed very markedly in appearance, the former being dark brown, and the other yellow. In spite of this fact, no difference could be detected between them as regards their suitability for this oxidation. It is well to mention however that this compound, of more or less indefinite composition, is somewhat unstable, and if preserved for over six months should be tested for its efficiency of oxidation.

(*l*) **Ferric Nitrate.**—In order to test the effect of ferric salts upon this method, it was necessary to obtain iron, or some salt of iron which was free or practically free from manganese. This proved to be a difficult task, and after testing American ingot iron, and a large number of ferrous and ferric salts, the only one found satisfactory was a sample of Merck's crystallized ferric chloride, which contained less than 0.001 per cent manganese. To convert this to nitrate, it was first converted to sulphate by evaporation to the appearance of fumes with an excess of sulphuric acid, and the sulphate was precipitated with ammonia, washed and dissolved in nitric acid. The resulting salt was free from chloride (of which traces interfere in the bismuthate method) and contained only a small amount of sulphate (which is without effect on this method).

(*m*) **Use of Weight Burettes.**—Simple weight burettes were made by drawing down the tips of cylindrical graduated separatory funnels (50 and 100 cc). The increased accuracy gained by the use of weight burettes is especially desirable in an operation involving a back titration, and also the ratio of the two solutions used. Weighings were usually made to 0.01g, except in the case of the smaller amounts of manganese sulphate solutions, which were weighed to 0.005g or in some cases 0.001g. The titrations were usually made in Erlemmeyer flasks of convenient size.

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[28] J. Chem. Soc., **101**, p. 131; 1912.

5. STANDARDIZATION OF PERMANGANATE WITH SODIUM OXALATE

Nothing is to be added to the conclusions of McBride,[24] except to emphasize their relation to the present problem. The conditions recommended by him for the standardization of 0.1 *N* permanganate are briefly as follows: Volume of 250 cc; acidity 2 per cent sulphuric acid by volume; initial temperature, 80° to 90°; slow addition of permanganate, especially at beginning and end; final temperature not less than 60°; and end point correction by comparison with a blank containing a known amount of the permanganate. His statement that the variation in results over a wide range of conditions does not exceed 1 part in 1000, applies to titrations involving the use of about 50 cc of 0.1 *N* permanganate. If, however, 0.03 *N* permanganate, commonly used in the bismuthate method, is standardized with sodium oxalate, slight variations in the conditions may cause a relatively much larger error, especially if as is not uncommon, only about 25 cc of permanganate is employed. For standardization of 0.03 *N* permanganate, the conditions of McBride were employed, except that the initial volume was 75 cc instead of 250 cc, i. e., the oxalate concentration was about the same as for 0.1 *N* permanganate. In this way the uncertainty in the end point caused by titrating in a large volume with weak permanganate, can be reduced to a minimum. For accurate work, however, the end point correction should be made, since the object of this titration is to determine the absolute oxidizing power of the permanganate. With so small a volume of solution it is usually necessary to reheat it to 60° to 70° before completing the titration. These conditions, as shown by McBride, represent a minimum consumption of permanganate, i. e., the iron or manganese values are a maximum. Any deviation from these conditions will tend to lower the iron or manganese values, which it is believed accounts in part for the discrepancy noted by Brinton and others between values derived from sodium oxalate and from manganese sulphate. For calculation of the manganese value from the sodium oxalate, the factor 0.16397 was employed.

[24] J. Am. Chem. Soc., 34, p. 415; 1912.

6. STANDARDIZATION OF PERMANGANATE WITH MANGANOUS SULPHATE

(*a*) **Standardization of Manganous Sulphate Solutions.**—The two methods commonly used for determining the strength of a manganous sulphate solution are (*a*) precipitation as manganese ammonium phosphate and ignition to pyrophosphate, and (*b*) evaporation of the solution and heating the residue to a certain temperature. Unfortunately both of these will yield high results if the solution contains substances other than manganese sulphate; whether in the original salt or derived from the glass in which the solution is preserved. But even with pure solutions the results are of uncertain accuracy, especially in the case of the pyrophosphate as above mentioned (p. 717). Weighing as sulphate was therefore adopted as a means of securing a known amount of manganese. The chief source of uncertainty here is the temperature of the final heating, a point upon which the evidence is rather uncertain and contradictory. Volhard [25] was able to obtain constant weight with a special burner, but not with a Bunsen burner. Marignac [26] determined the atomic weight of manganese by heating the sulphate "nearly to red heat." Meineke [27] determined this element as the sulphate, which after being heated to a temperature not stated, was completely soluble in water. Friedheim [28] heated the salt to 360° to 400°, while Gooch and Austin [29] obtained constant weight by heating in double crucibles, 1 cm apart, the outer one being at red heat, a procedure since recommended by Treadwell.[30] In determining the water of crystzallization of the various hydrates of manganous sulphate, Thorpe and Watts [31] heated the salt to 280°, Linebarger [32] to 170° to 180° and Cottrell [33] to 270° to 280°, though the latter found that no decomposition took place at 350°. Richards and Fraprie [34] showed however that as much as 0.1 per cent H_2O remained in the salt after

[25] Ann. Chem., 198, pp. 318–364; 1879.
[26] Arch. Sci. phys. et Nat. [3] 10, p. 25; 1883.
[27] Chem. Ztg., 9, pp. 1478, 1787; 1885.
[28] Z. anal. Chem., 38, p. 687; 1899.
[29] Am. J. Sci., 5, p. 209; 1898.
[30] Treadwell and Hall: Quant. Analysis, II, p. 104.
[31] J. Chem. Soc., 37, p. 113; 1880.
[32] Am. Chem. J., 15, p. 225; 1893.
[33] J. Phys. Chem., 4, p. 637; 1900.
[34] Am. Chem. J., 26, p. 75; 1901.

heating for one half hour at 350°, but that five minutes heating at 450° produced complete dehydration without decomposition. Classen [35] and Blair [36] recommend heating to dull red; while Fresenius [37] declares that accurate results can be obtained only by chance, as it is impossible to expel all excess sulphuric acid without decomposing the salt.

The following experiments were conducted to determine the temperature to which manganous sulphate may and must be heated, to expel all the water or excess sulphuric acid and to obtain the normal anhydrous salt. About 2 g of the salt was heated in an open platinum crucible in a small electrically heated muffle, temperatures of which up to 400°, were measured with a 450° nitrogen-filled thermometer, and above 400° with a platinum-rhodium thermocouple calibrated at this Bureau. The crucible was kept covered in the desiccator and upon the balance, where it was weighed against a similar crucible as a tare. The results of three series of heatings are shown in Table II, the figures in the last column being calculated from the weight which remained practically constant from 450° to 500°.

TABLE II

Temperature of decomposition of manganous sulphate

A

(Manganous sulphate prepared as on p. 723)

Temperature (°C)	Time (hours)	Weight of $MnSO_4$ (g)	Per cent of constant weight	Remarks
300	1	2.3655	100.10	
350	16	49	100.07	
420	2	42	100.04	
420	2	40	100.03	
420	17	37	100.02	
480	4	33	100.00	
480	17	32	100.00	
540	4	31	100.00	
550	18	29	99.99	
620	4	25	99.97	Slight darkening

[35] Ausgew. Meth. d. Analytischen Chem., I, p. 363.
[36] Chem. Analysis of Iron, 7th ed., p. 126.
[37] Fresenius-Cohn: Quant. Analysis, I, p. 297.

The final product was dissolved in water, and the insoluble residue filtered out, washed and ignited, yielding 0.0006 g Mn_3O_4, equivalent to 0.0004 g Mn, or 0.0011 g $MnSO_4$. The filtrate was evaporated for series B.

B

Temperature (°C)	Time (hours)	Weight $MnSO_4$ (g)	Per cent of constant weight	Remarks
300	3	2.3659	100.15	
340	18	44	100.10	
400	4	28	100.03	
440	18	23	100.00	
480	5	22	100.00	
480	18	22	100.00	
570	5	17	99.98	Slight darkening.
570	18	09	99.95	Decided darkening.

A few drops $H_2SO_3 + H_2SO_4$ were added to the final product, which was then reheated for series C.

C

Temperature (°C)	Time (hours)	Weight $MnSO_4$ (g)	Per cent of constant weight	Remarks
300	4	2.3717	100.40	
320	17	690	100.28	
400	5	41	100.08	
400	17	34	100.05	
460	5	23	100.00	
460	17	23	100.00	
525	5	21	99.99	
525	17	19	99.98	
580	5	17	99.97	Slight darkening.
580	17	08	99.94	Decided darkening.

From Table II the following conclusions may be drawn:

(1) Manganous sulphate does not undergo any appreciable decomposition upon prolonged heating to temperatures up to 500°. (2) At temperatures from 550° to 600° (from incipient to dull redness) this salt decomposes slowly. (3) The anhydrous normal salt can be obtained only by heating for considerable periods at 450° to 500°, especially if an excess of sulphuric acid be originally

present. (4) Attempts to obtain the pure salt by heating directly over a flame, or even in a double crucible, without temperature regulation or measurement, must be subject to considerable uncertainty.

Having now a means for obtaining a known weight of manganous sulphate, solutions of known strength (from 0.002 to 0.005 g manganese per g of solution) were prepared by dissolving a known weight of the pure salt, heated to constant weight at 450° to 500°, and making up to a definite weight of solution, the manganese content of which was calculated by the use of the factor $MnSO_4 \rightarrow Mn = 0.3638$. In one case, for example, 5.749 g pure anhydrous $MnSO_4$ was dissolved in water and the solution made up to exactly 1000 g; producing a solution 1 g of which contained 0.002091 g Mn, which value was confirmed by evaporation of a weighed portion of the solution and heating to 475° to constant weight. Determinations made by another chemist upon this solution, by evaporation and heating for a short time to "dull redness", yielded the values 0.002100, 2092, 2103, and 2101; the mean value 0.002099 being therefore 0.38 per cent too high, i. e. an error of about 1 part in 250. Upon another solution prepared in the above manner, and containing 0.002000 g Mn per solution, the same chemist obtained by direct heating to dull redness 0.002004, 2006 and 2005; i. e. the results were high, in spite of the fact that in the latter series at least, very slight decomposition had evidently taken place in the bottom of the crucible. Apparently therefore those parts of the salt on the sides of the crucible had not been heated to the necessary temperature for a sufficient length of time to expel all water or excess acid. In view of these facts, the desirability of substituting for the manganous sulphate, some other standard, such as sodium oxalate, is very evident.

(*b*) **Effect of Conditions upon Standardization with Manganous Sulphate**—(1) *Ferrous sulphate-permanganate ratio.*—This ratio, which is fundamental for the accuracy of the method, is usually determined by means of a blank experiment; that is, a determination is run through in the absence of manganese, under the conditions to be used in the regular analyses. This procedure, which was evidently devised for the purpose of eliminating errors due to

impurities in the reagents, has been found to be unnecessary, i. e. the ratio so obtained is the same as that obtained by direct titration of the ferrous sulphate in the same volume. This is due to the fact that on the one hand the bismuthate oxidizes readily any traces of nitrous acid which may be present in the nitric acid, and that on the other hand nitric acid of the strength present in the final solution does not have any effect upon the ferrous salt in the short time necessary for a titration. If, however, the ferrous salt be titrated in the presence of nitric acid containing small amounts of nitrous acid, which has not been treated with bismuthate, an excessive amount of permanganate will be consumed, due to the reducing action of the nitrous acid upon the permanganate, which takes place more rapidly in the presence of ferrous salt than in its absence. It must be clearly understood that conducting the blank experiment in the usual way does not obviate the necessity of avoiding the presence of nitrous acid in the solutions of manganese used in the standardizations or analyses; since, as indicated by Blair [38], nitrous acid will reduce part of the permanganic acid, precipitating manganese peroxide, which is not reoxidized by the bismuthate.

While not strictly necessary, the determination of this ratio by means of a blank affords a convenient means of testing the efficacy of the filter, and has therefore been followed in all this work. The conditions found most satisfactory are as follows: To 50 cc of nitric acid (25 per cent by volume), add a small amount of bismuthate; shake and allow to stand a few minutes, dilute with 50 cc of 3 per cent nitric acid; filter through the asbestos filter and wash with 100 cc of 3 per cent nitric acid. To the filtrate, which should be perfectly clear, add a volume of ferrous sulphate approximately equal to that to be used in the subsequent determinations (25 to 50 cc) and titrate at once to the first visible pink. Even for the most accurate work, no end point correction is required for this titration, provided only that the solutions are always titrated to the same color, and that about the same volumes are used in the standardization and analyses.

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[38] 7th ed., p. 128.

(2) *Amount of manganese present in a determination.*—One of the serious limitations of this method is the small amount of manganese generally determined, making it somewhat unreliable for high grade materials. Blair recommends the presence of from 0.01 to 0.02 g Mn, involving the use of a sample of manganese ore of only 0.02 g, obtained by taking an aliquot of the solution of 1 g of the ore. Ibbotson and Brearley[39] state that the method is equally applicable for large or small amounts of manganese without, however, giving the evidence for this conclusion. Since with 0.03 *N* permanganate, 0.015 Mn is the largest amount that can be conveniently determined, the following experiments were conducted with approximately 0.1 *N* $KMnO_4$ and $FeSO_4$. The results are expressed in terms of the manganese value of 1 g of the permanganate solution. It should be noted that a high result indicates incomplete oxidation of the manganese by the bismuthate. In these and the following series the following conditions were tentatively employed, and the variation produced by a change of one condition was noted in each series of experiments. The manganese sulphate was oxidized at room temperature in a volume of about 50 cc, containing 25 per cent nitric acid by volume. An excess of bismuthate (about 0.5 g) was added, the solution was agitated for one minute, the sides of the flask were rinsed down with 50 cc of 3 per cent nitric acid, and the solution at once filtered with suction through the asbestos filter, previously coated with bismuthate. The flask and filter were washed several times with 3 per cent nitric acid of which about 100 cc was used. The filtration and washing required from one to three minutes. To the filtrate ferrous sulphate was added immediately in slight excess, which was at once titrated with permanganate.

[39] Chem. News, 82, p. 269; 1900.

TABLE III

Effect of Amount of Manganese in the Presence of Variable Amounts of Iron

$KMnO_4$ Solution I

Manganese values calculated from sodium oxalate	Manganese values calculated from manganous sulphate		
Values determined over a period of three weeks	Grams manganese present	Grams iron present	1 g $KMnO_4$ solution = g Mn
	A		
0.001090	0.03	...	0.001088
91	0.03	...	88
92	0.03	1.0	89
92	0.03	1.0	89
91	0.03	1.0	91
91	0.03	1.0	92
91			
92	B		
91	0.05	...	90
92	0.05	...	92
90	0.05	...	90
90	0.05	...	87
89	0.05	...	90
92	0.05	1.0	89
91	0.05	2.0	90
91	0.05	3.0	89
90			
90	C		
	0.10	...	89
	0.10	...	89
	0.10	...	1161
	0.10	...	1089
	0.10	...	97
	0.10	...	96
	0.10	...	1107
	0.10	...	1088
Av. 0.001091	Av. of A & B.....0.001089		

From Table III it is evident that for amounts of manganese up to 0.05 g the method is accurate within the limits of error, i. e. about 1 part in 500, while results obtained with as much as 0.10 g Mn are decidedly erratic, only one-half approaching the correct values. It is apparent therefore, that about 0.05 g Mn is the practical limit under these conditions. This amount is, however, far more satisfactory than only 0.01–0.02 g, and permits the use of 0.10 g of high grade manganese ore, a decided advantage. As seen in series A and B, the results with as much as 3 g iron present, are

entirely satisfactory. The agreement of the sodium oxalate and manganese sulphate values will be discussed later.

(3) *Acidity, Volume, Time of Standing, etc.*—The results of several series of experiments to determine the effect of various conditions upon the bismuthate method are summarized in the following table:

TABLE IV

Effect of Conditions upon Bismuthate Standardization

($KMnO_4$ Solution K)

Series	Method	Modification	Determinations	1 g $KMnO_4$ solution—g Mn
A 1	$Na_2C_2O_4$	Standard, p. 726	6	0.001098
B 1	$MnSO_4$	Standard, p. 732	9	1098
C 1	$MnSO_4$	Initial conc. HNO_3, 10 per cent	3	1114
2	$MnSO_4$	Initial conc. HNO_3, 40 per cent	3	1098
D 1	$MnSO_4$	Initial volume, 150 cc	3	1098
E 1	$MnSO_4$	Shaken with bismuthate 15 sec	3	1097
F 1	$MnSO_4$	Stood before filtration 10 min	3	1097
2	$MnSO_4$	Stood before filtration 30 min	1	1097
G 1	$MnSO_4$	Stood after filtration 10 min	1	1097
2	$MnSO_4$	Stood after filtration 20 min	1	1097
3	$MnSO_4$	Stood after filtration 30 min	1	1098
H 1	$MnSO_4$	Stood after addition of $FeSO_4$ 10 min	1	1096
2	$MnSO_4$	Stood after addition of $FeSO_4$ 30 min	1	1084
I 1	$MnSO_4$	Addition of H_3PO_4—5 cc	3	1099
		Mean of all $MnSO_4$ values except C_1 and H_2		1098

Of the $MnSO_4$ values given in Table IV, the individual determinations of all except those in C_1 and H_2 varied less than 1 part in 500 from the mean, showing that accurate results can be obtained over a very wide range of conditions. The only conditions found to produce appreciable errors were (a) deficiency of nitric acid and (b) allowing the solution to stand more than 10 minutes after the addition of the ferrous sulphate, of which about 10 cc excess was present. Since there is no occasion for either of these conditions to arise in good practice, the method may be considered accurate under all ordinary conditions of procedure, an important criterion for a standard method of analysis.

(4) *Use of Phosphoric Acid.*—The addition of this reagent as recommended by Dudley, was found convenient though not necessary, since with 0.1 *N* solutions there was no difficulty in obtain-

ing a sharp end point within 0.03 cc of permanganate, without its use. If used, it should be added to the ferrous sulphate solution beforehand, rather than during the titration, since in the latter case a white precipitate, probably consisting of basic bismuth phosphate separates, rendering the end point slightly less distinct. With very large amounts of iron, e. g. 3–5 g, such as would have to be used if Mn in steel were determined with 0.1 *N* permanganate, it was found that addition of phosphoric acid possesses no advantage, since it tends to produce a pink color, due probably to the formation of an acid ferric phosphate,[40] which obscures the end point as much as does the ferric nitrate. The use of 0.1 *N* solutions is therefore recommended only for manganese ores and similar high-grade products, in which the highest accuracy is desired.

(*c*) **Probable Course of Reactions.**—From Table IV, some light may be thrown upon the probable course of the reactions when manganese is oxidized by bismuthate. At least two reactions are probable (a) direct oxidation to Mn^{VII} and (b) interaction of unoxidized Mn^{II} with the Mn^{VII}, precipitating Mn^{IV}, which is then removed from the oxidizing influence of the bismuthate. If these two reactions may take place, the problem resolves itself into a determination of the conditions under which reaction (a) will be accelerated and (b) will be retarded, so that (a) goes practically to completion before (b) can take place to an appreciable extent. The favorable conditions for (b) as conducted in the Volhard method, for example, are slight acidity and high temperature; which should therefore be avoided in the bismuthate oxidation. That this explanation is plausible is shown by a comparison of C_1 and C_2. That complete oxidation may be effected in a short time is indicated in E_1; in which connection the necessity for thorough agitation must be emphasized. Other experiments, not recorded here, showed that with 0.05 g or more of Mn, complete oxidation could not be effected if the solution was not thoroughly agitated. In the earlier experiments in this investigation, the solutions were artificially cooled to about 5°; but after it was found that results at room temperature, 20° to 25°, were entirely satisfactory, artificial cooling was dispensed with.

[40] Erlenmeyer and Heinrich: Ann. Chem., 190, p. 191: 1877.

(*d*) **Conditions Recommended.**—Correct results can be obtained under the following conditions: To the manganese solution containing 20 to 40 per cent nitric acid (free from nitrous acid) in a volume of 50 to 150 cc, add a slight excess of bismuthate (usually 0.5 to 1.0 g), agitate thoroughly for about one-half minute, wash down the sides of the flask with 3 per cent nitric acid, filter through asbestus, wash with 100 cc of 3 per cent nitric acid, add a slight excess of ferrous sulphate, and titrate at once with permanganate. For iron and steel, 0.03 *N* solutions as described by Blair are satisfactory.

For ores and ferromanganese 0.1 *N* permanganate solution may be employed, and an amount of material containing about 0.05 g manganese. For the rapid solution of ores, a method recommended by Blair[41] has been found convenient. One g of the ore is fused in a large platinum crucible with 10 g potassium bisulphate, 1 g of sodium sulphite and 0.5 g sodium fluoride. The heating should be very slow till effervescence ceases. After complete fusion the product is cooled, then heated carefully with 10 cc concentrated sulphuric acid, cooled, dissolved in water, and made up to a definite volume. The slight precipitate of barium sulphate usually present will not influence the manganese determination.

7. AGREEMENT OF VALUES DERIVED FROM SODIUM OXALATE AND MANGANOUS SULPHATE

TABLE V

Comparison of $Na_2C_2O_4$ and $MnSO_4$ values

Permanganate solution	Manganese values derived from—			
	Sodium oxalate		Manganese sulphate	
	Determinations	1 g $KMnO_4$=g Mn	Determinations	1 g $KMnO_4$=g Mn
A_1	3	0.0003465	3	0.0003469
A_2	4	0.0003462	3	0.0003462
B	6	0.0003454	5	0.0003458
E	9	0.001096	4	0.001094
G	7	0.001091	5	0.001090
I	18	0.001091	28	0.001089
K	6	0.001098	29	0.001098

[41] Private communication.

Consideration of the values in Table V shows plainly that no greater difference than 1 part in 500 exists between the results derived from the sodium oxalate and manganese sulphate, respectively, instead of the former values being 1 per cent lower, as claimed by Brinton and others. In fact in the case of the 0.1 *N* solutions, the only ones in which an accuracy of more than 1 part in 500 is realizable, the manganous sulphate values show a tendency to be from one to two parts per thousand lower than the sodium oxalate results. It is at least interesting though perhaps not significant, that if the value 55.00 instead of 54.93 be used for the atomic weight of manganese, the results with the 0.1 *N* solutions agree in every case to within 1 part in 1000.

8. ANALYSIS OF PURE PERMANGANATE CRYSTALS

Additional evidence of the correctness of the above values was found in the analysis of the pure permanganate prepared as described on page 721, which contained 0.38 per cent water. The salt should therefore contain 34.63 per cent manganese, instead of 34.76 per cent, the theoretical content for pure anhydrous $KMnO_4$. This difference with specially purified permanganate indicates clearly the probable presence of water as well as manganese peroxide in C. P. permanganate, rendering it unsuitable as a primary standard. Manganese was determined gravimetrically by precipitation with ammonium sulphide; the manganese sulphide being washed with dilute ammonium sulphide, ignited in a weighed crucible, treated with sulphrous and sulphuric acids, evaporated, heated to 450° to constant weight, and weighed as $MnSO_4$. The manganese in the filtrates was determined colorimetrically. Results of duplicate analyses were 34.70 per cent and 34.66 per cent, the mean 34.68 per cent agreeing closely with the theoretical value 34.63 per cent. The oxidizing value of this permanganate was determined by means of solutions A_1, A_2, and E (Table V) which were prepared by the solution of an exact weight of the salt in a definite weight of solution. In A_1 and A_2, exactly 1 g $KMnO_4$ was dissolved and diluted to 1 kg; yielding solutions having an oxidizing value equivalent to 34.65 per cent Mn (average of all sodium oxalate and manganous sulphate values for A_1 and

A_2). Solution E contained 3.1606 g of the salt per kilogram, and possessed an oxidizing power equivalent to 34.65 per cent Mn, (derived from the average of all sodium oxalate and manganous sulphate values for solution E). Solutions B, G, I, and K were prepared of only approximately the desired strength, and the results have no relation to the composition of the solid permanganate employed.

9. ANALYSIS OF MANGANESE ORES

Analyses of the Bureau of Standards Manganese Ore (Standard Sample No. 25) by means of permanganate I, gave as the average of nine determinations, 56.30 per cent Mn upon the basis of the sodium oxalate standardization, and 56.20 per cent if calculated from the manganous sulphate. These results are in good agreement with the mean value 56.36 per cent derived from all determinations upon the certificate, and with the value 56.33 formerly found by the author with the bismuthate method, using sodium oxalate as the standard. Unfortunately comparisons based upon this sample are not necessarily conclusive, since the mean value 56.36 per cent is derived from results ranging from 56.15 to 56.63, obtained by eight chemists using a variety of methods, the lack of agreement of which is illustrated. If the bismuthate results by the author are correct, a conclusion made highly probable by the work here described, the value of the ore lies between 56.20 and 56.30 per cent manganese; and many of the values found by other methods, by the author and others, are too high. That the tendency of many commercial methods is to yield results higher than those by the method here recommended, is shown in the results of analyses of three manganese ores by the author and two well known commercial chemists.

TABLE VI

Analyses of Manganese Ores

Analyst	Method	Ore I	Ore II	Ore III
A	Bismuthate	52.47	52.53	50.50
B	Modified acetate	52.40	52.29	50.52
Author	Bismuthate	51.98	52.08	50.12

The differences here shown, amounting to 0.8 to 1.0 per cent of the manganese present, are by no means insignificant. The discrepancy between the results by A and the author, both using the bismuthate method, was found to be due mainly to differences in the method of standardizing the manganous sulphate solution. (See p. 730.) These results show clearly the necessity for a thorough investigation of other methods for determining manganese, in order that accurate results may be uniformly obtained.

10. SUMMARY

1. To obtain normal anhydrous manganous sulphate, the salt may and must be heated for a considerable time at 450° to 500°, i. e., just below red heat.

2. Standardizations of permanganate solutions (both 0.03 *N* and 0.1 *N*) by means of sodium oxalate, manganous sulphate, and solid permanganate agree within the experimental error, which in the bismuthate method could not be reduced much below 1 part in 500. Taken together with the agreement of sodium oxalate and iron values, and the experiments upon the reduction and reoxidation of permanganate, the absolute accuracy of the above results, within the experimental limits, is rendered almost certain.

3. In view of the difficulties attending the use of manganous sulphate, standardization by means of sodium oxalate, under definite, but easily realizable conditions, is recommended.

4. Results by the bismuthate method are accurate over a very wide range of conditions, for amounts of manganese up to 0.05 g.

5. For accurate determinations on rich ores, etc., the use of 0.1 *N* permanganate is recommended, while for iron and steel the method described by Blair is entirely satisfactory.

6. The statement of Blair that "this method for materials containing small amounts of manganese, say up to 2 per cent, is more accurate than any other method, volumetric or gravimetric," may be extended to include materials containing large amounts of manganese.

7. Filtered permanganate solutions preserve their strength when exposed to diffused light, if protected from dust and reducing substances. In the presence of the latter, alkaline permanganate solutions decompose less rapidly than do neutral solutions.

The author desires to express his thanks to Dr. W. F. Hillebrand for valuable suggestions and advice during the course of this investigation.

WASHINGTON, June 21, 1912.

INDEX TO VOLUME 8

A

Acidimetric standard, Benzoic acid as an, 643.

Agnew, P. G., Tubular electrodynamometer for heavy currents, 651.

Alternating - current work, Resistance coils for, 495.

Ampere, Determination of the international, in absolute measure, 269.

B

Benzoic acid as an acidimetric standard, 643.

Blum, William, Determination of manganese as sulphate and by the sodium bismuthate method, 715.

Hydrolysis of sodium oxalate and its influence upon the test for neutrality, 519.

Brooks, H. B., Deflection potentiometers for current and voltage measurements, 395.

Outline of design of deflection potentiometers, with notes on the design of moving-coil galvanometers, 419.

Buckingham, Edgar, Correction for emergent stem of the mercurial thermometer, 239.

Deduction of Wien's displacement law, 545.

C

Correction for emergent stem of the mercurial thermometer, 239.

Curtis, H. L., Frederick W. Grover and, measurement of the inductances of resistance coils, 455.

Curtis, H. L., F. W. Grover and, Resistance coils for alternating-current work, 495.

D

Deduction of Wien's displacement law, 545.

Deflection potentiometers for current and voltage measurements, 395.

Determination of the international ampere in absolute measure, 269.

Determination of manganese as sulphate and by the sodium bismuthate method, 715.

Determination of total sulphur in india rubber, 445.

Dorsey, N. E., E. B. Rosa, and J. M. Miller, Determination of the international ampere in absolute measure, 269.

E

Electrodynamometer, A tubular, for heavy currents, 651.

F

Formulas and tables for the calculation of mutual and self-inductance (revised), 1.

Four-terminal conductor and the Thomson bridge, 559.

G

Grover, F. W., E. B. Rosa and, Formulas and tables for the calculation of mutual and self-inductances (revised), 1.

Grover, F. W., and H. L. Curtis, Measurement of the inductances of resistance coils, 455.

Resistance coils for alternating-current work, 495.

H

Harper, D. R., Thermometric lag, 659.

Helium gas, electrically conducting, Luminous properties of, 487.

Hydrolysis of sodium oxalate and its influence upon the test for neutrality, 519.

I

Inductance, mutual and self-, Formulas and tables for the calculation of, 1.
Inductances of resistance coils, Measurement of the, 455.

L

Luminous properites of electrically conducting helium gas. II. Reproducibility, 487.

M

Manganese as sulphate and by the sodium bismuthate method, Determination of, 715.
McBride, R. S., Standardization of potassium permanganate solution by sodium oxalate, 611.
Measurement of the inductances of resistance coils, 455.
Miller, J. M., N. E. Dorsey, E. B. Rosa and, Determination of the international ampere in absolute measure, 269.
Morey, G. W., Benzoic acid as an acidimetric standard, 643.

N

Neon, Wave lenghts of, 539.
Nutting, P. G., Luminous properties of electrically conducting helium gas. II. Reproducibility, 487.

O

Outline of design of deflection potentiometers, with notes on the design of moving-coil galvanometers, 419.

P

Potassium permanganate solution by sodium oxalate, Standardization of, 611.
Potentiometers, Deflection, for current and voltage measurements, 395.
Potentiometers, Outline of design of deflection, with notes on the design of moving-coil galvanometers, 419.
Priest, I. G., Wave lengths of neon, 539.

R

Resistance coils for alternating-current work, 495.
Resistance coils, Measurement of the inductances of, 455.
Rosa, E. B., N. E. Dorsey, and J. M. Miller, Determination of the international ampere in absolute measure, 269.
Rosa, E. B., and F. W. Grover, Formulas and tables for the calculation of mutual and self-inductance (revised), 1.
Rubber, Determination of total sulphur in, 445.

S

Sodium oxalate and its influence upon the test for neutrality, Hydrolysis of, 519.
Standardization of potassium permanganate solution by sodium oxalate, 611.
Sulphur, Determination of total, in india rubber, 445.

T

Thermometer, mecurial, Correction for emergent stem of the, 239.
Thermometric lag, 659.
Thomson bridge, Four-terminal conductor and the, 559.
Tubular electrodynamometer for heavy currents, 651.
Tuttle, J. B., C. E. Waters and, Determination of total sulphur in india rubber, 445.

W

Waters, C. E., and J. B. Tuttle, Determination of total sulphur in india rubber, 445.
Wave lengths of neon, 539.
Wenner, Frank, Four-terminal conductor and the Thomson bridge, 559.
Wien's displacement law, Deduction of, 545.

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69. On the Standard Scale of Temperature in the Interval 0° to 100° C. . . . . *C. W. Waidner and H. C. Dickinson*
70. Clark and Weston Standard Cells . . . . . *F. A. Wolff and C. E. Waters*
71. The Electrode Equilibrium of the Standard Cell . *F. A. Wolff and C. E. Waters*
72. A Comparative Study of Plain and Frosted Lamps . *E. P. Hyde and F. E. Cady*
73. The Variation of Resistances with Atmospheric Humidity . . . . . *E. B. Rosa and H. D. Babcock*

74. On the Self-Inductance of a Toroidal Coil of Rectangular Section . . . . . *Edward B. Rosa*
75. On the Self-Inductance of Circles . . . . *Edward B. Rosa and Louis Cohen*
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108. Errors in Magnetic Testing with Ring Specimens . . . . *Morton G. Lloyd*
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123. The Theory of the Hampson Liquefier . . . . . *Edgar Buckingham*
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125. The Daylight Efficiency of Artificial Illuminants . . . . . *H. E. Ives*
126. Coupled Currents in which the Secondary has Distributed Inductance and Capacity . . . . . *Louis Cohen*
127. Effect of Phase of Harmonics upon Acoustic Quality . . . . . *M. G. Lloyd and P. G. Agnew*
128. White Light from the Mercury Arc and its Complementary . *Herbert E. Ives*
129. The Regulation of Potential Transformers and the Magnetizing Current . . . . . *M. G. Lloyd and P. G. Agnew*
130. The Determination of the Constants of Instrument Transformers . . . . . *P. G. Agnew and T. T. Fitch*
131. Selective Radiation from Various Solids, II . . . . . *W. W. Coblentz*
132. Luminous Efficiency of the Firefly . . *Herbert E. Ives and W. W. Coblentz*
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DUE NOV 12 1917
DUE DEC 12 1917
FEB 14 1929
DUE JUN 1 1928
DUE MAR 30 1928
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